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Heuristic Methods of Construction of Sequential Questionnaire

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The paper concerns a method for a rapid detection of unknown states. A tool for the detection is a sequential questionnaire, whose new definition along with heuristic methods of construction are presented.

INTRODUCTION

A generally formulated problem is the following one: We look for an unknown state in which there is an object under observation. We suppose the knowledge of all alternative states $\{s_i\}_{i=1}^n$ $(n > 1)$ in which the object could occur. It is assumed that the determination of the unknown state could be done only through a set of observation functions $\{q_j\}_{j=1}^m$ $(m > 0)$ which are defined in the set of states of the object. The observation functions will be called *questions*. Every question $q \in \{q_j\}_{j=1}^m$ need not be defined for all alternative states. Thereby the question q is a partial mapping

$$q : \{s_i\}_{i=1}^n \to P(q) ,$$

where $P(q)$ is a finite set of values $q(s_i)$. The subset of all questions q_j for which $q_j(s)$ is defined is denoted by $I(s)$. Therefore

$$I(s) \subseteqq \{q_j\}_{j=1}^m .$$

The equality in the previous inclusion holds *iff* (if and only if) $q_j(s)$ is defined for all $j = 1, \ldots, m$. In practice the set $(\{q_j\}_{j=1}^m - I(s))$ includes all questions which are irrelevant to the state s.

We assume that the set of states $\{s_i\}_{i=1}^n$ and the set of questions $\{q_j\}_{j=1}^m$ satisfy the following condition

(1)
$$(\forall i, i' \in \{1, \ldots, n\};\ i \neq i')\,(\exists j \in \{1, \ldots, m\})\,(q_j \in I(s_i) \cap I(s_{i'})\ \&\ q_j(s_i) \neq q_j(s_{i'})) .$$

 Satisfaction of this condition guarantees that the states $\{s_i\}_{i=1}^n$ are *distinguishable.*

Simplification in the used formulations is enabled by the following restrictions.

We suppose, that each of the mentioned questions has exactly α different answers ($\alpha \geqq 2$). Thus $|P(q_j)| = \alpha$ for each $j = 1, \ldots, m$ ($|A|$ denotes the number of elements of the set A). Moreover, we suppose, that the elements of $P(q_j)$ are arranged in sequence, and $\varrho(q_j; k)$ ($k = 1, \ldots, \alpha$) denotes the k-th element of the $P(q_j)$.

In spite of this restriction, the following algorithm is applicable also if $P(q_j)$ have different numbers of elements for different j.

This paper propounds an instrument as well as instructions for construction of it that enables to determinate the state of the observed object. The tool is a *sequential questionnaire* (further referred as *questionnaire*) defined according to [2] with supplementary mappings defined on its nodes.

By a questionnaire we shall call an ordered quadruple (G, π, f, g) in which G is an orientated graph, V denotes the set of all nodes of G. The set of terminal, non-terminal nodes is denoted by W, U respectively. G has to meet the following conditions:

1) there are starts of α and only α edges in each the node $u \in U$;

2) only in one node $u_0 \in U$ there are no terminals of edges; u_0 will be called the *root*;

3) there is one and only one path from u_0 to each node $v \in (V - \{u_0\})$ in the graph G;

π is a function from V into $\langle 0, 1 \rangle$ which satisfy

$$\pi(u) = \sum_{v \in T_d(u)} \pi(v)$$

(the definition of the symbol $T_d(u)$ is below). Since $\pi(v)$ represents a probability of reaching the node v it holds $\pi(u_0) = 1$ usually.

f is a mapping assigning one state to each terminal node.

g is a mapping assigning one question to each nonterminal node.

For sets of nodes the following symbols are used:

$$T_d(u) = \{v \in V : \text{there is an edge from } u \text{ to } v\} \quad \text{for} \quad u \in U\,,$$

$$T_p(v) = \{u \in U : \text{there is a path from } u \text{ to } v\} \quad \text{for} \quad v \in (V - \{u_0\})\,,$$

$$T_p(u_0) = \emptyset\,,$$

$$T_t(u) = \{w \in W : u \in T_p(w)\} \quad \text{for} \quad u \in U\,,$$

$$T_t(w) = \{w\} \quad \text{for} \quad w \in W\,.$$

The *average length* $\bar{L}$ of a questionnaire reflects the average cost of answering the questionnaire (i.e. an average cost required to determine the state). It is defined by the relation:

$$\bar{L} = \bar{L}(G, \pi, f, g) = \sum_{u \in U} t(g(u))\, \pi(u)\,,$$

where $t(g(u))$ describes the *cost* of answering the question which is assigned to the node u by the mapping g.

Naturally, $t(q_j) > 0$ for $j = 1, \ldots, m$ is assumed. If $t(q_j) = 1$ for each $j = 1, \ldots, m$ then the above introduced definition of the average length corresponds to the definition introduced in [2].

For each $v \in V$, $F(v)$ will denote the set of states

$$F(v) = \bigcup_{w \in T_t(v)} \{f(w)\}\,.$$

Thus for terminal nodes w, $F(w) = \{f(w)\}$.

