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A Probabilistic Approach to Automaton-Environment Systems

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There are many papers dealing with the automaton-environment systems from deterministic point of view. The aim of this paper is to introduce a probabilistic aspect into this field. The presented model supposes that the actions of the automaton can involve certain unexpected consequences in the environment. Frame axioms are in this model replaced by a new deduction rule allowing to assume that some assertion, describing the situation of the environment in a past situation is valid also in the present situation under the condition that no explicit information about changes concerning this assertion is at disposal. It is proved that there is a positive probability that the system of assertions the aim of which is to describe the situation in the environment will become inconsistent. Using the apparatus of random walks we are able to prove that under certain conditions this probability of inconsistency tends to one when the number of actions performed by the automaton increases.

1. INTRODUCTION

There are many books and papers dealing with automaton-environment systems of various types. Such an investigation is closely connected with some problems arising in robotics, cybernetics and automatic problem-solving and it is very difficult to define precise borders between these fields of applied science.

The basis idea on wich an investigation of automaton-environment systems is based is the following one: an automaton is situated in certain medium or world surrounding it. The automaton is able to observe the medium, to measure some of the environment properties or parameters, in general, to obtain some information from the medium. Moreover, the automaton is also able to intervene in an active form in the medium, to perform some operations and, in this way, to change somehow and in certain measure the situation in the environment. In this paper we shall assume, moreover, that the automaton can move, i.e. that it is able to change its own position in the environment. The automaton activity is not goalless, there is a goal, defined usually by the human user, which is to be reached; usually the goal is certain situation of the environment which is to be reached. Being able to achieve the goal the automaton constructs, in its memory, certain formal description of the environment derived from the empirical data, based on an appropriate logical calculus and modified during the automaton activity according to its actions. On the other hand this formal representation is used in order to derive, which next steps would lead to the goal, in other words, the formal representation is used in order to plan automaton further activity or behaviour.

The greatest part of mathematical theories trying to describe this relation between the automaton and the environment model the situation in deterministic way. The environment is considered to be static or to develop according to some deterministic laws known to the user, the way of the reasoning of the automaton is also deterministic and all the automaton actions lead to consequences which are again supposed to be deterministic and to be known to the user as well as to the automaton itself.

In this paper we propose a model of automaton-environment systems which is of statistical and probabilistic nature. Here we shall mention only briefly the basic aspects in which our model differs from the usual ones; a more detailed description can be found in further chapters.

(1) The properties of various objects in the environment are supposed to be random events which are not known a priori neither to the user nor to the automaton.

(2) The operations which are at the automaton disposal are supposed to be of "fuzzy" character, i.e. the operations can lead to some unexpected consequences (besides the expected ones). These unexpected consequences are supposed to be of statistical nature, i.e. they may accur or need not occur with some probability.

(3) The frame problem is solved not by the mean of the frame axioms but by introducing a new deduction rule, called frame deduction rule. This rule enables the following operation (which can be called actualization): If the automaton knows that something was valid in some past situation, it assume that the same is valid now, in the present situat'on, supposing in the automaton memory is not present a formula saying that the assertion ceased to be valid in some past situation between those two situations.

It is shown in this paper that these three assumptions lead to the following principal conclusions. First, there is a positive probability that the formalized theory describing the environment becomes inconsistent. As an inconsistent theory cannot serve as an appropriate model of the environment this event forces some intervention and it is why it deserves attention. Second, it is possible, if some slight simplifications admitted, to describe the automaton behaviour using the apparatus that is developed in the random walk theory. Here, our intention is not to describe the internal aspects of automaton behaviour, just the picture seen by an external observer. This apparatus enables also to derive some further results concerning the inconsistency of the formal description in question. Namely, with the probability one, sooner or later this formalized theory becomes inadequate in such a sense that there is at least one formula derivable from this representation but not valid in the environment. If, moreover, the random walk describing the automaton behaviour is symmetric, then, with the probability one, the formal representation in question becomes, sooner or later, inconsistent.

There is a connection between this paper and the author's former works [8]; [9]. In those work we investigated the probability with which some conclusions derived from axioms are valid supposing we knew the probability with which the axioms are valid. The problem how to derive the probability for axioms was not taken into consideration. In this paper the probability of the validity of our formal representation is not supposed to be known a priori, but it is derived from other parameters of the probabilistic model proposed to describe the investigated automaton-environment system.

All the chapters of this work contain, in our opinion, a rather detailed intuitive description of the problems investigated in the chapter in question, so it seems rather not necessary to give some intuitive explanation here.

As already mentioned the inconsistency of the formal representation of the environment is an event which cannot be passively accepted as it menaces all the automaton-environment system in its deep grounds. The author's aim is to develop a procedure which would either eliminate or at least minimize the probability of inconsistency mentioned above. Another way, which seems to be hopefull is to use the so called "almost consistent" or "quasi-consistent" theories, however, this field of mathematical logic requests, first of all, some more theoretical investigation.

2. A PROBABILISTIC MODEL OF AUTOMATON BEHAVIOUR

In this work we intend to develop and to study a probabilistic model of the so called automaton-environment systems. Hence, we shall start with a formal description of what *environment* means in the following considerations.

As the basic space we consider the two-dimensional space I^2 of the points with integer coordinates, i.e.

$$I^2 = \{\langle i, j\rangle : i, j \in I\}, \quad I = \{\ldots, -1, 0, 1, 2, \ldots\}.$$

Elements of I^2 are called *points*, denoted by a, b, c, x, y etc. and, moreover, if $a \in I^2$ then a_1 (a_2 resp.) denotes, without a special mentioning, the first (the other, resp.) coordinate of a, i.e. $a = \langle a_1, a_2\rangle$.

Let $n \in N = \{0, 1, 2, \ldots\}$, let $a \in I^2$. The *n-th neighbourhood* $\mathcal{O}_n(a)$ of the point a is defined as follows

$$\mathcal{O}_0(a) = \{a\},$$
$$\mathcal{O}_1(a) = \{b : b \in I^2, |b_1 - a_1| \leqq 1, |b_2 - a_2| \leqq 1\},$$
$$\mathcal{O}_n(a) = \bigcup_{y \in \mathcal{O}_{n-1}(a)} \mathcal{O}_1(y).$$

If $A \subset I^2$, then $\mathcal{O}_n(A) = \bigcup_{y \in A} \mathcal{O}_n(y)$. Instead of 1-st neighbourhood only the term *neighbourhood* will be used.

The points in I^2 can have various properties. In order to be able to describe these properties and to discuss them we must have at our disposal an appropriate formalized language $\mathscr{L}$. In all the rest of this paper we shall speak only about properties of points in I^2, not about properties of sets of points or about properties of relations among points etc. Hence, the first-order predicate calculus seems to be quite adequate for this sake, supposing appropriate individual, predicate and functional constants are chosen.

Moreover, we limit ourselves to properties of particular points and we shall not take into consideration relations between properties of points. Only exception from this rule will be the so called dependence axioms, which will be described later. This means, that our language contains only unary predicates the only variable ranging over I^2.

In this way, however, we should be able to speak about points from I^2 and their properties only from the static point of view, i.e. we should not be able to express somehow the fact, that our space and properties of its points can change as the time passes. In this paper we consider the automaton to be the only source of changes in the environment, but, as will be shown later, we shall suppose that some consequences of the automaton actions can be of random character.

In order to adapt our language to be able to describe the dynamicity of the environment we enriche $\mathscr{L}$ by a new kind of indeterminates – so called *situation indeterminates*. The situation indeterminates range a set S of situations; this set wil be described later, here S can be considered just to be an abstract non-empty set. So we obtain a two-sorted language $\mathscr{L}^*$ (S), to any formula $P(x) \in \mathscr{L}$ corresponds the set $\{P(x, s) : s \in S\} \subset \mathscr{L}^*(S)$ of formulas. We shall see later that the set of situations and that of situation indeterminates need not be distinguished (at lrast from practical point of view) and we may allow ourselves this small logical incorrectness for the sake of more simple denotation.

It is a well-known fact that all the formulas of a formalized language based on the first-order predicate calculus can be effectively numbered (so called Gödelization or Gödel numbers theory, cf., e.g. [7]). This gives, when applied to $\mathscr{L}$ and to $\mathscr{L}^*$, that we can introduce a new, ternary predicate V such that

$$V : I^2 \times N \times S \to \{true, false\}$$

Said in another way $V(a, i, s)$ means: the point a possesses, in the situation s, the i-th property, where i-th property is that one, formally defined by the predicate (unary) of $\mathscr{L}$ having obtained the Gödel number i during the process of Gödelization. Here, every natural number is supposed to be the Gödel number of a unuary predicate from $\mathscr{L}$, the usual process of Gödelization can be effectively modified to satisfy this condition. The use of situation indeterminates ensures, clearly, that formulas V

(a, i, s) and $\neg V(a, i, s')$ need not be contradictory, if $s \neq s'$, hence, our description is capable to reflex some changes in the environment.

Using only unuary predicates (not taking into consideration situation indeterminates) our apparatus is not able to describe relations between points and their properties. As we want to profit from the advantages of the fact that $\mathscr{L}$ contains just unary predicates we must try to express these relations between points and their properties in another way, inside our formalism. For this reason the so called *dependence axioms* will be introduced. The set S of situations is supposed to be countable and effectively enumerable.

Definition 2.1. Dependence axiom of size $n (n \in N)$ is a triple (f, g, h) of recursive functions defined on Cartesian product $(I^2)^n \times N^n \times S^n$ such that

f takes its values in I^2,
g takes its values in N and
h takes its values in S.

For any $a_1, a_2, \ldots, a_n \in I^2$, any $i_1, i_2, \ldots, i_n \in N$ and any $s_1, s_2, \ldots, s_n \in S$ the implication

$$(2.1) \quad [V(a_1, i_1, s_1) \& \ldots \& V(a_n, i_n, s_n)] \to V(fa_1, \ldots, a_n, i_1, \ldots, i_n, s_1, \ldots, s_n), \\ g(a_1, \ldots, a_n, i_1, \ldots, i_n, s_1, \ldots, s_n), h(a_1, \ldots, a_n, i_1, \ldots, i_n, s_1, \ldots, s_n))$$

is called an *instance* of the dependence axiom (f, g, h); the sense of dependence axioms is to describe in a concentrated form the set of all their instances.

