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A Note on the Exponential Stability of a Matrix Riccati Equation of Stochastic Control

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A general matrix Riccati equation of stochastic control is considered. It is proved by a new method that, under certain assumptions, the solution of such equation tends to the equilibrium point exponentially fast.

1. INTRODUCTION

The object of this paper is to prove a theorem concerning the asymptotic behaviour of the solution of the following matrix Riccati equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = A^*P + PA + \pi_1(P) + Q - PB[R + \pi_2(P)]^{-1} B^*P , \quad t \geqq 0 , \tag{1}$$

with the initial condition $P_0 \geqq O$. Here π_1, π_2 are linear and monotonic transformations, which map the space $\mathscr{E}_n$ of all symmetric $n \times n$ matrices into the spaces $\mathscr{E}_n$ and $\mathscr{E}_m$ respectively of all $n \times n$ and $m \times m$ symmetric matrices. A and B are respectively $n \times n$ and $n \times m$ matrices. We shall assume that $n \times n$ matrix Q and $m \times m$ matrix R are positive definite. This assumption is rather restrictive (see, for instance, the appropriate theorems in [1]), but the aim of the paper is to present a new method rather than to prove the strongest theorem. The point (b) of the Theorem below is new.

2. STATEMENT AND PROOF OF THE BASIC RESULT

Let $\mathscr{K}_n$ denote the cone of all positive semi-definite $n \times n$ matrices and let S be a transformation $S : \mathscr{K}_n \to \mathscr{E}_n$ given by the following formula:

$$S(P) = A^*P + PA + \pi_1(P) + Q - PB[R + \pi_2(P)]^{-1} B^*P .$$

We are going to prove the following theorem:

Theorem. *Let* $Q > 0$, $R > 0$.

a) *There exists at least one solution* $\bar{P} > 0$ *of the equation*:

$$S(P) = 0\,. \tag{2}$$

b) *If* $\bar{P} > 0$ *is the solution to* (2) *and* $P_0 > 0$ *then the solution* $\{P_t; t \geqq 0\}$ *of* (1) *tends to* $\bar{P}$ *exponentially fast as* $t \to +\infty$.

Remark 1. The point a) of Theorem is a special case of well known results (see [1]), it is also a consequence of b), therefore it remains to prove the point b).

Lemma 1. *Thre transformation S is concave.*

Proof. We have to prove that if $\alpha \geqq 0$, $\beta \geqq 0$, $\alpha + \beta = 1$, $U \geqq 0$, $V \geqq 0$, then

$$S(\alpha U + \beta V) \geqq \alpha\, S(U) + \beta\, S(V)\,.$$

For any $m \times n$ matrix K let us define the transformation $\Psi_K : \mathcal{E}_n \to \mathcal{E}_n$ by the following formula:

$$\Psi_K(P) = (A - BK)^* P + P(A - BK) + Q + \pi_1(P) + K^*[R + \pi_2(P)]\, K\,.$$

Then, see [1, identity 3.2],

$$S(P) = \Psi_K(P) - (K - K_P)^* \left[R + \pi_2(P)\right] (K - K_P)\,,$$

where

$$K_P = [R + \pi_2(P)]^{-1}\, B^* P\,.$$

From this we obtain that for all K and $P \geqq 0$: $S(P) \leqq \Psi_K(P)$, and that $S(P) = \Psi_{K_P}(P)$. Thus

$$S(\alpha U + \beta V) = \Psi_{K_{\alpha U + \beta V}}(\alpha U + \beta V) = \alpha \Psi_{K_{\alpha U + \beta V}}(U) + \beta \Psi_{K_{\alpha U + \beta V}}(V) \geqq$$
$$\geqq \alpha \Psi_{K_U}(U) + \beta \Psi_{K_V}(V) \geqq \alpha\, S(U) + \beta\, S(V)\,.$$

Remark 2. A special case of the above lemma ($\pi_1 \equiv 0$, $\pi_2 \equiv 0$) has been proved by many authors (see for instance [2]) but by different methods.

The proof of the lemma below was given (implicitely) in [1].

Lemma 2. *If* $P_0^1 \geqq P_0^2 \geqq 0$, *then the solutions* $\{P_t^1; t \geqq 0\}$, $\{P_t^2; t \geqq 0\}$ *of* (1) *subject to the initial conditions* P_0^1, P_0^2 *satisfy*

$$P_t^1 \geqq P_t^2 \geqq 0 \quad \textit{for all} \quad t \geqq 0\,.$$

Proof of Theorem, b). Let us fix a number $\bar{t}$, $0 < \bar{t} < 1$ and let $\{\tilde{P}_t;\ t \geqq 0\}$ be the solution to (1) with the initial condition: $\tilde{P}_0 = \bar{t}\bar{P}$. Let us define the function ϱ $\varrho : [0, +\infty) \to [0, 1]$ by the formula:

$$\varrho(t) = \sup\{s : s\bar{P} \leqq \tilde{P}_t\}, \quad t \geqq 0 .$$

Evidently ϱ is a continuous function, and $\varrho(0) = \bar{t}$. Moreover the function $\lambda = 1 - \varrho$ satisfies the differential inequality (4) below. Namely let $t \geqq 0$, and let $\{P_{t,u};\ u \geqq 0\}$ be the solution of (1) subject to the initial condition $\varrho(t)\,\bar{P}$. That means

$$P_{t,u} = \varrho(t)\,\bar{P} + \int_0^u S(P_{t,v})\,\mathrm{d}v .$$

Since $\tilde{P}_t \geqq P_{t,0}$ we have (see Lemma 2) that

$$\tilde{P}_{t+u} \geqq P_{t,u} = \varrho(t)\,\bar{P} + \int_0^u S(P_{t,v})\,\mathrm{d}v . \tag{3}$$

The concavity of the function S implies:

$$S(\varrho(t)\,\bar{P}) = S(\varrho(t)\,\bar{P} + (1 - \varrho(t))\,0) \geqq \varrho(t)\,S(\bar{P}) + (1 - \varrho(t))\,S(0) \geqq$$
$$\geqq (1 - \varrho(t))\,Q \geqq (1 - \varrho(t))\,\gamma\bar{P} ,$$

where γ is a positive number such that $Q \geqq \gamma\bar{P}$. From (3)

$$\frac{1}{u}\,(\tilde{P}_{t+u} - \varrho(t)\,\bar{P}) \geqq \frac{1}{u}\int_0^u S(P_{t,v})\,\mathrm{d}v .$$

But

$$\frac{1}{u}\int_0^u S(P_{t,v})\,\mathrm{d}v \to S(\varrho(t)\,\bar{P}) \geqq (1 - \varrho(t))\,\gamma\bar{P} \quad \text{as} \quad u \downarrow 0.$$

Therefore, for sufficiently small $u > 0$

$$\frac{1}{u}\int_0^u S(P_{t,v})\,\mathrm{d}v \geqq (1 - \varrho(t))\,\bar{\gamma}\bar{P}$$

where $0 < \bar{\gamma} < \gamma$.

Thus for small $u > 0$

$$\tilde{P}_{t+u} - \varrho(t)\,\bar{P} \geqq u(1 - \varrho(t))\,\bar{\gamma}\bar{P}$$

and the definition of the function ϱ implies

$$\varrho(t + u) \geqq \varrho(t) + u\bar{\gamma}(1 - \varrho(t)).$$

Consequently

$$\overline{\lim_{u \downarrow 0}} \frac{\lambda(t + u) - \lambda(t)}{u} \leqq -\bar{\gamma}\,\lambda(t) \tag{4}$$

and Theorem 4.1 together with Remark 2 of the monograph [3] imply

$$1 - \varrho(t) \leqq (1 - \bar{t})\,\mathrm{e}^{-\bar{\gamma}t}, \quad t \geqq 0.$$

Applying the same method as above we obtain that if $\bar{s} > 1$, $\{\hat{P}_t; t \geqq 0\}$ is the solution to (1) with the initial condition $\bar{s}\bar{P}$ and $\mu(t) = \inf\{s; \hat{P}_t \leqq s\bar{P}\}$ then

$$\mu(t) - 1 \leqq (\bar{s} - 1)\,\mathrm{e}^{-\bar{\gamma}t}, \quad t \geqq 0.$$

To finish the proof let $P_0 > 0$ and let $\bar{t}$, $\bar{s}$ be numbers such that $\bar{t}\bar{P} < P_0 < \bar{s}\bar{P}$, $0 < \bar{t} < 1$, $1 < \bar{s}$. Then

$$0 \leqq \tilde{P}_t \leqq P_t \leqq \hat{P}_t \quad \text{for all} \quad t > 0,$$

because of Lemma 2, and

$$|P_t - \bar{P}| \leqq |\hat{P}_t - \tilde{P}_t| \leqq |\bar{P} - \tilde{P}_t| + |\hat{P}_t - \bar{P}| \leqq$$
$$\leqq [(1 - \bar{t}) + (\bar{s} - 1)]\,|\bar{P}|\,\mathrm{e}^{-\bar{\gamma}t}.$$

This completes the proof of Theorem.

Remark 3. The analogous method of the proof was applied first in the paper [4], in which discrete time systems were considered. The definitions of the functions ϱ and μ, were borrowed from [5, Theorem 6.7]

Remark 4. For the different proof valid only in the case $\pi_1 \equiv 0$, $\pi_2 \equiv 0$ and based on the method of Liapunov function, we refer to [6, pp. 73–74].

Remark 5. The Theorem is true in the case of infinite dimensions under the condition that A is a bounded operator (the same proof as above).

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