Example. Let us take 5 states s_1, s_2, s_3, s_4, s_5 and 3 observation functions (questions) q_1, q_2, q_3, each of which has $\alpha = 2$ values. Let they be defined in the following way:

$$\begin{array}{lll} q_1(s_1) \text{ undefined}\,, & q_1(s_2) = 1\,, & q_1(s_3) = 0\,, \\ q_2(s_1) = 0\,, & q_2(s_2) \text{ undefined}\,, & q_2(s_3) = 1\,, \\ q_3(s_1) = 0\,, & q_3(s_2) = 1\,, & q_3(s_3) \text{ undefined}\,, \end{array}$$

$$\begin{array}{ll} q_1(s_4) = 0\,, & q_1(s_5) = 1\,, \\ q_2(s_4) = 0\,, & q_2(s_5) = 1\,, \\ q_3(s_4) = 1\,, & q_3(s_5) = 0\,. \end{array}$$

Immediately it can be seen that the sets $\{s_i\}_{i=1}^5$ and $\{q_j\}_{j=1}^3$ satisfy the condition (1). An example of a questionnaire distinguishing these states is given in Fig. 1. In the

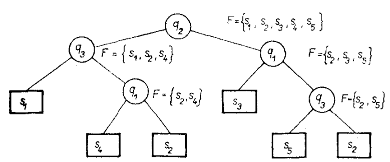

Fig. 1.

graph the circles mark the nonterminal nodes. The questions which are assigned to nonterminal nodes are written in the circles. In squares denoting terminal nodes there are written states which are assigned to those nodes by the mapping f.

Let us consider a questionnaire (G, π, f, g). If we leave out the root u_0 from the graph G we obtain α separated subgraphs G_k $(k = 1, \ldots, \alpha)$ of the graph G. Each of these subgraphs G_k, together with mappings π, f and g (defined ony on the nodes of the subgraph G_k), will be regared as a questionnaire* and will be described by (G_k, π, f, g). These questionnaires (G_k, π, f, g) will be called *main subquestionnaires* of the questionnaire (G, π, f, g). It is evident that

$$(2) \qquad \bar{L}(G, \pi, f, g) = \sum_{u \in U} t(g(u))\, \pi(u) = t(g(u_0)) + \sum_{k=1}^{\alpha} \bar{L}(G_k, \pi, f, g)\,.$$

If we proceed in the same manner leaving out the roots from all subquestionnaires composed from more then one node, we obtain the set of all subquestionnaires of the questionnaire under observation.

An algorithm presented in the next section yields all possible questionnaires which distinguish the set of states $\{s_i\}_{i=1}^{n}$ using questions $\{q_j\}_{j=1}^{m}$. Properties of the constructed questionnaire strongly depend on the way in which the choice of the question (in step (vi) of the algorithm) is practised. The main property, we shall dealt with, is the value of the average length of the questionnaire which is to be minimize. One does not know an universal choosing rule whose application yields questionnaires with the minimal average length. There are two choosing rules presented in this paper. Both of this are advisable only under some additional conditions. Moreover, none of them guaranttee that the constructed questionnaire achieves the short est possible average length.

Both presented choosing rules are based on the same heuristic thought. There are deduced lower bounds (expressions (4) and (12)) for an average length of any questionnaire solving the problem under observations. These lower bounds implicative depend on the choice of the question assigned to the root of the questionnaire. Such question is assigned to the root of the constructed questionnaire which minimizes the respective lower bound. The assigning of a question to any other node of a graph is proceeded in the same way, as each nonterminal node may be conside-as a root of some subquestionnaire.

ALGORITHM

We always assume, that the set of states $\{s_i\}_{i=1}^{n}$ and the set of questions $\{q_j\}_{j=1}^{m}$ fulfil the condition (1). Questionnaires enabling to distinguish these states can be constructed according to the following simple algorithm:

(i) Define the root u_0 of a graph G and define $\tilde{F}(u_0) = \{s_i\}_{i=1}^{n}$. Continue according to the step (ii).

* If some subgraph G_k consists only of one node, we shall regard this subgraph with its respective mappings as a questionnaire although it does not comply with the definition. For these cases we shall define the average length as equal to 0.

(ii) When there is a defined node to which no value of the mappings f or g has been assigned, continue according to the step (iii). Discontinue in the opposite case.

(iii) Let v be a node complying with the condition sub (ii).
If $|\tilde{F}(v)| = 0$, continue according to the step (iv).
If $|\tilde{F}(v)| = 1$, continue according to the step (v).
If $|\tilde{F}(v)| > 1$, continue according to the step (vi).

(iv) Define $f(v) = s$, where s is an arbitrary element of $\tilde{F}(u)$. u now denotes the node with an edge from u to v. Continue according to the step (ii).

(v) Define $f(v) = s \in \tilde{F}(v)$. Continue according to the step (ii).

(vi) Define $g(v) = q$, where q is a question selected in some way from

$$(\{q_j\}_{j=1}^{m} - \bigcup_{u \in T_p(v)} \{g(u)\}) .$$

Define α new nodes $v_1, \ldots, v_\alpha$ and α edges from v to $v_1, \ldots, v_\alpha$ of the graph G. Define the set

$$\tilde{F}(v_k) = \{s : q(s) = \varrho(q; k) \text{ or } q \notin I(s)\} \cap \tilde{F}(v)$$

for each v_k. Continue according to the step (ii).

The constructed questionnaire is defined by the graph G and mappings f a g which are step by step defined by the alfgorithm. The function π is not defined, nevertheless, its knowledge is necessary only for the exact specification of the average length of the questionnaire. If the knowledge of probabilities $p(s_i)$ of an occurence of the state s_i is assumed then it is possible to define the function π so as to fulfil the equality

$$p(s_i) = \sum_{\{w : f(w) = s_i\}} \pi(w)$$

for each state s_i.

It is advisable to define $\pi(w) = 0$ for such terminal nodes w with assigned value of mapping f by the step (iv).

The questionnaires obtained according to the preceding algorithm will be called *questionnaires pertaining to the set of states* $\{s_i\}_{i=1}^{n}$.