Intuitively, a dependence axiom expresses our knowledge about some relations among points and their properties which are valid in the environment. Namely, having to our disposal the dependence axiom (f, g, h) and knowing that the point a possesses the i_1-th property in the situation s_1 and, at the same time, that the point a_2 possesses the i_2-th property in the situation s_2 and, at the same time ... we can be sure that the point $a_0 = f(a_1, \ldots, a_n, i_1, \ldots, i_n, s_1, \ldots, s_n)$ possesses the property $i_0 = g(a_1, \ldots, a_n, i_1, \ldots, i_n, s_1, \ldots, s_n)$ in the situation $h(a_1, \ldots, a_n, i_1, \ldots, i_n, s_1, \ldots \ldots, s_n) = s_0$. Moreover, a_0, i_0 and s_0 can be effectively computed.

During the process of Gödelization all the properties of points from I^2, i.e. all the unary predicates of $\mathscr{L}$ were numbered, including also the contradictory properties, as. e.g. "to have just one colour & to be red & to be green". The Gödel numbers of such predicates will be called *degenerated*.

Definition 2.2. Denote by P_n the unary predicate of $\mathscr{L}$ with the Gödel number n. Set

$$(2.2) \quad Deg = \{i : i \in N, \{(\exists x)[(x \in I^2) \& P_i(x)]\} \leftrightarrow \{(\exists x)[(x \in I^2) \& P_i(x)]\} \& \\ \& (\forall x)((x \in I^2) \to \neg P_i(x))\} \textit{ is valid implication}\}.$$

Natural numbers from Deg are called degenerated.

Definition 2.3. A dependence axiom (f, g, h) is called *consistent* if the function g satisfies: $g(a_1, \ldots, a_n, i_1, \ldots, i_n, s_1, \ldots, s_n) \in Deg$ if and only if some $i_j, j \geqq n$ belongs to Deg.

This means that if the property proclaimed by the dependence axiom (f, g, h) to be valid for the point $f(a_1, \ldots, a_n, i_1, \ldots, i_n, s_1, \ldots, s_n)$ is contradictory, then some of the properties occuring in the premises must be also contradictory and, on the other side, from premises containing at least one contradictory property nothing positive can be deduced.

Definition 2.4. A finite set $\{(f_1, g_1, h_1), (f_2, g_2, h_2), \ldots, (f_m, g_m, h_m)\}$ of dependence axioms (not necessarily of the same size) is called *consistent*, if for any $j \leqq m$ (f_j, g_j, h_j) is consistent in the sense of the last definition and if for any $k_1 < k_2 < \ldots$ $\ldots < k_M \leqq m$ and any n (k_j)-tuples $(a_1^{k_j}, \ldots, a_{n(k_j)}^{k_j})$, $(i_1^{k_j}, \ldots, i_{n(k_j)}^{k_j})$, $(s_1^{k_j}, \ldots, s_{n(k_j)}^{k_j})$, $j = 1, 2, \ldots, m$, such that

$$f_{k_1}(a_1^{k_1}, \ldots, a_{n(k_j)}^{k_1}, i_1^{k_1}, \ldots, i_{n(k_1)}^{k_1}, s_1^{k_1}, \ldots, s_{n(k_1)}^{k_1}) =$$

$$= f_{k_2}(a_1^{k_2}, \ldots, a_{n(k_2)}^{k_2}, i_1^{k_2}, \ldots, i_{n(k_1)}^{k_2}, s_1^{k_2}, \ldots, s_{n(k_2)}^{k_2}) = \ldots =$$

$$= f_{k_M}(a_1^{k_M}, \ldots, a_{n(k_M)}^{k_M}, i_1^{k_M}, \ldots, i_{n(k_M)}^{k_M}, s_1^{k_M}, \ldots, s_{n(k_M)}^{k_M}$$

hold and, at the same time, the same equalities are valid if f replaced by h the following holds:

the Gödel number of the predicate

$$Pg(a_1^{k_1}, \ldots, a_{n(k_1)}^{k_1}, i_1^{k_1}, \ldots, i_{n(k_1)}^{k_1}, s_1^{k_1}, \ldots, s_{n(k_1)}^{k_1})(X)\ \&$$

$$\&\ Pg(a_1^{k_2}, \ldots, a_{n(k_2)}^{k_2}, i_1^{k_2}, \ldots, i_{n(k_2)}^{k_2}, s_1^{k_2}, \ldots, s_{n(k_2)}^{k_2})(X)\ \&\ \ldots\ \&$$

$$\ldots\ \&\ Pg(a_1^{k_M}, \ldots, a_{n(k_M)}^{k_M}, i_1^{k_M}, \ldots, i_{n(k_M)}^{k_M}, s_1^{k_M}, \ldots, s_{n(k_M)}^{k_M})(X)$$

belongs to $N - Deg$.

This, from the first sight rather complicated definition, has the following intuitive sense: if we are able, using two or more dependence axioms to deduce more properties concerning the same point in the same situation, then these properties do not contradict each other.

Now two special cases of dependence axioms will be introduced, both of them being important in the following explanation.

Definition 2.5. A dependence axiom (f, g, h) is called *limited*, if there is a natural R such that for any n-tuple $\langle a_1, a_2, \ldots, a_n \rangle \in (I^2)^n$ containing at least one a_j, $j \leqq n$ with $|a_1| + |a_2| > R$ and for any n-tuples $\langle i_1, i_2, \ldots, i_n \rangle \in N^n$, $\langle s_1, s_2, \ldots, s_n \rangle \in S^n$ there exists $k \leqq n$ with the property:

$$f(a_1, \ldots, a_n, i_1, \ldots, i_n, s_1, \ldots, s_n) = a_k ,$$
$$g(a_1, \ldots, a_n, i_1, \ldots, i_n, s_1, \ldots, s_n) = i_k ,$$
$$h(a_1, \ldots, a_n, i_1, \ldots, i_n, s_1, \ldots, s_n) = s_k .$$

This gives, that if for some $j \leqq n$ $|a_1| + |a_2| > R$, i.e., a is "rather far from the beginning" then all the instances of dependence axiom (f, g, h) are of the type

$$[V(a_1, i_1, s_1) \,\&\, V(a_2, i_2, s_z) \,\&\, \ldots \,\&\, V(a_n, i_n, s_n)] \to V(a_k, i_k, s_k),\ k \leqq n ,$$

i.e. it is trivial. Hence, a limited dependence axiom enables to make some conclusions only for the points not being too far from the beginning (i.e. from $\langle 0, 0\rangle$). This notion expresses the fact, that our knowledge concerning the environment is usually limited to a sphere of objects and events being observable by ourmeans of observation.

The next definition introduces the notion of the *similarity dependence axiom*. Such an axiom expresses the fact that there is a point such that all the points in its certain neighbourhood have just the same properties as the points in the corresponding neighbourhood of $\langle 0, 0\rangle$.

Definition 2.6. A similarity dependence axiom with the centre a, $a \in I^2$, and radius $R \in N$ is a dependence axiom (f, g, h) of the size 1 with the following properties: If $b \in I^2$, $i \in N$, $s \in S$, then

$$f(b, i, s) = \langle b_1 + a_1, b_2 + a_2\rangle \quad \text{in case} \quad |b_1| + |b_2| \leqq R ,$$
$$f(b, i, s) = b, \quad \text{if} \quad |b_1| + |b_2| > R ,$$
$$g(b, i, s) = i, \quad h(b, i, s) = s \quad \text{in all cases.}$$

Let us finish, at least for now, the developing of the apparatus being to our disposal in the rest of this paper in order to describe and investigate the environment and let us concern our attention to the other side or aspect of the automaton – environment system, i.e. to the automaton.

It is not the aim of these lines to try to define what the automaton may be. We shall limit ourselves to the assumptions, that the automaton is able:

to change its position in the environment,

to observe points and their properties – at least some of them,

to change some properties of points into another ones,

to make a formal representation of the observed facts, e.g. using some symbolic representation in a computer storage,

to make some conclusions from the information being to its disposal.

It is beyond our intentions to investigate how all tasks and activities could be ensured from the technical and practical point of view. In the rest of this chapter we shall try to formalize somehow, what the automaton is supposed to be able to do and what it does in various situations.

All the kinds of activities which the automaton is capable to perform can be divided into two great groups, namely:

(a) *actions*, representing the "physical" or "dynamical" parts of the automaton behaviour. There are the following kinds of actions: *movements*, *operations*, *observations*, *deductions*, which will be described below,

(b) *representation*, which means that there is, in a storage of the automaton, certain set or supply of formulas, either given a priori by the user of the automaton as valid, or formed on the basis of observations or derived from the foregoing ones using some deduction rules. This set of formulas is dynamically modified according to the new observations or other activities of the automaton and the intention is that these formulas should represent or describe the environment "in the best possible way" in certain sense. There is also the close connection between the representation and the action in the sense that with every action certain transformation of the representation is associated.

A detailed description of actions:

(a) *movements*: there are eight of them, namely the automaton, being situated in a point $a \in I^2$ is supposed to be able to move itself into any point in $\mathcal{O}_1(a)$. We shall use the geographical convention and the particular moves will be denoted by $N, NE, E, SE, S, SW, W, NW$, respectively.

(b) *operations*: operation expresses the ability of the automaton to change some properties of points under conditions. For the sake of simplicity we shall consider, in all this paper, only the situation, when the operation changes some property of just the point in which the automaton is actually situated and when the conditions, under which the operation can be performed, concern only the point in question. If the properties of points are, e.g., their colours, then an example of an operation can be: "if the point is red, change its colour to green". In such a case any operation is defined by a pair $\langle j, k \rangle$ of naturals in this way: $Op\,(j, k)$ means that if $V(a, j, s)$ holds, then it is changed to $V(a, k, s')$ where a is the actual position of the automaton, s is the actual situation and s' is the situation resulting from s by the application of $Op(j, k)$ to s.

(c) *observations*: express the ability of the automaton to observe the environment and to find or investigate, whether the points have or have not certain properties. Any observation is defined by a pair $\langle a, j \rangle$, $a \in I^2$, $j \in N$ in this way: $Ob(a, j)$ means that the automaton investigates, whether the point a possesses the j-th property or not, i.e. whether $V(a, j, s)$ is valid or not in the actual situation s.

(d) *deductions* express the ability of the automaton to use the dependence axioms being to its disposal in order to deduce some new formulas of the type $V(a, i, s)$. Deduction is defined by a dependence axiom (f, g, h) and three n-tuples $\langle a_1, \ldots, a_n \rangle$, $\langle i_1, \ldots, i_n \rangle$, $\langle s_1, \ldots, s_n \rangle$ to which the axiom should be applied. It is why deduction will be denoted by

$$Ded(f, g, h, \langle a_1, \ldots, a_n \rangle, \langle i_1, \ldots, i_n \rangle, \langle s_1, \ldots, s_n \rangle)\,.$$

As the functions f, g, h are recursive, it can be easily seen that there are infinitely countably many possible deductions.