Now the problem to be solved is: what method is to be used to select the questions from the set

$$(\{q_j\}_{j=1}^{m} - \bigcup_{u \in T_p(v)} \{g(u)\})$$

in order to receive a questionnaire with the shortest possible average length.

Remark. The auxiliary mapping $\tilde{F}(v)$, which is gradually defined on the nodes of the graph G during construction of the questionnaire is joined with the formerly defined mapping $F(v)$ by the relationship:

$$\tilde{F}(v) \neq \emptyset \Rightarrow \tilde{F}(v) = F(v) .$$

In this part we shall consider that $t(q_j) = 1$ for every $j = 1, \ldots, m$. Therefore, according to [2],

$$\bar{L} = \sum_{u \in U} \pi(u) = \sum_{w \in W} \pi(w)\, r(w)\,,$$

where $r(w)$ denotes the length of the path from u_0 to w.

Since we suppose that there are α answers to every question there are also α edges which run out of every nonterminal node. From this it can be deduced that

$$\sum_{w \in W} \alpha^{-r(w)} = 1\,.$$

According to this equality it is possible to use the well-known inequality* in the following calculation

$$\bar{L} = \sum_{w \in W} \pi(w)\, r(w) = \frac{1}{\log \alpha} \Big(- \sum_{w \in W} \pi(w) \log \alpha^{-r(w)} \Big) \geqq$$

$$\geqq \frac{1}{\log \alpha} \Big(- \sum_{w \in W} \pi(w) \log \pi(w) \Big) = \frac{H(\pi)}{\log \alpha}\,.$$

Since distribution π is defined so that

$$p(s_i) = \sum_{\{w : f(w) = s_i\}} \pi(w)$$

for each $i = 1, \ldots, n$, the following relations hold**

$$- \sum_{i=1}^{n} p(s_i) \log p(s_i) = H(p) \leqq H(\pi)$$

therefore

$$\bar{L} \geqq \frac{H(p)}{\log \alpha}\,. \tag{3}$$

This inequality has been deduced just under the condition that

$$\sum_{i=1}^{n} p(s_i) = 1$$

and therefore also

$$\sum_{w \in W} \pi(w) = 1\,.$$

* Cf. e.g. lemma 1 of Chapter 2 in [1].

** Cf. e.g. lemma 4 of Chapter 1 in [1].

In order to be able to apply inequality (3) also to main subquestionnaires (G_k, π, f, g), it is necessary to consider the conditional distribution π_k instead of distribution π. The conditional distribution is defined as it follows:

$$\pi_k(w) = \pi(w) / \sum_{v \in W_k} \pi(v) \quad \text{for} \quad w \in W_k ,$$

$$\pi_k(w) = 0 \quad \text{for} \quad w \in (W - W_k) ,$$

$$\pi_k(u) = \sum_{w \in T_t(u)} \pi_k(w) \quad \text{for} \quad u \in U ,$$

where W_k denotes the subset of such nodes of W which are nodes of the graph G_k. In this case also

$$\bar{L}(G_k, \pi, f, g) / \sum_{w \in W_k} \pi(w) = \bar{L}(G_k, \pi_k, f, g) \geqq H(\pi_k) / \log \alpha .$$

If the probability distribution p_k is defined by the relation

$$p_k(s_i) = \sum_{\{w : f(w) = s_i\}} \pi_k(w)$$

it also holds

$$\bar{L}(G_k, \pi_k, f, g) \geqq H(p_k) / \log \alpha .$$

From this point of view it is possible to deduce

$$\bar{L}(G, \pi, f, g) = 1 + \sum_{k=1}^{\alpha} \bar{L}(G_k, \pi, f, g) = 1 + \sum_{k=1}^{\alpha} \bar{L}(G_k, \pi_k, f, g) \sum_{w \in W_k} \pi(w) \geqq$$

$$\geqq 1 + \frac{1}{\log \alpha} \sum_{k=1}^{\alpha} H(p_k) \sum_{w \in W_k} \pi(w) . \tag{4}$$

From the definition of the probability distribution p_k it may be seen that $p_k(s_i)$ is a conditional probability of occurence of the state s_i under the condition that the answer to the question $q = g(u_0)$ is $\varrho(q; k)$.

In order to minimize the expression on the right side of the inequality (4), we select such a question which yields the smallest possible average value of the entropy of the conditional distribution. Proceeding in this way at every nonterminal node of the questionnaire we obtain the first founded algorithm.

This often used algorithm (e.g. in [3]) usually gives good results. However, it may be used only under the condition that $t(q_j) = const$ for every question. Moreover, one cannot decide if there exists a questionnaire pertaining to the same group of states having a shorter average length. Also, there is no better estimate of difference between the average length of the constructed questionnaire, and the average length of the optimal questionnaire (the *optimal questionnaire* is a questionnaire pertaining to the given group of states with the shortest possible average length) than

$$H(p)/\log \alpha \leqq L_{\text{OPT}} \leqq \bar{L}(G, \pi, f, g),$$

where L_{OPT} denotes the average length of the optimal questionnaire.

Remark. An example contained in the appendix of this paper exhibits that this algorithm may not yield the optimal questionnaire.

COMPLETE GROUP OF STATES

Since this moment we cease to suppose that $t(q_j) = const$ and another special case will be studied.

Let us consider an ordered m-tuple

$$(\varrho_1, \varrho_2, \ldots, \varrho_m),$$

where ϱ_j denotes an arbitrary element of $P(q_j)$. So $(\varrho_1, \ldots, \varrho_m)$ is a *complex of answers* to all questions.

We shall say that the complex of answers $(\varrho_1, \ldots, \varrho_m)$ complies to the state s if for all $j = 1, \ldots, m$

$$q_j \notin I(s) \quad \text{or} \quad q_j(s) = \varrho_j .$$

The group of states $\{s_i\}_{i=1}^n$ is called *complete* if to all possible complexes of answers $(\varrho_1, \ldots, \varrho_m)$ there is a state $s \in \{s_i\}_{i=1}^n$ with complying complex $(\varrho_1, \ldots, \varrho_m)$. The following theorem gives the way in which one can find whether the given group of states is complete.

Theorem 1. *Let $\{s_i\}_{i=1}^n$ and $\{q_j\}_{j=1}^m$ fulfil the condition* (1). *$\{s_i\}_{i=1}^n$ is complete iff*

$$\sum_{i=1}^{n} \alpha^{-|I(s_i)|} = 1 . \tag{5}$$

Proof. It is obvious that the number of different complexes of answers which comply to the state s is exactly $\alpha^{m-|I(s)|}$.

Since $\{s_i\}_{i=1}^n$ and $\{q_j\}_{j=1}^m$ fulfil the cindition (1), an arbitrary complex of answers complies to not more than one state $s \in \{s_i\}_{i=1}^n$. From this it follows that the number of all different complexes of answers which comply to some of the states $\{s_i\}_{i=1}^n$ is equal to

$$\sum_{i=1}^{n} \alpha^{m-|I(s_i)|} . \tag{6}$$

Since the number of all possible different complexes of answers is α^m, it follows from condition (1) that

$$(7) \qquad \sum_{i=1}^{n} \alpha^{m-|I(s_i)|} \leqq \alpha^m .$$

According to the definition of complete group of states $\{s_i\}_{i=1}^n$, this group fulfil the relation

$$(8) \qquad \sum_{i=1}^{n} \alpha^{m-|I(s_i)|} \geqq \alpha^m ,$$

From the inequalities (7) and (8) it follows that for a complete group of states $\{s_i\}_{i=1}^n$ the following equality is true

$$\sum_{i=1}^{n} \alpha^{m-|I(s_i)|} = \alpha^m$$

and the condition to be proved is obtained by dividing both sides of this equality by α^m.

On the other hand, let the equality (5) be valid. Hence also the above equality is valid. As condition (1) is fulfilled, for α^m different complexes of answers $(\varrho_1^{(k)}, \ldots, \varrho_m^{(k)})$ $(k = 1, \ldots, \alpha^m)$ there exists a state $s \in \{s_i\}_{i=1}^n$ to which the complex $(\varrho_1^{(k)}, \ldots, \varrho_m^{(k)})$ complies. Since the number of all possible different complexes of answers is α^m, it was proved that for each complex $(\varrho_1, \ldots, \varrho_m)$ there is a state $s \in \{s_i\}_{i=1}^n$ to which the complex $(\varrho_1, \ldots, \varrho_m)$ complies. Thus the validity of the other side of the proved equivalence was confirmed. Q.E.D.

The choosing rule, which will be described below, is based on the property contained in Theorem 2.

Theorem 2. *The following implication is valid for every questionnaire pertaining to the complete group of states*:

$$q_j \in I(f(w)) \Rightarrow (\exists u \in T_p(w)) (g(u) = q_j) .$$

Proof. Let us consider an arbitrary complex of answers $(\varrho_1, \ldots, \varrho_m)$ complying to the state $f(w)$. Let us defined the complex of answers $(\bar{\varrho}_1, \ldots, \bar{\varrho}_m)$ as follows: Define $\bar{\varrho}_j \neq \varrho_j$ arbitrarily. For each $k \neq j$ for which there exists $v \in T_p(w)$ such that $q_k = g(v)$ and simultaneously

$$R(v) = \bar{F}(v') \cap \{s : q_j \in I(s)\} \neq \emptyset \quad \text{where} \quad v' \in (T_p(w) \cup \{w\}) \cap T_d(v)$$

define $\bar{\varrho}_k = q_k(s)$ for some $s \in R(v)$ (according to the algorithm it is obvious that $q_k(s) = q_k(s')$ for $s, s' \in R(v)$). In all the other cases define $\bar{\varrho}_k$ arbitrarily.

Henceforth let s denotes the state $f(w)$ and $\bar{s}$ that state from $\{s_i\}_{i=1}^n$ to which the complex $(\bar{\varrho}_1, \ldots, \bar{\varrho}_m)$ complies. Since it is assumed that $q_j \in I(s)$ it is obvious that the states s and $\bar{s}$ are different.

From the definition of $(\bar{\varrho}_1, \ldots, \bar{\varrho}_m)$ and from the way of defining the sets $\tilde{F}(u)$ by the algorithm, the following can be deduced. If $s, \bar{s} \in \tilde{F}(u)$, $g(u) \neq q_j$ then for $v \in T_d(u) \cap (T_p(w) \cup \{w\})$

$$s \in \tilde{F}(v) \Leftrightarrow \bar{s} \in \tilde{F}(v) .$$

Let us assume that $q_j \notin \bigcup_{v \in T_p(w)} \{g(v)\}$. The terminal node is defined by the algorithm iff $|\tilde{F}(w)| \doteq 1$ or 0. However, $|\tilde{F}(v)| \neq 1$ according to the fact that from $s = f(w) \in \tilde{F}(w)$ it follows that $\bar{s} \in \tilde{F}(w)$, and, hence, $|\tilde{F}(w)| \geqq 2$. Neither the other possibility $|\tilde{F}(w)| = 0$ may occur. Since it is assumed that $\{s_i\}_{i=1}^n$ is complete, for no node v of the graph constructed by the algorithm can be $\tilde{F}(v) = \emptyset$. By this contradiction the proof is accomplished. Q. E. D.