Now, we are in a position to explain and to define precisely what will be understood under the notion "situation" used, till now, only in the abstract sense. In the next chapter this definition will be slightly modified.

Definition 2.7. The set S of situations is the minimal set satisfying the following conditions:

(a) s_0 is a situation (the so called initial or starting situation), i.e. $s_0 \in S$.
(b) If $s \in S$, then Ns, NEs, Es, SEs, Ss, SWs, Ws, $NWs \in S$.
(c) If $s \in S$ and $\langle j, k \rangle$ is a pair of natural numbers, then $Op(i, j)\, s \in S$.
(d) If $s \in S$, $a \in I^2$, $j \in N$, then $Ob(a, j)\, s \in S$.
(e) If $s \in S$, $s_1, s_2, \ldots, s_n \in S$, $a_1, \ldots, a_n \in I^2$, $i_1, \ldots, i_n \in N$ and (f, g, h) is a dependence axiom of size n, then

$$Ded(f, g, h, \langle a_1, \ldots, a_n \rangle, \langle i_1, \ldots, i_n \rangle, \langle s_1, \ldots, s_n \rangle)\, s \in S\,.$$

Denote by $\mathscr{A}$ the set of all actions defined above, i.e.

$$\mathscr{A} = \{N, NE, E, SE, S, SW, W, NW,\ Op(i, j)\,(i, j \in N),\ Ob(a, j)\,(a \in I^2, j \in N),$$
$$Ded(f, g, h,\ \langle a_1, \ldots, a_n \rangle,\ \langle i_1, \ldots, i_n \rangle,\ \langle s_1, \ldots, s_n \rangle)$$
$$((f, g, h) \text{ is a dependence axiom, } a_j \in I^2,\ i_j \in N, s_j \in S,\ j = 1, 2, \ldots, n)\}\,.$$

Using the notation A^* for the set of all finite sequences of elements from a set A and denoting by $s < s'$, $s, s' \in S$ the fact that s' can be constructed from s using finitely many times Definition 2.7 we can state:

Theorem 2.1. The set S of situations is the subset of $(\mathscr{A} \cup \{s_0\})^*$, satisfying the following conditions:

(a) If $s = \alpha_1,\alpha_2, \ldots, \alpha_m \in S$, then $\alpha_m = s_0$, $\alpha_j \neq s_0$ for $j < m$.
(b) If $s = \alpha_1\alpha_2, \ldots, \alpha_m \in S$, if some α_j, $j < m$ is a deduction and if s' is a situation occuring in α_j, then $s' < s$.

Proof. Immediately from the definition of S.

Having described precisely what the action means we can come back to the problem of formal representation of the environment in the automaton. We shall be able, now, to describe also in which way the actions reflect in this formal model of the environment.

In our paper we do not investigate, how the formal representation and an appropriate logical calculus on which it is based serve in order to plan the future actions and behaviour of the automaton. This means that we do not investigate which theorem-proving algorithms are to the automaton's disposal or how to deduce, from a formal proof of an appropriate theorem, the sequence of actions leading to the goal. Some papers or monographies dealing with this problems were mentioned in the introductory part of this paper, some more can be found in the references. The only aspect to which we concern our attention in this paper is to study, in which measure the set of formulas, being storaged in the automaton memory, together with their logical consequences describes in an adequate way the environment. Our special attention will be concerned to the problem of frame axioms and to the possibility of inconsistency of the formalized theory resulting from our formal description when the frame axioms are replaced by a new deduction rule, the so called *frame deduction rule.* More details concerning this rule will be given later.

In the title of this chapter we promised to give a probabilistic approach to the description and modelling of the automaton behaviour. Now, we are in a position to explain the probabilistic aspects of our model.

As it is usual in great part of problem-solving models, also in our description to every action certain transformation of the set of valid formulas representing the environment is associated. As in the so called STRIPS method such a transformation is described by a triple

$$\langle Cond\ \varphi, Out\ \varphi, In\ \varphi\rangle\ , \quad \varphi \in \mathscr{A}\ ,$$

where $Cond\ \varphi$, $Out\ \varphi$ and $In\ \varphi$ are finite sets of formulas of the form $V(a, i, s)$. The meaning of these sets can be roughly described as follows: if the formulas from $Cond\ \varphi$ are valid in some situation, then the action φ is applicable in this situation and, as the result of this action, formulas from the set $Out\ \varphi$ are not more valid in the new situation and must be eliminated from the set of valid formulas, on the other side formulas from $In\ \varphi$, not having been, in general, valid before, are valid in the new situation and must be joined with the set of valid formulas.

The usual models for describing the automaton behaviour are based on the presumption that this triple of sets describes and reflects precisely and correctly all the consequences of an action in the environment. In this paper we leave this presumption. We assume, now, that the changes in the environment, involved by an action, are of probabilistic character. Namely, if the formulas from $Cond\ \varphi$ are valid, the action φ is applicable and if it is applied, again, formulas from $Out\ \varphi$ are not more valid in the new situation. However, now we assume, that, with some probability, even some other formulas, having been valid before and not contained in $Out\ \varphi$ are not more valid in the new situation. And the same assumption is connected with the set $In\ \varphi$, it is possible, with a positive probability, that there are formulas not having been valid before, which are valid after the application of φ and which are not contained in $In\ \varphi$.

To describe this situation we shall assume, in what follows, that $Cond\ \varphi$ is again a finite set of formulas, but not $Out\ \varphi$ and $In\ \varphi$. Now, $Out\ \varphi$ is a pair $\langle Out_0\ \varphi, (f\varphi)\rangle$ and $In\ \varphi$ is a pair $\langle In_0\ \varphi, (g\varphi)\rangle$ such that if $\mathscr{W}$ is the set of all formulas of the form $V(a, i, s)$, then $Out_0\ \varphi \subset \mathscr{W}$ and $(f\varphi)$, $(g\varphi)$ are real functions defined on $\mathscr{W}$, taking their values in $\langle 0, 1\rangle$ and such that if $V(a, i, s) \in Out_0\, \varphi(V(a, i, s) \in In_0\, \varphi$, resp.), then $(f\varphi)(V(a, i, s)) = 1$ $((g\varphi)(V(a, i, s)) = 1$, resp.). For the readers familiar with the notation of fuzzy-set this definition can be translated in a very simple form: $(f\varphi)$ and $(g\varphi)$ are fuzzy-sets over the set $\mathscr{W}$ of all formulas of the form $V(a, i, s)$. (To find some information about fuzzy-sets see, e.g. [6] or [14].)

Clearly, if $Out_0\, \varphi$ contains all the formulas for which $(f\varphi)$ takes the value 1 and $In_0\, \varphi$ contains all the formulas for which $(g\varphi)$ takes the value 1, then the pair $\langle Out_0\, \varphi, In_0\, \varphi\rangle$ describes precisely all the *necessary* or *deterministic* consequences of action φ and for other formulas the value $(f\varphi)(V(a, i, s)$ expresses the probability that $V(a, i, s)$ was valid before the application of φ and it is not valid now (analogously for $(g\varphi)(V(a, i, s))$. If there is a formula $V(a, i, s) \notin Out_0\, \varphi$, for which $(f\varphi)(V(a, i, s)) = 1$, or $V(a, i, s) \notin In_0\, \varphi$ and $(g\varphi)(V(a, i, s)) = 1$, then the pair $\langle Out_0\, \varphi, In_0\, \varphi\rangle$ does not describe precisely nor the deterministic consequences of action φ. The non-probabilistic approach, e.g. that of STRIPS (as mentioned above) is, simply, a special example of our approach, namely if $(f\varphi)$ is the characteristic function of $Out_0\, \varphi$ and $(g\varphi)$ is the characteristic function of $In_0\, \varphi$, i.e. $(f\varphi)(V(a, i, s)) = 0$, if $V(a, i, s) \notin Out_0\, \varphi$, $(g\varphi)(V(a, i, s)) = 0$, if $V(a, i, s) \notin In_0\, \varphi$.

Before describing in details the sets $Cond\, \varphi$, $Out_0\, \varphi$ and $In_0\, \varphi$ for particular actions $\varphi \in \mathscr{A}$ we must come to an agreement how to describe the set of all formulas the validity of which is storaged in a particular situation. The set of all formulas which are valid in situation s and whose validity is known or can be verified by the automaton on the base of its storage contain will be denoted by $H(s)$. This means, that $H(s)$ contains formulas of the form $V(a, i, s')$ obtained from some observations, contained in $In_0\, \varphi$ of an action φ which has been applied, deduced from another formulas e.t.c. Here s' may be any situation, i.e. $H(s)$ contains formulas of various "age"; "actualization" of these formulas to the actual situation s can be done by the frame deduction rule as will be described later. Moreover, $H(s)$ contains all the consequences of the formulas mentioned above for which the automaton is able to deduce that they follow from some others. What does mean "the automaton is able to deduce" may depend on the case, in general we suppose the automaton to have to its disposal a theorem prover enabling at least for some formulas following from others to prove this fact. In this case the set $H(s)$ can be described by the use of a smaller set $H_0(s) \subset H(s)$ and the consequence operation Cn^* corresponding to the actual theorem – prover being to our disposal in the way:

$$H(s) = Cn^*(H_0(s)) \subset Cn(H_0(s))$$

As can be easily seen the fact of consistency or inconsistency of the set $H(s)$ does not depend on the way in which $H(s)$ is described in the form $Cn^*(H_0(s))$. The same holds if we ask, whether there is a formula in $H(s)$ which is not valid in the environment (inadequacy of $H(s)$). So we shall assume in this paper, for the sake of simplicity, that only the formulas contained in $H(s)$ can be proved from $H(s)$, i.e. that $H_0(s) =$

$= H(s)$ and Cn^* is the most trivial identity operation. The question how to sample, in practice, Cn^* and $H_0(s)$ is very important from the point of view of economization, however, this matter will not be investigated here.

Let us finish this chapter by giving the form of the sets $Cond_0\ \varphi$ and $In_0\ \varphi$ for various $\varphi \in \mathscr{A}$. The functions $(f\varphi)$ and $(g\varphi)$ may differ in various cases according to the actual conditions causing or involving the uncertainty of the automaton action, it is why we do not define them here. Some simplifying assumptions concerning $(f\varphi)$ and $(g\varphi)$ are introduced in Chapter 5.