Using Theorem 2 we shall estimate the average length of a questionnaire which is pertaining to the complete group of states $\{s_i\}_{i=1}^n$ so that the estimate would not include the need for function π.

$$\bar{L}(G, \pi, f, g) = \sum_{u \in U} t(g(u))\, \pi(u) = \sum_{u \in U} t(g(u)) \sum_{w \in T_t(u)} \pi(w) .$$

Let us denote

$$T(u) \;\; = T_t(u) \cap \{w : g(u) \in I(f(w))\} ,$$

$$T_*(u) = T_t(u) \cap \{w : g(u) \notin I(f(w))\} .$$

Thus

$$\bar{L}(G, \pi, f, g) = \sum_{u \in U} t(g(u)) \sum_{w \in T(u)} \pi(w) + \sum_{u \in U} t(g(u)) \sum_{w \in T_*(u)} \pi(w) = E_1 + E_2 .$$

Each of these two expressions will be estimated separately.

It is obvious that in the expression E_2 it will suffice to add only the terms pertaining to the nodes for which $T_*(u) \neq \emptyset$, or, nodes from the set

$$U^* = \{u \in U : g(u) \notin \bigcap_{s_i \in F(u)} I(s_i)\} . \tag{10}$$

Since $t(q_j) > 0$ is assumed for all $j = 1, \ldots, m$ and $\pi(w) \geqq 0$, the following inequality is valid for all $U_1 \subseteqq U^*$.

$$E_2 \geqq \sum_{u \in U_1} t(g(u)) \sum_{w \in T_*(u)} \pi(w) .$$

In order to eliminate the function π and replace it by the probability of occurence of the states $p(s_i)$, the subset U_1 has to meet the condition

$$u \in U_1 \Rightarrow (w \in T_t(u) \;\&\; w' \notin T_t(u) \Rightarrow f(w) \neq f(w')) . \tag{11}$$

This condition can be verbally expressed in the following way:

If the state $f(w) = s$ is assigned to a terminal node $w \in T_t(u)$, then all nodes to which the state s is assigned by the mapping f belong to the set $T_t(u)$.

For such subsets U_1 the following equality is valid

$$\sum_{u \in U_1} t(g(u)) \sum_{w \in T_*(u)} \pi(w) = \sum_{u \in U_1} t(g(u)) \sum_{s_i \in F_*(u)} p(s_i)\,,$$

where $F_*(u) = F(u) \cap \{s : g(u) \notin I(s)\}$. Hence

$$E_2 \geqq \sum_{u \in U_1} t(g(u)) \sum_{t_i \in F_*(u)} p(s_i)\,.$$

The expression E_1 will be calculated easily.

$$E_1 = \sum_{u \in U} t(g(u)) \sum_{w \in T(u)} \pi(w) = \sum_{w \in W} \pi(w) \sum_{u} t(g(u))\,,$$

where the last sum is over the set $T_p(w) \cap \{u : g(u) \in I(f(w))\}$. Since, according to Theorem 2.

$$|T_p(w)| \geqq |I(f(w))|$$

and, according to the algorithm, one question may be assigned at most to one node from $T_p(w)$, it is valid

$$E_1 = \sum_{w \in W} \pi(w) \sum_{q_j \in I(f(w))} t(q_j) =$$

$$= \sum_{i=1}^{n} \sum_{\{w : f(w) = s_i\}} \pi(w) \sum_{q_j \in I(f(w))} t(q_j) = \sum_{i=1}^{n} p(s_i) \sum_{q_j \in I(s_i)} t(q_j)\,.$$

Thus, if U_1 meets the condition (11) we get

$$\bar{L} \geqq \sum_{i=1}^{n} p(s_i) \sum_{q_j \in I(s_i)} t(q_j) + \sum_{u \in U_1} t(g(u)) \sum_{s_i \in F_*(u)} p(s_i)\,. \tag{12}$$

CHOOSING RULE

Let us consider a node u, to which we are to assign a question

$$q \in (\{q_j\}_{j=1}^{m} - \bigcup_{v \in T_p(u)} \{g(v)\}) = N(u)$$

according to step (vi) of the algorithm. When

$$M(u) = (\bigcap_{s_i \in \bar{F}(u)} I(s_i) - \bigcup_{v \in T_p(u)} \{g(v)\}) \neq \emptyset$$

we can define $g(u) = q \in M(u)$ arbitrarily. If $M(u) = \emptyset$, then to each question $q \in N(u)$ we compute the value

$$C(q) = t(q) \sum_{s_i} p(s_i),$$

where the last sum is over the $s_i \in \tilde{F}(u) \cap \{s : q \notin I(s)\}$. Finally we define $g(u)$ equal to such $q \in N(u)$ for which the value $C(q)$ is minimal.

THE PROPERTIES OF THE CHOOSING RULE

We shall now study the properties of the questionnaires which have been constructed by the algorithm using the choosing rule. We continue to assume that the set of states $\{s_i\}_{i=1}^n$ is complete under the given set of questions $\{q_j\}_{j=1}^m$.

Let us consider the set U_1^* of nonterminal nodes of questionnaire which is defined

$$U_1^* = \{u \in U^* : T_p(u) \cap U^* = \emptyset\},$$

where U^* describes the set which was defined earlier by (10). **Thus,** U_1^* is a set of those nodes u of U^* for which there is no node from $T_p(u)$ which is simultaneously in U^*.

Theorem 3. *Let (G, π, f, g) be a questionnaire constructed by the algorithm using the choosing rule. If $(\bar{G}, \bar{\pi}, \bar{f}, \bar{g})$ is an arbitrary questionnaire pertaining to the same complete set of states $\{s_i\}_{i=1}^n$, then it is valid*

$$\bar{L}(\bar{G}, \bar{\pi}, \bar{f}, \bar{g}) \geqq \sum_{i=1}^{n} p(s_i) \sum_{q_j \in I(s_i)} t(q_j) + \sum_{u \in U^*_1} t(g(u)) \sum_{s_i \in F_*(u)} p(s_i).$$

Proof. Throughout this proof we shall denote all elements of the questionnaire (G, π, f, g) in the usual way and analogical symbols pertaining to the questionnaire $(\bar{G}, \bar{\pi}, \bar{f}, \bar{g})$ will be differentiated by a bar (e.g. $\bar{G}$, $\bar{u}_0$).

First, we shall find a mapping $\mathcal{M}$ from U_1^* into $\bar{U}^*$ having important properties. This mapping enables us to accomplish the proof.

Let us consider an arbitrary node $u \in U_1^*$. It will be shown that there exists a node $\bar{u} \in \bar{U}^*$, so that

(14) $$\bar{f}(\bar{w}) \in F(u) \Rightarrow \bar{w} \in T_t(\bar{u})$$

and simultaneously

(15) $$\bar{g}(\bar{u}) \notin \bigcup_{v \in T_p(u)} \{g(v)\}.$$

This $\bar{u}$ will be defined as a value of the mapping $\mathcal{M}(u)$.

Let us consider an arbitrary state $s \in F(u)$. There is at least one terminal node $\bar{w} \in \bar{W}$ of the graph $\bar{G}$ for which $\bar{f}(\bar{w}) = s$. Let us denote the path from $\bar{u}_0$ to $\bar{w}$

$$\bar{u}_0, \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_t = \bar{w}\,.$$

Now it will be shown that if the condition (14) is valid for a $\bar{u}_k \notin \bar{U}^*$ then

$$\bar{g}(\bar{u}_k) \in \bigcup_{v \in T_p(u)} \{g(v)\}\,.$$

Let $\bar{u}_k \notin \bar{U}^*$ fulfils the condition (14). From $\bar{u}_k \notin \bar{U}$ follows that

$$\bar{g}(\bar{u}_k) \in \bigcap_{s_i \in \bar{F}(\bar{u}_k)} I(s_i) \subseteqq \bigcap_{s_i \in F(u)} I(s_i)\,.$$

The last relation is deduced from the validity of

$$F(u) \subseteqq \bar{F}(\bar{u}_k)$$

for $\bar{u}_k$ which fulfils the condition (14). Since $u \in U_1^* \subseteqq U^*$, then

$$\left(\bigcap_{s_i \in F(u)} I(s_i) - \bigcup_{v \in T_p(u)} \{g(v)\}\right) = \emptyset$$

and therefore there is $v \in T_p(u)$ for which $g(v) = \bar{g}(\bar{u}_k)$.

Let $\bar{u}_k$ fulfil the condition (14). If

$$q_j = \bar{g}(\bar{u}_k) \in \bigcup_{v \in T_p(u)} \{g(v)\} \tag{16}$$

then there are only such states s_i in $F(u)$, for which

$$q_j(s_i) = q_j(\bar{f}(\bar{w}))\,.$$

In the graph $\bar{G}$ all terminal nodes to which there are assigned states from $F(u)$ are located behind the node $\bar{u}_k$ according to the condition (14). From condition (16) all states (and only these states) $s_i \in \bar{F}(\bar{u}_k)$ for which

$$q_j \notin I(s_i) \quad \text{or} \quad q_j(s_i) = q_j(\bar{f}(\bar{w}))$$

are from the set $\bar{F}(\bar{u}_{k+1})$. That is why all terminal nodes with assigned states from $F(u)$ are located in the graph $\bar{G}$ behind the node $\bar{u}_{k+1}$. In this way, the condition (14) is also fulfilled for the node $\bar{u}_{k+1}$. Thus it was shown that if (14) and (16) are fulfilled for $\bar{u}_k$ then (14) is fulfilled also for $\bar{u}_{k+1}$.

The proof, that there is a node $\bar{u} \in \{\bar{u}_k\}_{k=0}^{t} \cap \bar{U}^*$ fulfilling the conditions (14) and (15) will be performed by a contraversy.

It is obvious that the root $\bar{u}_0$ fulfils the condition (14). Since we suppose that

$$\bar{u}_0 \notin \bar{U}^* \quad \text{or} \quad \bar{g}(\bar{u}_0) \in \bigcap_{v \in T_p(u)} \{g(v)\}$$

the condition (14) is fulfilled also for $\bar{u}_1$ according to the foregoing part of the proof. In this way the process may be continued untill it is proved that the condition (14) is fulfilled also for $\bar{u}_t = \bar{w}$, which is in contradiction to

$$1 = |\bar{F}(\bar{w})| < |F(u)|$$

because from (14) it also follows that $\bar{F}(\bar{w}) \supseteqq F(u)$.

Thus we can define the mapping $\mathcal{M}$ from U_1^* into $\bar{U}^*$ so that $\mathcal{M}(u) = \bar{u}$ fulfils the conditions (14) and (15) for $u \in U_1^*$.