We assume to have in $H(s)$ a special unary predicate describing the automaton position in a situation s. Let the Gödel number of this predicate be I_0, i.e. $V(a, I_0, s)$ means that in the situation s the automaton is in the point $a \in I^2$. At the beginning, i.e. in the situation s_0, the automaton is supposed to be in $\langle 0, 0\rangle$ and, moreover, it is supposed to know about it, i.e.

$$V(\langle 0, 0\rangle, I_0, s_0) \in H(s_0)$$

There are no more presumptions concerning the set $H(s_0)$, it may contain, of course, some more formulas given to it a priori. The automaton is also supposed to know, from the very beginning, the dependence axioms being to its disposal, however these axioms are not supposed to be in the sets $H(s)$ and are not subjected to changes when s changes.

We shall not investigate the most general situation and we adopt some assumptions for the sake of simplicity of the following reasoning. The moves and observations are supposed to be applicable in every case, i.e., there are no special conditions to which these actions should be subjected. For an operation φ the set $Cond(\varphi)$ is supposed to be one-element.

Denote, for any $a \in I^2 : N(a) = \langle a_1, a_2 + 1\rangle$, $NE(a) = \langle a_1 + 1, a_2 + 1\rangle$, $E(a) = \langle a_1 + 1, a_2\rangle$, $SE(a) = \langle a_1 + 1, a_2 - 1\rangle$, $S(a) = \langle a_1, a_2 - 1\rangle$, $SW(a) = \langle a_1 - 1, a_2 - 1\rangle$, $W(a) = \langle a_1 - 1, a_2\rangle$, $NW(a) = \langle a_1 - 1, a_2 + 1\rangle$ (the sense should be clear), denote for any $s \in S$ the point of I^2, where the automaton is situated in the situation s, by $R(s)$ (e.g. $R(s_0) = \langle 0, 0\rangle$, $R(NEs_0) = \langle 1, 1\rangle$ etc.). Now, the sets $Cond\ \varphi$, $Out\ \varphi$ and $In\ \varphi$ for various actions follow (actions are supposed to be applied in a situation $s \in S$):

(a) moves: For any move φ

$$Cond(\varphi) = Out_0(\varphi) = \emptyset \quad \text{(the empty set)}.$$

$$In_0(NE) = \{V(NE(R(s)), I_0, NEs)\},$$

analogously for other moves. The intuitive sense is clear.

(b) observations: $Ob(a, j)$

$Cond(Ob(a, j)) = \emptyset$,

$Out_0\,(Ob(a, j)) \;= \{\neg\, V(a, j, s)\}$, if $V(a, j, s)$ is valid and $\neg\, V(a, j, s) \in H(s)$,
$Out_0\,(Ob(a, j)) \;= \emptyset$ else,
$In_0\,(Ob(a, j)) \;= \{V(a, j, Ob(a, j)\, s), V(R(s), I_0, Ob(a, j)\, s)\}$, if $V(a, j, s)$ is valid,
$In_0\,(Ob(a, j)) \;= \{V(R(s), I_0, Ob(a, j)\, s)\}$ else.

Intuitively: $Ob(a, j)$ means that the automaton observes, whether the point $a \in I^2$ possesses the j-th property. If the automaton is able to observe that it is the case, it puts $V(a, j, s)$ into its storage and removes from it $\neg\, V(a, j, s)$ supposing this formula was in $H(s)$. The fact that $V(a, j, s)$ is not removed from $H(s)$ supposing the automaton is not able to verify $V(a, j, s)$ by the application of $Ob(a, j)$ follows from the idea that the automaton failure to verify $V(a, j, s)$ can be caused not only by non-validity of this formula, but also by limited means of observation. $V(a, j, s)$ would be removed from $H(s)$ if the observation $Ob(a, k)$ (k is the Gödel number of the negation of j-th property) were successful and $V(a, k, s)$ were verified.

(c) deductions: $Ded\,(f, g, h, \langle a_1, \ldots, a_n\rangle, \langle i_1, \ldots, i_n\rangle, \langle s_1, \ldots, s_n\rangle)$

$$Cond_0\,(Ded\,(f, g, h, \langle a_1, \ldots, a_n\rangle, \langle i_1, \ldots, i_n\rangle, \langle s_1, \ldots, s_n\rangle)) = \\ = \{V(a_j, i_j, s_j),\ j = 1, \ldots, n\}\,,$$

$$Out_0\quad (Ded\,(f, g, h, \langle a_1, \ldots, a_n\rangle, \langle i_1, \ldots, i_n\rangle, \langle s_1, \ldots, s_n\rangle) = \emptyset\,,$$

$$In_0\quad (Ded\,(f, g, h, \langle a_1, \ldots, a_n\rangle, \langle i_1, \ldots, i_n\rangle, \langle s_1, \ldots, s_n\rangle) = \\ \{V(f(a_1, \ldots, a_n), g(i_1, \ldots, i_n),\ h(s_1, \ldots, s_n))\,, \\ V(R(s), I_0, Ded\,(f, g, h, \langle a_1, \ldots, a_n\rangle, \langle i_1, \ldots, i_n\rangle, \langle s_1, \ldots, s_n\rangle)\, s)\}\,.$$

(d) operations: $Op(j, k)$

$$Cond\,(Op(j, k)) = \{V(R(s), j, s)\}\,,$$
$$Out_0\,(Op(j, k)) = \{V(R(s), j, s)\}\,,$$
$$In_0\,(Op(j, k)) \;= \{V(R(s), I_0, Op(j, k)\, s), V(R(s), k, Op(j, k)\, s)\}\,.$$

This means: If the point in which the automaton is situated possesses the j-th property, then $Op(j, k)$ is applicable and, when applied, the point will posses the k-th property. The fact that $V(R(s), j, s)$ is removed does not mean that this formula is not more valid, it is possible that in the situation $Op(j, k)\, s$ the point $R(s)$ possesses both the j-th and the k-th properties, however, we are not justified to claim it in general.

In all the cases mentioned above the presence of the formula $V(R(s), I_0, \varphi s)$ in $In_0\, \varphi$, where φ is not a move, is necessary to express explicitly that during an observation, deduction or operation the automaton is supposed not to change its position in I^2. It would be possible to remove, in all the cases, the formula $V(R(s), I_0, s)$ from $H(s)$, as the sequence of automaton positions in I^2, from s till the actual situation φs can be extracted from φs. I.e., in all the cases above we could, instead of $Out_0\, \varphi$, consider the set $Out_0\, \varphi \cup \{V(R(s), I_0, s)\}$.

From the point of view of storage space saving this may be useful and important, but from the point of view of consistency or adequancy of $H(s)$, which is dominant in this paper, this difference is irrelevant.

For all action the new set H is defined:

$$H(\varphi s) = (H(s) - Out_0(\varphi)) \cup In_0(\varphi), \quad \varphi \in \mathscr{A}, \varphi \text{ applied in } s \in S .$$

In order to simplify our position we shall suppose, in all the rest of this paper, that if φ is a move, an observation or a deduction, then $(f\varphi)$ is the characteristic function of $Out_0(\varphi)$ and $(g\varphi)$ is the characteristic function of $In_0(\varphi)$. On the other hand, if φ is an operation, then we suppose that for any $V(a, i, s) \in \mathscr{W}$

$$(f\varphi)(V(a, i, s)) > 0 , \quad (g\varphi)(V(a, i, s)) > 0 .$$

I.e. only operations can have some unexpected consequences, but if an operation is applied, then any property of any point has a positive probability to occur in $Out(\varphi)$ or in $In(\varphi)$. Of course, there are some properties of points which cannot change, e.g. tautological properties, which are always valid or contradictory properties, which are never valid. This situation can be described in our model in such a way that formulas $V(a, i, s)$, corresponding to such properties, occur or do not occur simultaneously in $Out(\varphi)$ and $In(\varphi)$ and both the cases have positive probabilities. Here $Out(\varphi)$ and $In(\varphi)$ are random sets describing the formulas which cease to be valid or began to be valid in particular cases, i.e. $Out(\varphi)$ and $In(\varphi)$ contain also the random consequences of φ.

The assumptions introduced above represent an important simplification of our basic general model and it is possible to construct automaton-environment systems for which these simplifications are not justified. However, it would be very difficult and maybe impossible to bring some analytical and numerical results, which we intend to bring in Chapter 5, for the most general model. We hope, that even including the introduced simplifications our model represents an interesting alternative to the usual deterministic model. These simplifications, moreover, do not influence, in decisive way, the problems concerning the consistency and adequacy of the formal representation, which will be investigated below.

3. FRAME DEDUCTION RULE

When describing the possible actions of the automaton we have introduced a special type of automaton activity which will be called *actualization*. It is caused by the fact that this activity cannot be described by a triple $\langle Cond\ \varphi, Out\ \varphi, In\ \varphi \rangle$ as the actions, mentioned above, nor in case these sets are supposed to be of fuzzy nature as we supposed in the foregoing chapter. The reason is the following: the actualization can be applied only if some formula or formulas are *not* valid in the actual situation. As far no problem occurs, because the non-validity of a formula can be written as the validity of its negation, however, if the validity of a formula can be proved only by finding this formula in $H(s)$, then the non-validity would be provable only by finding the negation of the checked formula in $H(s)$ and this would be too weak condition for our purposes. So we are forced, as will be shown later, to formalize the conditions of actualization operation in such a way that actualization is applicable if certain formula or formulas does or do not occur in $H(s)$ – and this condition cannot be expressed in the form *Cond* φ, described above. Before giving the details we would like to remark that this actualization operation is very close to the UNLESS – operator introduced by Sandewall in [12]. We shall see that difficulties connected with both the operations are very similar, namely they both can lead to the inconsistency of a system to which they are added as a new deduction rule. However, our probabilistic approach enables to admit this possibility supposing its probability is not "to great" in certain sense.

The frame deduction rule, enabling the procedure of actualization is suggested in this chapter as a way how to solve or avoid the so called frame axioms problem. This problem can be roughly described as follows:

The sets of formulas *Out* φ, *In* φ (*Out*$_0$ φ, *In*$_0$ φ in our probabilistic approach) describe explicitly what changes in the environment when action φ applied. In practice, when an action is described in this way, we usually suppose, without special mentioning, that everything else, not mentioned explicitly in *Out* φ or *In* φ is the same as before also in the new situation resulting from the foregoing one by application of the action in question. For example, if it is not mentioned explicitly, that an object changes its colour during an action of the automaton, the colour of that object is supposed not to be subjected to any change when the situation changes. Formally, if for an object $a \in I^2$ $V(a, i, s)$ holds, then also $V(a, i, \varphi s)$ holds. However, to be able to joint also $V(a, i, \varphi s)$ with $H(\varphi s)$ we should be forced to have to our disposal a large number of axioms, connected with every action and expressing explicitly the fact that nothing not having been mentioned in *Out* φ and *In* φ is subjected to a change. These axioms, introduced for the first time in [4] and called here *frame axioms* are usually too numerous to be acceptable as a part of the formal representation of the environment. The problem how to avoid this difficulty is called *frame axiom problem.* A description of this problem and a brief survey of some solutions, their advantages and disadvantages can be found in [5].