Unfortunately, the set of nodes

$$\mathcal{M}(U_1^*) = \{\mathcal{M}(u)\}_{u \in U_1^*} \subseteqq \bar{U}^*$$

need not fulfil the condition (11) and therefore the estimate (12) may not be used. However, according to (14) the set $\mathcal{M}(U_1^*)$ fulfils the condition:

$$u \in U_1^* \Rightarrow (\bar{w} \in T_t(\mathcal{M}(u)) \cap \{\bar{w} : \bar{f}(\bar{w}) \in F(u)\}\ ;\ \bar{w}' \notin T_t(\mathcal{M}(u)) \Rightarrow \bar{f}(\bar{w}) \neq \bar{f}(\bar{w}'))\,.$$

Therefore the expression

$$\bar{E}_2 = \sum_{\bar{u} \in \bar{U}^*} t(\bar{g}(\bar{u})) \sum_{\bar{w} \in \bar{T}_*(\bar{u})} \bar{\pi}(\bar{u})$$

is not less than

$$\sum_{u \in U_1^*} t(\bar{g}(\mathcal{M}(u))) \sum_{\bar{w}} \bar{\pi}(\bar{w})\,, \tag{17}$$

where the second sum is for

$$\bar{w} \in T_t(\mathcal{M}(u)) \cap \{\bar{w} : g(\mathcal{M}(u)) \notin I(\bar{f}(\bar{w}))\} \cap \{\bar{w} : \bar{f}(\bar{w}) \in F(u)\}\,.$$

From the validity of the condition (14) for nodes $\mathcal{M}(u)$ ($u \in U_1^*$) it follows that

$$T_t(\mathcal{M}(u)) \supseteqq \{\bar{w} : \bar{f}(\bar{w}) \in F(u)\}$$

and therefore the sum (17) is equal to

$$\sum_{u \in U_1^*} t(\bar{g}(\mathcal{M}(u))) \sum_{s_i} p(s_i)\,,$$

where the last sum is over the set $F(u) \cap \{s : g(\mathcal{M}(u)) \notin I(s)\}$.

Using the choosing rule, to each node $u \in U_1^*$ such a question $g(u)$ is assigned for which

$$t(g(u)) \sum_{s_i \in F_*(u)} p(s_i)$$

is of the minimal value. Thus

$$\bar{E}_2 \geqq \sum_{u \in U_1^*} t(g(u)) \sum_{s_i \in F_*(u)} p(s_i)\,, \tag{18}$$

Since the value of the expressions E_1 is identical for both the questionnaires (G, π, f, g) and $(\bar{G}, \bar{\pi}, \bar{f}, \bar{g})$,

$$E_1 = \sum_{i=1}^{n} p(s_i) \sum_{q_j \in I(s_i)} t(q_j)$$

the proof is accomplished by adding the expression E_1 to both sides of the inequality (18). Q.E.D.

If $U_1 = U_1^* = U^*$ then the equality is valid in the relation (12) and the following corollary follows immediately from Theorem 3.

Corollary. *If $U^* = U_1^*$ for the questionnaire (G, π, f, g) constructed by the algorithm using the choosing rule, then (G, π, f, g) is optimal.*

According to Theorem 3 it is possible to obtain an upper bound of the difference between the average length of questionnaire constructed by means of choosing rule, and the average length of the optimal questionnaire. The corollary of this theorem provides a sufficient (though unnecessary) condition of the optimality of the constructed questionnaire. Another interesting property is expressed by the theorem given below.

Theorem 4. *Provided there exists a questionnaire pertaining to the complete set of states $\{s_i\}_{i=1}^n$ (assuming that $\{s_i\}_{i=1}^n$ and $\{q_j\}_{j=1}^m$ fulfil the condition (1)), so that*

$$\bar{L} = \sum_{i=1}^{n} p(s_i) \sum_{q_j \in I(s_i)} t(q_j)$$

then the questionnaire constructed by the algorithm using the choosing rule is optimal.

Proof. To prove this theorem it will suffice to show that for the constructed questionnaire $U^* = \emptyset$ and therefore $E_2 = 0$.

The proof will be performed in the way of coming to a contradictory. Let us consider an arbitrary $u \in U_1^*$. According to the definition of U^*

$$\bigcap_{s_i \in F(u)} I(s_i) - \bigcup_{v \in T_p(u)} \{g(v)\} = \emptyset . \tag{19}$$

Let us denote by $(\bar{G}, \bar{\pi}, \bar{f}, \bar{g})$ the questionnaire pertaining to the same set of states to which

$$\bar{L}(\bar{G}, \bar{\pi}, \bar{f}, \bar{g}) = \sum_{i=1}^{n} p(s_i) \sum_{q_j \in I(s_i)} t(q_j) .$$

From the estimate (12) follows that $\bar{U}^* = \emptyset$, under the condition that $t(q_j) > 0$ and $p(s_i) > 0$ for all j and i.

Let us consider $\bar{w} \in \bar{W}$ such that $\bar{f}(\bar{w}) \in F(u)$ and the path

$$\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_t = \bar{w} .$$

If $\bar{U}^* = \emptyset$ then

$$\bar{g}(\bar{u}_k) \in \bigcap_{s_i \in \bar{F}(\bar{u}_k)} I(s_i)$$

for $k = 0, \ldots, t - 1$.

It will be proved that

$$\bar{F}(\bar{u}_k) \supseteq F(u) \Rightarrow \bar{F}(\bar{u}_{k+1}) \supseteq F(u)$$

for $k = 0, \ldots, t - 1$.