Our idea can be called "principle of conservation" or "stability presumption" and consists in this way of reasoning: we suppose that a property of a point valid in some past situation has not changed if we have no explicit information about such a change. This means that if a formula $V(a, i, s') \in H(s)$, $s' < s$ and if there is no formula $V(a, i, s'')$ in $H(s)$, $s' < s'' < s$, such that $V(a, i, s')$ & $Va, i, s'')$ is a contradiction (i.e. the Gödel number of this formula belongs to *Deg*). then $V(a, i, s)$ is also supposed to be valid and is joined with $H(s)$.

To describe this rule in a form applicable to our purposes we must take into consideration that the automaton is not supposed to be able to decide, for any contradictory formula $V(a, i, s')$ & $V(a, i, s'')$ that it is actually a contradiction. There is just a subset $D_0 \subset Deg$ such that the formulas with Gödel numbers ftom D_0 can be tested to be contradictory. According to our assumption that only the formulas being in $H(s)$ are derivable from $H(s)$ we shall suppose that the automaton is able to find that $V(a_0, i, s'')$ contradicts to $V(a_1, i, s')$ only if $V(a_0, i, s'')$ is just the negation of $V(a_1, i, s')$ and $a_1 = a_0$. Now, the frame deduction rule can be described as follows:

Definition 3.1. Frame deduction rule sounds: If $V(a, i, s') \in H(s)$, $s' < s$ and if for no s'', $s' < s'' < s(s', s'', s \in S)$ $\neg V(a, i, s'') \in H(s)$, then $V(a, i, s)$ can be joined with $H(s)$. An application of frame deduction rule is called *actualization* and denoted *Act* (a, i, s', s).

Denote

$$\mathscr{A}' = \mathscr{A} \cup \{Act\,(a, i, s', s), a \in I^2, i \in N, s', s \in S\}$$

the elements of $\mathscr{A}'$ will be called *steps*, so step is either an action ar an actualization.

As already mentioned the frame deduction rule stands very close to Sandewall's UNLESS operator in its basic idea that the condition for application of such a rule is *non-validity* of certain formula. However, our frame deduction rule does not suffer from logical and philosophical difficulties connected with UNLESS-operator and mentioned briefly in [12]. It is caused by the fact that the negative condition for application of the frame deduction rule is not of the form that a formula must not be derivable, the only request is that it has not been derived (from observations or other formulas) until the moment the rule is to be applied. On the other hand, our frame deduction rule as well as the UNLESS-operator do not possess the extension property possessed by all the deduction rules in usual formalized theorirs. This property consists in fact that enlarging the set of premises the set of their consequences is also larger or at least the same. Clearly, when using the frame deduction rule it is possible that, joining a new formula with the set of premises a formula, having been derivable before, will not be more derivable. A more careful and detailed investigation of the deduction rules of this type seems to be very interesting and justified from the point of view of "pure" mathematical logic as well as from the point of view of their various applications, however, such an investigation is beyond the limits of this paper.

In the foregoing chapter we described all the actions which the automaton is supposed to be able to perform, in this chapter we enriched the list of the automaton abilities by a new one — actualization. We also investigated in which form the application of particular steps is reflected in the automaton formal representation of the world.

However, the common interaction between the automaton and the environment is not an individual event, it is a process. From the side of the automaton this means that its behaviour is described by a sequence of actions, not by a particular action. Consequently, the automaton must be supposed not only to perform a particular step but also to decide, after performing a step, which one should be the next.

There are two main aspects influencing the choosing of the next step. First, the automaton is given some goal and its aim is to change the state of the environment in such a way, using the operations, that the goal should be reached. The other aspect is given by the fact that the automaton takes into consideration the information about the environment being at its disposal, i.e. the set $H(s)$. We recall that thanks to a formula of the type $V(a, I_0, s)$ (or more formulas of this type) being always contained in $H(s)$ the automaton has at its disposal the complete information about its behaviour since the beginning situation s_0.

The decision taken by the automaton about its next step can be seen as plan-making activity of the automaton. In general, it is possible to distinguish two forms of planning – one – step planning, when the automaton chooses just the next step and more – step planning, when the automaton decides to perform a finite sequence of steps and as late as after their performing its chooses again what to do now. It is also possible to admit that the automaton chooses at least some of the next steps on a random base, it means on the base of a random experiment (compare with the so called mixture strategies in game theory, e.g. in [1]). The notion of the action random function of the automaton, as introduced below, seems to be general enough to cover all these possibilities.

Definition 3.2. Let $(\Omega, \mathscr{S}, P)$ be a probability space, let $Fin(\mathscr{W})$ be the set of all finite subsets of the set $\mathscr{W}$ (let us recall that $\mathscr{W}$ is the set of all formulas of the form $V(a, i, s)$, $a \in I^2$, $i \in N$, $s \in S$). Then *action random function* M of the automaton is a mapping from the Cartesian product $Fin(\mathscr{W}) \times \Omega$ into the set $\mathscr{A}'$ of steps such that for all $X \in Fin(\mathscr{W})$ the mapping $M(X, .)$ is a random variable, i.e. for any $r \in \mathscr{A}'$

$$\{\omega : M(X, \omega) = r\} \in \mathscr{S} .$$

Supposing, there is, for any $X \in Fin(\mathscr{W})$ an $r = r(X) \in \mathscr{A}'$ such that

$$P(\{\omega : M(X, \omega) = r(X)\}) = 1 , \tag{3.1}$$

then M is called *action function* of the automaton. The demand (3.1) can be replaced by a slightly stronger one, namely

$$M(X, \omega) = r(X) \quad \text{for all} \quad \omega \in \Omega . \tag{3.2}$$

(3.1) as well as (3.2) express the fact that the next step of the automaton is deterministically (or at least with the probability 1) given when known the image $H(s)$ of the environment being at automaton disposal in a situation.

In our approach all the automaton capability to search for the goal, to choose subgoals, to make appropriate plans etc. is hidden somehow in the action (random) function M. This description seems to be very general and in an actual situation it really will be, however, as will be shown below, for our reasons this approach seems to be quite justifiable. Our intention is to model somehow (namely by the use of random walks) the automaton behaviour as seen from the point of view of an external observer, observing only what the automaton does do and which is its formal representation of the environment. The observer does not know and is not interested in the internal automaton activity and neglects it supposing his model of automaton behaviour is, in a sense, "good enough" to describe the external features of the automaton behaviour and to enable to the observer to make some conclusions and hypotheses concerning the automaton (e.g. its future moves, actions or behaviour). We ask the reader to keep in mind this remark because its aim is to serve as an intuitive justification of our further way of reasoning.

First of all we shall suppose that properties of points in I^2 are random events. This assumption does not mean that somebody makes random experiments and, according to their results, defines the properties of points. We would like just to say, that we do not know the properties of points a priori and that we consider the probabilistic apparatus to be appropriate for expressing this ignorance. Formally, let us consider a mapping $\mathscr{V}$ defined on the Cartesian product $I^2 \times N \times S \times \Omega$ and taking its values in two-element set $\{truth, false\}$ such that for any fixed $a \in I^2$, $i \in N$, $s \in S$ the mapping $\mathscr{V}(a, i, s, .)$ is a random variable, i.e.

$$\{\omega : \mathscr{V}(a, i, s, \omega) = true\} \in \mathscr{S} . \tag{3.3}$$

The random event in (3.3.) is that one consisting in the fact that the point $a \in I^2$ posseses, in the situation s, the i-th property. Of course, for different triples $\langle a_1, i_1, s_1 \rangle$, $\langle a_2, i_2, s_2 \rangle$ the random variables $V(a_1, i_1, s_1, .)$ and $V(a_2, i_2, s_2, .)$ need not to be independent, in fact our dependence axioms described just the situation when the dependence among those random variables is of deterministic nature and various types of statistical dependence are not excluded as well.

This assumption involves the following consequence. If, e.g. the observation $Ob(a, j)$ is to be executed in a situation s, then the fact, whether $V(a,j, s)$ is joined with $H(Ob(a, j)\, s)$ or not is a random event, as it dependens on the value taken by $\mathscr{V}(a, i, s)$. Hence, the set $H(s)$ is, in general, a random set, which can be formally described as a value taken by a random variable H, defined on $(\Omega, \mathscr{S}, P)$ and taking its values in $Fin(\mathscr{W})$. Moreover, as the next action of the automaton, given by the action (random) function M depends on $H(s) = H(s, \omega)$, also this next action or step can be considered to be a value of an appropriate random variable. Finally, the situation s, being a finite sequence of steps, depends also on ω, so we can write $s(\omega)$ and $H(s(\omega), \omega)$ instead of s and $H(s)$. The fact, that the sets $\mathscr{A}'$, S and $Fin(\mathscr{W})$ are countable assures the correctness of what we have just said and justifies our intention to work only within the scope of discrete probability distributions. We can also define, for any $x \in \mathscr{A}'$, the real $p(x)$ as follows:

$$p(x) = P(\{\omega : M(H(s(\omega), \omega), \omega) = X\}) . \tag{3.4}$$

The countability of $Fin(\mathscr{W})$ and $\mathscr{A}'$ and the factorization proves the correctness of this definition. Hence, $p(x)$ is the probability, in general, that the step $x \in \mathscr{A}'$ will be applied. Now, we shall assume that this probability is the same and positive in every case and in every situation and that the probabilities of steps in different situations are statistically independent. Said in other word, we assume that the automaton behaviour can be considered to be a sequence of random samples, mutually independent and equally distributed, sampling and performing steps from $\mathscr{A}'$ with respect to the probability distribution $p(x)$, $x \in \mathscr{A}'$. Formally, the automaton behaviour is described by a random variable α, defined on the probability space $(\Omega, \mathscr{S}, P)$ and taking its values in $\mathscr{A}'$ supposing the following two conditions are satisfied:

(3.5) for any $s \in S : s = s(\omega) = \alpha_n(\omega)\, \alpha_{n-1}(\omega), \ldots, \alpha_2(\omega)\, \alpha_1(\omega)\, s_0$,

where α_i, $i = 1, 2, \ldots$, are copies of the random variable α, (3.6.) for any $s = \varphi_n\varphi_{n-1}, \ldots, \varphi_2\varphi_1 s_0 \in S$ holds:

$$P(\{\omega : s(\omega) = s\}) = \prod_{i=1}^{n} P(\{\omega : M(H(s_{i-1}(\omega), \omega), \omega) = \varphi_i\}) = \prod_{i=1}^{n} p(\varphi_i),\ \varphi_i \in \mathscr{A}',$$
$$i = 1, 2, \ldots, n\,.$$

For any situation $s \in S$ define $lh(s)$ as follows:

$$lh(s_0) = 0\,,$$
$$lh(\varphi s) = lh(s) + 1\,, \quad s \in S\,, \quad \varphi \in \mathscr{A}'\,.$$

This gives that for $s = \varphi_1\varphi_2, \ldots, \varphi_n s_0 \quad lh(s) = n$.