Provided that $\bar{F}(\bar{u}_k) \supseteq F(u)$ and $\bar{U}^* = \emptyset$ it follows that

$$\bar{g}(\bar{u}_k) \in \bigcap_{s_i \in \bar{F}(\bar{u}_k)} I(s_i) \subseteq \bigcap_{s_i \in F(u)} I(s_i)$$

and according to (19)

$$\bar{g}(\bar{u}_k) \in \bigcup_{v \in T_p(u)} \{g(v)\} .$$

By similar consideration as was adopted in the proof of Theorem 3 it is possible to obtain $\bar{F}(\bar{u}_{k+1}) \supseteq F(u)$. It is obvious that $\bar{F}(\bar{u}_0) \supseteq F(u)$ and, therefore, by the induction

$$\bar{F}(\bar{u}_t) = \bar{F}(\bar{w}) \supseteq F(u)$$

which is again a contradiction to $|F(u)| > 1$. Q. E. D.

APPENDIX

The procedure for the construction of questionnaires will be now illustrated by an example.

Let us consider the set of states $\{s_i\}_{i=1}^4$ and the set of questions $\{q_j\}_{j=1}^4$ which are defined as follows:

$$\begin{array}{llll} q_1(s_1) \text{ undefined}, & q_2(s_1) \text{ undefined}, & q_3(s_1) = 1, & q_4(s_1) = 1, \\ q_1(s_2) = 1, & q_2(s_2) \text{ undefined}, & q_3(s_2) = 0, & q_4(s_2) \text{ undefined}, \\ q_1(s_3) \text{ undefined}, & q_2(s_3) = 1, & q_3(s_3) = 0, & q_4(s_3) \text{ undefined}, \\ q_1(s_4) = 1, & q_2(s_4) \text{ undefined}, & q_3(s_4) = 1, & q_4(s_4) = 0. \end{array}$$

Therefore in this example $\alpha = 2$. It may be seen that the states s_2 and s_3 do not meet the condition (1). For that reason we shall define other new states s_{23}, s_{2-3} and s_{3-2}

in the following way

$$q_1(s_{23}) = 1\,, \quad q_2(s_{23}) = 1\,, \quad q_3(s_{23}) = 0\,,$$
$$q_1(s_{2-3}) = 1\,, \quad q_2(s_{2-3}) = 0\,, \quad q_3(s_{2-3}) = 0\,,$$
$$q_1(s_{3-2}) = 0\,, \quad q_2(s_{3-2}) = 1\,, \quad q_3(s_{3-2}) = 0\,,$$

q_4 is not defined for each of these states.

The state s_{23} meets the reality that the observated object is at the state s_2 and s_3 simultaneously. The state s_{2-3}, s_{3-2} indicates that the object is only at the state s_2, s_3 respectively.

Let probabilities of occurence of states be

$$p(s_1) = p(s_4) = 0{\cdot}4\,, \quad p(s_{23}) = 0{\cdot}1\,, \quad p(s_{2-3}) = p(s_{3-2}) = 0{\cdot}05\,.$$

Let us suppose that the costs of answering the considered questions are equal in order to enable illustration of both the manners of choosing of questions. Therefore let $t(q_j) = 1$ for all j.

If we want to assign a question to a root of a questionnaire by the method based on an enumeration of the value of information we have to enumerate values of the expression on the right side of the inequality (4) for each question q_j, $j = 1, \ldots, 4$. These values (with the exception of an additive and nonnegative multiplicative constants) are contained in Table 1. Thus the question q_4 is assigned to the root of the graph G. If we proceed according to the algorithm, we obtain the questionnaire shown in Fig. 2 whose average length is $\bar{L} = 2{\cdot}35$.

Table 1.

Question	The value of the expression $\sum_{k=1}^{2} H(p_k) \sum_{w \in W_k} \pi(w)$
q_1	0·4251
q_2	0·4060
q_3	0·3311
q_4	0·3076

If we use the choosing rule for the construction of a questionnaire, we assign the question q_3 to the root of the graph G because

$$M(u_0) = I(s_1) \cap I(s_{23}) \cap I(s_{2-3}) \cap I(s_{3-2}) \cap I(s_4) = \{q_3\}\,.$$

Let us define nodes u_1 and u_2 and sets

$$\tilde{F}(u_1) = \{s_{23}, s_{2-3}, s_{3-2}\}\,, \quad \tilde{F}(u_2) = \{s_1, s_4\}$$

and according to the algorithm we obtain the questionnaire shown in Fig. 3 whose average length is $\bar{L} = 2.15$

Remark. During the counting the conditional probabilities we assumed that

$$p(s = s_i \mid q_j(s) = 0) = p(s = s_i \mid q_j(s) = 1) = \tfrac{1}{2}p(s_i)$$

when $q_j \notin I(s_i)$.

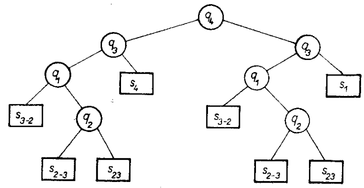

Fig. 2.

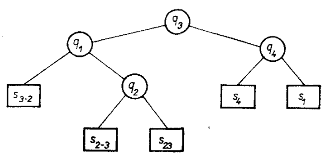

Fig. 3.

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REFERENCES

[1] A. Feinstein: Foundation of Information Theory. McGraw Hill, New York 1958.

[2] C. F. Picard: Quasi-questionnaire, codes and Huffman's length. Kybernetika *6* (1970), 6.

[3] L. N. Landa: Opit primenenija matematičeskoj logiki i teorii informacii k nekotorym problemam obučenija. Voprosy psichologii (1962), 2, 19—40.

[4] D. E. Knuth: Optimal Binary Search Trees. Acta informatica *1* (1971), 1.

[5] A. Renyi: On the enumeration of search-codes. Acta Math. Acad. Sci. Hung. *21* (1970), 27—33.

[6] S. Guiasu: On the most rational algorithm of recognition. Kybernetik *5* (sept. 1968), 109—113.

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