Theorem 3.1. Consider an automaton-environment system described above. Let $a \in I^2$, $i \in N$ be such that in every situation s the automaton is able to decide whether $V(a, i, s)$ is valid or not. Let $b \in I^2$ be a point having the i-th property just if a has (i.e. the automaton is able to derive, using a dependence axiom, $V(b, i, s)$ from $V(a, i, s)$ and $\neg\, V(b, i, s)$ from $\neg\, V(a, i, s)$). Let there be at least one operation, let no operation changes the i-th property. Let all the dependence axioms be limited. Then, if $a \neq b$,

$$P(\{\omega : Cn(H(s(\omega), \omega)) = \mathscr{W}\} \mid \{\omega : lh(s(\omega)) \geqq 5\}) > 0\,.$$

Remark. Intuitively said, the theorem sounds: If the automaton performs at least five steps, then there is a positive probability that its formal representation of the environment is inconsistent.

Proof. The assumption that $p(x) > 0$ for any $x \in \mathscr{A}'$ and (3.6) give that

$$P(\{\omega : s(\omega) = s\}) > 0$$

for any $s \in S$. Hence, our theorem will be proved supposing we find, for any $n \geqq 5$ a situation s with $lh(s) = n$ and such that $H(s, \omega)$ is inconsistent.

Consider the following steps $\xi_1, \ldots, \xi_6$:

ξ_1: $Ob(a, i, s_0)$. Here we may suppose that $V(a, i, s_0)$ holds (if not, take instead of i the Gödel number of the negation of the i-th property, say $neg(i)$). This gives that $V(a, i, s_0) \in H(\xi_1 s_0)$.

ξ_2: $Ded(f, g, h, \langle a \rangle, \langle i \rangle, \langle s_0 \rangle)$, where $\langle f, g, h \rangle$ is the dependence axiom enabling to transpose the i-th property from a to b. This dependence axiom is applicable and $V(b, i, s_0)$ enters the set $H(\xi_2 \xi_1 s_0)$.

ξ_3: $Act(b, i, s_0, s_1)$, where

$$s_1 = Op(k, l)\, \xi_2 \xi_1 s_0 .$$

Hence, $V(b, i, s_1) \in H(\xi_2 \xi_1 s_0)$, as the actualization may be performed.

ξ_4: Any operation $Op(k, l)$ such that $\{k, l\} \cap \{i, neg(i)\} = \emptyset$. With a positive probability, namely $(g\ Op(k, l))(V(a, neg(i), \xi_2 \xi_1 s_0))$, now the formula $V(a, neg(i), \xi_2 \xi_1 s_0)$ is valid. Which formula or formulas enter the set $H(\xi_4 \xi_3 \xi_1 s_0)$ is irrelevant, our assumptions assure that none of the formulas $V(a, i, s_0)$, $V(b, i, s_0)$, $V(b, i, s_1)$ is removed from $H(\xi_4 \xi_3 \xi_1 s_0)$. $\xi_5 = Ob(a, neg(i))$: Consider the case mentioned in the foregoing step, that $V(a, neg(i), \xi_4 \xi_2 \xi_1 s_0)$ holds. Then $V(a, neg(i), \xi_4 \xi_2 \xi_1 s_0)$ enters $H(\xi_5 \xi_4 \xi_2 \xi_1 s_0)$.

$\xi_6 = Ded(f, g, h, \langle a \rangle, \langle neg\ i \rangle, \langle \xi_4 \xi_2 \xi_1 s_0 \rangle)$. This dependence axiom be applied and $V(b, neg(i), \xi_4 \xi_2 \xi_1 s_0)$ occurs in $H(\xi_6 \xi_5 \xi_4 \xi_2 \xi_1 s_0)$. However, $V(b, i, s_1) \in \in H(\xi_6 \xi_5 \xi_4 \xi_2 \xi_2 s_0)$ as well and, because $s_1 = \xi_4 \xi_2 \xi_1 s_0$, the set $H(\xi_6 \xi_5 \xi_4 \xi_2 \xi_1 s_0)$ is proved to be inconsistent. Nowe because of the fact that all the dependence axioms are limited, and there is only a finite number of such axioms, there exists an integer R_0 such that any instance $Ded(f, g, h, \langle a_1, \ldots, a_n \rangle, \langle i_1, \ldots, i_n \rangle, \langle s_1, \ldots, s_n \rangle)$ of any dependence axiom, containing at least one a_j, $j \leqq k$, such that $|a_{j1}| + |a_{j2}| > R$, is just of the trivial form.

$$(3.7) \qquad [V(a_1, i_1, s_1) \,\&\, V(a_2, i_2, s_2) \,\&\, \ldots \,\&\, V(a_n, i_n, s_n)] \to V(a_k, i_k, j_k), \quad j \leqq n .$$

Therefore, for any $n \geqq 5$ we can take the situation s_n,

$$s_n = \varphi_{n-5} \varphi_{n-6} \varphi_{n-7}, \ldots, \varphi_2 \varphi_1 \xi_6 \xi_5 \xi_4 \xi_2 \xi_1 s_0 ,$$

where φ_i, $i \leqq n - 5$, are "trivial" deductions of the type (3.7). These deductions do not change the set H, hence

$$H(s_n) = H(\xi_6 \xi_5 \xi_4 \xi_2 \xi_1 s_0) .$$

With the probability

$$(g\xi_4)(V(a, neg(i), \xi_2 \xi_1 s_0)) > 0 ,$$

this set is inconsistent; with the probability

$$\Big(\prod_{j=1}^{n-5} p(\varphi_j)\Big)\, p(\xi_6)\, p(\xi)_5\, p(\xi_4)\, p(\xi_3)\, p(\xi_2)\, p(\xi_1) > 0$$

the relation

$$H(s(\omega)) = H(s_n)$$

holds under the condition that $lh\,(s(\omega)) = n$. This proves, that if $lh\,(s(\omega)) \geqq 5$, then there is a positive probability that $H(s(\omega))$ is inconsistent. The theorem is proved.

The construction leading to an inconsistent formal representation may seem to be rather artificial and the assumptions rather special. It would be possible to find another sequences of actions leading to inconsistent $H(s)$, may be more closely to the real positions. However, it is not the aim of the foregoing theorem to do it. There are the two following important aspects of the theorem, namely

(a) The possibility of inconsistency is proved. As an inconsistent formal representation is useless (at least from the usual point of view which we adopt here; we do not take into consideration the so called quasi-consistent theories, see, e.g. [11]), we are forced to do something if this representation is inconsistent. Or, accepting a statistical approach, we must be sure that the probability of such an inconsistency is small enough to be acceptable.

(b) Notice the interesting fact that using only the formulas $V(a, i, s')$, resulting from observations contained in $H(s)$ we are not able to say which of the two contradictory formulas proved to be in $H(s)$ is valid, as both of them result from foregoing deductions, none of them is a direct result of an observation. The reader may remember that when the negation of an observation was in $H(s)$, this negation was removed from $H(s)$ and the observation was joined, i.e. the automaton prefers the observations to hypotheses deduced from other formulas. Nevertheless, the theorem proves that such a selection does not save from inconsistency. This gives that the problem of inconsistency of the set $H(s)$ is rather deep and deserves a more detailed investigation. To this goal Chapter 5 of this paper will be devoted and we shall profit from our assumption that the automaton behaviour can be modeled by a probability distribution on the set $\mathscr{A}'$ of steps. This assumption, together with other ones, give us the idea to use the notion of random walk and the apparatus being at the disposal of this field or probability theory to our purposes. For this sake we explain in the next chapter some notions and results concerning the random walks and we shall also prove some assertions being useful for us in the following part of this paper.

4. MODIFIED RANDOM WALKS

The notion of random walk is one among the most important in this work because of our aim to simulate by an appropriate random walk the position changes of our automaton, i.e. its movements in two dimensional discrete space. This chapter gives some information about random walks necessary for our purposes. Some elementary explanation of this subject can be found in [2], the reader demanding a more detailed study can be referred to [13] or to another monography of this kind.

The expression "random walk" is connected with the following intuitive idea (we consider first the one-dimensional case): a particle is situated in a point on the real line. This point is supposed to have an integer coordinate, i.e. its position is described by an integer.

The particle has a possibility to change its position; in every step it changes its position and goes to one of the two neighbour points with integer coordinates. Which of the two possibilities occurs – it is a random event; with probability q the particle moves to the left, with the probability $p = 1 - q$ it moves to the right. This probabilities are supposed to be constant during the random walk and the movements in different steps are supposed to be mutually independent.

Suppose that the particle is situated in the point $z > 0$, let there exist barriers in the points 0 and $a > z$. Denote by q_z the probability that the particle reaches, sooner or later, the left barrier (in 0) not having reached, before, the right one in a, by p_z denote the probability that the particle reaches the right barrier not having reached, before, the left one. Then the following assertions are valid:

(4.1) $$g_z + p_z = 1\,,$$

(4.2) $$\text{if } p \neq q\,, \quad \text{then} \quad g_z = \frac{(g/p)^q - (q/p)^z}{(q/p)^a - 1}\,,$$

(4.3) $$\text{if} \quad p = q = 1/2, \text{ then } q_z = 1 - \frac{z}{a}$$

(proofs can be found in [2]).

Let $p_0, p_1, \ldots$ be a discrete probability distribution, i.e. every $p_i \geqq 0$ and $\sum_{i=0}^{\infty} p_i = 1$. Clearly, the series

$$P(s) = \sum_{i=0}^{\infty} p_i \,.\, s^i$$

converges at least if $-1 \leqq s \leqq 1$. The function $P(s)$ is called *creative function* of the probability distribution $p_0, p_1, \ldots$. Clearly, having to our disposal $P(s)$ we are able to derive p_i using the formula

(4.4) $$p_i = \frac{1}{i!}\left[\frac{d^i}{ds^i} P(s)\right]_{s=0}\,.$$

So the creative function represents a very simple and a very concentrated representation of the probability distribution in question even if an explicit and numerical calculation of particular values p_i may be sometimes a rather peculiar matter. It is why we shall limit ourselves, sometimes, to the deriving of the creative function.

Denote by $\bar{u}_{z,n}$ the probability that the particle, starting in the point z comes, at the n-th step, for the first time to the point 0 not having come before the point a. Consider the creative function

(4.5) $$U_z(s) = \sum_{i=0}^{\infty} \bar{u}_{z,i} s^i\,.$$

This function can be expressed as follows. Denote

$$\lambda_1(s) = \frac{1 + (1 - 4pqs^2)^{1/2}}{2ps},$$

$$\lambda_2(s) = \frac{1 - (1 - 4pqs^2)^{1/2}}{2ps}$$

then the relation

$$U_z(s) = \left(\frac{q}{p}\right)^z \frac{(\lambda_1(s))^{a-z} - (\lambda_2(s))^{a-z}}{(\lambda_1(s))^a - (\lambda_2(s))^a} \tag{4.6}$$

is valid (c.f. [2]).

The case $a = \infty$ is not excluded, in this case the movement of the particle in the right direction is not limited. In this case can be proved that

$$U_z(s) = (\lambda_2(s))^z . \tag{4.7}$$

The probability that the particle reaches the point 0 is for $a = \infty$ equal to 1 if $q \geqq p$ and is equal to $(q/p)^z$ if $q < p$.

For our reasons we need a slight modification of the above concept of random walk. We shall admit also the possibility that the particle may rest at the same position in a particular step, i.e. that it does not move. Hence, this *modified random walk* is described by three non-negative reals p, p_0, q, $p_0 + p + q = 1$, such that p represents the probability of the move to the right, q that to the left and p_0 is the probability that the particle does not move.

Theorem 4.1. Consider a random walk described by the triple $\langle p, p_0, q\rangle$ with no barriers. Let the particle be situated in 0. Then the probability that the particle leaves 0 in the first step and will never return back equals to $|p - q|$.

Proof. In the first step there are three following possibilities. First, the particle rests in 0 with the probability p_0. Second, it moves to 1 and, maybe, returns to 0 eventually. This return has the probability equal to 1, if $q \geqq p$ or equal to q/p, if $q < p$. This results from foregoing results, setting $z = 1$, as our particle returns to 0 if and only if it returns to 0 supposing it is subjected to usual random walk with probabilities $p/(1 - p_0)$, $q/(1 - p_0)$. Third. it moves to -1, but this case can be transformed into the foregoing one changing the roles played by p and q. So, denoting by D the probability that the particle will be in 0 sometimes in future, we obtain:

$$D = p_0 + p + q(p/q), \quad \text{if} \quad q \geqq p,$$
$$D = p_0 + q + p(q/p), \quad \text{if} \quad q \leqq p,$$

i.e., in both the cases

$$D = p_0 + 2 \min\{p, q\} = 1 - |p - q|$$

which proves the theorem.

Theorem 4.2. Consider the same situation as in the foregoing theorem. Denote by u_n the probability that the particle will be, at the n-th step, for the first time again in 0. Then the creative function $U(s)$ of these probabilities satisfies:

$$U_s = p_0 s + s^{-1}[(1 - p_0 s) - ((1 - p_0)^2 - 4pqs^2)^{1/2}]. \tag{4.8}$$

Proof. Clearly $u_0 = 0$, $u_1 = p_0$. If the particle is situated in a point $z > 0$ and is subjected to the modified random walk with probabilities p, p_0, q then the probability $u_{z,n}$ of the first reaching of 0 satisfies the difference equation,

$$u_{z,n+1} = pu_{z+1,n} + q_{z-1,n} + p_0 u_{z,n}. \tag{4.9}$$

Multiplying the both sides by s^{n+1} and summing we obtain

$$U_z(s) = ps\, U_{z+1}(s) + qs\, U_{z-1}(s) + p_0 s\, U_z(s) \tag{4.10}$$

and this equality can be transformed into the form

$$U_z(s) = \frac{ps}{1 - p_0 s} U_{z+1}(s) + \frac{qs}{1 - p_0 s} U_{z-1}(s) \tag{4.11}$$

similar to that derived in [2] for the case of usual random walk. Applying the same methods and reasonings as in [2] we derive that in the case $a = \infty$ $U_z(s)$ satisfies:

$$U_z(s) = \left[(1 - p_0 s)\frac{(1 - p_0 s) - ((1 - p_0 s) - 4pqs^2(1 - p_0 s)^2)^{1/2}}{2ps}\right]^z. \tag{4.12}$$

At the very beginning the particle is in 0. If it moves to 1, the probabilities of its coming back to 0 are described by the creative function just described setting $z = 1$. If the particle moves to -1, the probabilities of its coming back to 0 is described by the creative function $U_z'(s)$ for $z = 1$, where $U_z'(s)$ results from $U_z(s)$ by changing the roles of p and q. Hence, the creative function $U(s)$ satisfies:

$$U(s) = p\, U_1(s) + q\, U_1'(s) + p_0(s).$$

After an easy calculation we find that

$$U(s) = p_0 + s^{-1}[(1 - p_0 s) - ((1 - p_0 s)^2 - 4pqs^2)^{1/2}]$$

which completes the proof.

Considering a usual random walk with probabilities p and q, with barriers in 0 and a and with the starting point of the particle in z, $0 < z < a$, we have mentioned that the creative function of the probabilities of coming to 0 is given by (4.6) if

$a < \infty$ and by (4.7) if $a = \infty$. It is possible to prove that the probability $\bar{u}_{z,n}$ satisfies:

$$(4.13)\quad \bar{u}_{z,n} = a^{-1}\, 2^n p^{(n-z)/2} q^{(n+z)/2} \sum_{v=1}^{a-1} \cos^{n-1}(\pi v a^{-1}) \sin(\pi v a^{-1}) \sin(\pi z v a^{-1}) \quad \text{if} \quad a < \infty$$

and

$$(4.14)\quad \bar{u}_{z,n} = 2^n p^{(n-z)/2} q^{(n+z)/2} \int_0^1 \cos^{n-1} \pi x \sin \pi x \sin \pi z x \, dx \quad \text{if} \quad a = \infty\,.$$

Any modified random walk can be considered as a usual one if we consider only the actual moves to be steps. This gives that to n steps of the modified random walk corresponds a random number $m = m(\omega)$ moves of the usual random walk and, clearly, the probability that $m(\omega) = i$, $i \leqq n$, is equal to

$$\binom{n}{i} (1 - p_0)^i \, p_0^{n-i}\,.$$

Hence, denoting by $u\,(z, n, a, p, p_0, q)$ the probability that a particle, subjected to a modified random walk with probabilities p, p_0 and q, with barriers in 0 and a and starting from a point z between 0 and a will come to 0 in the n-th step, we can clearly obtain

$$u(z, n, a, p, p_0, q) = \sum_{i=0}^{n} \binom{n}{i} (1 - p_0)^i \, p_0^{n-i} u(z, i, a, p \mid 1 - p_0, 0, q \mid 1 - p_0)\,,$$

namely

$$u(z, n, \infty, p, p_0, q) = \sum_{i=0}^{n} \binom{n}{i} (1 - p_0)^i \, p_0^{n-i}\, 2^i\,.$$

$$\cdot \left(\frac{p_0}{1 - p_0}\right)^{(n-z)/2} \left(\frac{q}{1 - p_0}\right)^{(n+z)/2} \int_0^1 \cos^{n-1} \pi x \sin \pi x \sin \pi z x \, dx\,.$$

Let us recall that $u(z, n, \infty, p, p_0, q)$ are, for $n = 0, 1, 2, \ldots$, coefficients in the exponential development of (4.12) and, if $p_0 = 0$, then $u(z, n, a, p, 0, q)$ are coefficients in the exponential development of (4.6) or (4.7), i.e. $u(z, n, a, p, 0, q)$ is given by (4.13) or (4.14) respectively.

From this follows that the probability of coming back to 0 for an unlimited random walk starting from 0 (denoted in the last theorem by u_n or, more precisely, $u_n(p, p_0, q)$) satisfies:

$$u_n(p, p_0, q) = p_0 +$$

$$+\, p \sum_{i=1}^{n} \binom{n}{i} (1 - p_0)^i \, p_0^{n-i}\, 2^i \left(\frac{p}{1 - p_0}\right)^{(i-1)/2} \left(\frac{q}{1 - p_0}\right)^{(i+1)/2}\,.$$

$$\int_0^1 \cos^{i-1} \pi x (\sin \pi x)^2 \, dx +$$

$$+ q \sum_{i=1}^{n} \binom{n}{i} (1 - p_0)^i \, p_0^{n-i} \, 2^i \left(\frac{q}{1 - p_0}\right)^{(i-1)/2} \left(\frac{p}{1 - p_0}\right)^{(i+1)/2} .$$

$$\cdot \int_0^1 \cos^{i-1} \pi x (\sin \pi x)^2 \, dx = p_0 + \frac{2pg}{1 - p_0} \sum_{i=1}^{n} 2^i .$$

$$\cdot \left(\frac{pq}{(1 - p_0)^2}\right)^{(i-1)/2} \int_0^1 \cos^{i-1} \pi x (\sin \pi x)^2 \, dx .$$

In the following part of this paper we shall use only the expressions $u_n(p, p_0, q)$ or $u(z, n, a, p, p_0, q)$ (if a is omitted, always it means that $a = \infty$). All the further reasonings and computations will be done only in the terms of these probabilities and without their detailed explicit developing. Various methods how to compute or estimate these probabilities are purely a matter of random walks theory and will not be investigated here.

Keeping in mind our basic idea to model the automaton behaviour by an appropriate random walk we shall consider a particle subjected to two simultaneous random walks. We can see this situation as follows: the particle is situated in I^2 and it can move, with certain probabilities, to any point in $\mathcal{O}_1(a)$ or to stay in a. Denoting the corresponding probabilities by $p(N)$, $p(NE), \ldots, p(NW)$ we can easily see that the projection of the particle to the axe y can be understood as if it were subjected to the modified random walk with probabilities

(4.15) $\langle p(NW) + p(N) + p(NE), \bar{p}_0 + p(E) + p(W), p(SE) + p(S) + p(SW)\rangle .$

At the same time, the projection of the particle to the axis x can be understood as if subjected to the modified random walk with probabilities

(4.16) $\langle p(NE) + p(E) + p(SE), \bar{p}_0 + p(N) + p(S), p(SW) + p(W) + p(NW)\rangle .$

Here

$$\bar{p}_0 = 1 - (p(N) + p(NE) + p(E) + p(SE) + p(S) + p(SW) + p(W) + p(NW))$$

denotes the probability that the particle does not move, considering the automaton it means that some other step than moving is applied. In the following we denote

$$\begin{aligned} p &= p(NW) + p(N) + p(NE) , \\ p_0 &= \bar{p}_0 + p(E) + p(W) , \\ q &= p(SE) + p(S) + p(SW) , \end{aligned}$$

$$r \;= p(NE) + p(E) + p(SE)\,,$$
$$r_0 = \bar{p}_0 + p(N) + p(S)\,,$$
$$s \;= p(SW) + p(W) + p(NW)\,,$$

and we shall consider a particle subjected to two ortogonal and statistically independent random walks with probabilities $\langle p, p_0, q\rangle$ and $\langle r, r_0, s\rangle$.

Definition 4.1. Let $z \in I^2$, $a \in I^2$, let p_0, p, q, r_0, r, s be non-negative reals such that $p + p_0 + q = r + r_0 + s = 1$. Define a real

$$f_n(z, a, R, p, p_0, q, r, r_0, s)$$

(R is a non-negative integer) as follows:

(a) if $|a_1 - z_1| \leqq R$ and $|a_2 - z_2| \leqq R$ (i.e. $z \in \mathcal{O}_R(a)$), then

$$f_0(z, a, R, p, p_0, q, r, r_0, s) = 1\,, \tag{4.17}$$
$$f_n(z, a, R, p, p_0, q, r, r_0, s) = 0\,,$$

(in the following the parameters of f_n are always the same and will not be explicitly repeated),

(b) if $|z_1 - a_1| \leqq R$ and $a_2 > z_2 + R$ then

$$f_n = (1 - \sum_{i=1}^{n} [u(z_1 - a_1 + R - 1, i, 2R + 2, p, p_0, q) + \tag{4.18}$$
$$+ u(a_1 - z_1 + R - 1, i, 2R + 2, q, p_0, p)]) \, u(a_2 - z_2 - R, n, \infty, s, r_0, r)$$

(c) if $|z_1 - a_1| \leqq R$ and $z_2 > a_2 + R$, then

$$f_n = (1 - \sum_{i=1}^{n} [u(z_1 - a_1 + R - 1, i, 2R + 2, p, p_0, q) + \tag{4.19}$$
$$+ u(a_1 - z_1 + R - 1, i, 2R + 2, q, p_0, p)]) \, u(z_2 - a_2 - R, n, \infty, r, r_0, S)$$

(d) if $|z_2 - a_2| \leqq R$ and $a_1 > z_1 + R$, then

$$f_n = (1 - \sum_{i=1}^{n} [u(z_2 - a_2 + R - 1, i, 2R + 2, r, r_0, s) + \tag{4.20}$$
$$+ u(a_2 - z_2 + R - 1, 2R + 2, s, r_0, r)]) \, u(a_1 - z_1 - R, n, \infty, q, p_0, p)\,,$$

(e) if $|z_2 - a_2| \leqq R$ and $z_1 > a_1 + R$, then

$$f_n = (1 - \sum_{i=1}^{n} [u(z_2 - a_2 + R - 1, i, 2R + 2, r, r_0, s) + \tag{4.21}$$
$$+ u(a_2 - z_2 + R - 1, i, 2R + 2, s, r_0, r)]) \, u(z_1 - a_1 - R, n, \infty, p, p_0, q)$$

(f) if $z_1 < a_1 - R$ and $z_2 < a_2 - R$, then

(4.22) $f_n = u(a_1 - z_1 - R, n, \infty, q, p_0, p)\, u(a_2 - z_2 - R, n, \infty, s, r_0, r)$,

(g) if $z_1 > a_1 + R$ and $z_2 < a_2 - R$, then

(4.23) $f_n = u(z_1 - a_1 - R, n, \infty, p, p_0, q)\, u(a_2 - z_2 - R, n, \infty, s, r_0, r)$,

(h) if $z_1 > a_1 + R$ and $z_2 > a_2 + R$, then

(4.24) $f_n = u(z_1 - a_1 - R, n, \infty, p, p_0, q)\, u(z_2 - a_2 - R, n, \infty, r, r_0, S)$,

(i) if $z_1 < a_1 - R$ and $z_2 > a_2 + R$, then

(4.25) $f_n = u(a_1 - z_1 - R, n, \infty, q, p_0, p)\, u(z_2 - a_2 - R, n, \infty, r, r_0, s)$.

Theorem 4.3. Consider a particle subjected to the two-dimensional modified random walk with the probabilities p_0, $p(N)$, $p(NE)$, $p(E)$, $p(SE)$, $p(S)$, $p(SW)$, $p(W)$ and $p(NW)$, let the probabilities $\langle p, p_0, q\rangle$ and $\langle r, r_0, s\rangle$ define the two one-dimensional modified random walks resulting when the considered two-dimensional random walk projected the two axes. Denote by $P_n(z, a, R, \langle p, p_0, q\rangle, \langle r, r_0, s\rangle)$ the probability that the particle, starting from $z \in I^2$ enters in the n-th step for the first time the set $\mathcal{O}_R(a)$, $a \in I^2$, $R \in N$. Then

$$P_n(z, a, R, \langle p, p_0, q\rangle, \langle r, r_0, s\rangle) \geqq f_n(z, a, R, p, p_0, q, r, r_0, s).$$

Proof. Let f_n be defined according to (4.17). Then $z \in \mathcal{O}_R(a)$ hence $P_n = 1$ if $n = 0$ and $P_n = 0$ otherwise, so the assertion holds. Let f_n be defined according to (4.18). Here, $z_1 \in \langle a_1 - R, a_2 + R\rangle$ and $u(Z_1 - a_1 + R - 1, i, 2R + 2, p, p_0, q)$ is nothing else than the probability that the projection of the particle on the axis x reaches $a_1 - R - 1$ in the i-th step not having reached before the point $a_1 + R + 1$. Here we use the fact that when a random walk considered, the corresponding probabilities are the same in case the barriers are in 0 and a and starting position in z as in the case the barriers are in K and $a + K$ and starting position in $z + K$. To find the probability of reaching the right barrier ($a_1 + R + 1$ in our case) we can notice that it is just the probability of reaching the left barrier when p replaced by q, q by p and z by $a - z$. This way of reasoning gives that

$$u(z_1 - a_1 + R - 1, i, 2R + 2, p, p_0, q) + u(a_1 - z_1 + R - 1, i, 2R + 2, q, p_0, p$$

is just the probability that the projection of the particle on the axis x leaves in the i-th step for the first time the interval $\langle a_1 - R, a_1 + R\rangle$.

Using the same way of reasoning we obtain that $u(a_2 - z_2 - R, n, \infty, s, r_0, r)$ is just the probability of the entering the y – projection of the particle into the interval $\langle a_2 - R, a_2 + R\rangle$ first time at the n-th step. Combining these results we have that f_n, defined by (4.18), expresses the probability that the particle will never leave the belt defined by the interval $\langle a_1 - R, a_1 + R\rangle$ and at the n-th step for the first time enters $\mathcal{O}_R(a)$. This is a sufficient condition for the event the probability of which is denoted by P_n, hence also in this case the assertion is valid.

The validity of the assertion supposing f_n is defined according to (4.19), (4.20) or (4.21) follows from the same arguments as in the case (4.18) because of the evident symmetry of all the four cases, just the appropriate replacements among $z_1 \leftrightarrow a_1$, $z_2 \leftrightarrow a_2$, $g \leftrightarrow p$, $r \leftrightarrow s$ must be done.

Suppose that f_n is defined according to (4.22). Here f_n expresses the probability that the x-projection of the particle reaches at the n-th step for the first time the point $a_1 - R$ and, at the same time, the y-projection reaches for the first time $a_2 - R$. This conjunction of events is, of course, sufficient for the first reaching of the particle the set $\mathcal{O}_R(a)$ at the n-th step, so the assertion holds again. Because of the symmetry also in the cases (4.23), (4.24) and (4.25) the assertion holds. The theorem is proved.

Theorem 4.4. Let the conditions of Theorem 4.3 hold. If $p = q, r = s, p_0 < 1$ and $r_0 < 1$ then

$$\sum_{n=0}^{\infty} P_n(z, a, R, \langle p, p_0, q\rangle, \langle r, r_0, s\rangle) = 1$$

else

$$\sum_{n=0}^{\infty} P_n(z, a, R, \langle p, p_0, q\rangle, \langle r, r_0, s\rangle) < 1 .$$

Proof. It is a well-known fact (see [13]), that if $p = q$, $r = s$, $p_0 < 1$ and $r_0 < 1$ (symmetric random walk), then with probability one the particle reaches, sooner or later, any point in I^2. On the other hand, if these conditions are not satisfied, then there is for any point, excepting the starting one, a positive probability that the particle will never reach this point. Our theorem just expresses these facts using the terms of P_n which proves the theorem.

With this theorem we finish this chapter and in the next one we return again to our investigation of automaton-environment systems from the point of view of the consistency of its internal formal representation of the environment.

5. INCONSISTENCY AND INADEQUACY PROBABILITIES

In this chapter a number of theorems is given with the aim to describe in more details the probability that the formal representation of the environment becomes inconsistent (positivity of this probability was proved in Theorem 3.1). In all this chapter we assume that the automaton can observe only the properties of the point where it is situated in the actual situation. Formally: $p(Ob\,(a, j)) = 0$ if $V(a, I_0, s)$ does not hold.

Some more notation seems to be convenient:

$$\mathscr{X} = \{N. NE, E, SE, S, SW, W, NW\},$$

if $X \in \mathscr{X}$, then $\bar{X}$ is the reverse move (i.e. $\bar{N} = S, \overline{NE} = SW, \bar{E} = W$, etc.),

$$\tilde{f}_n(a, p, p_0, q, r, r_0, s) = \sum_{X \in \mathscr{X}} p(X) f_n(Xa, a, 0, p, p_0, q, r, r_0, s)$$

where $f_n(z, a, R, p, p_0, q, r, r_0, s)$ is the function defined in Definition 4.1,

$$\tilde{F}(a, p, p_0, q, r, r_0, s) = \sum_{n=0}^{\infty} z^n \tilde{f}^n(a, p, p_0, q, r, r_0, s).$$

Theorem 5.1. Consider an automaton-environment system as described above. Denote by s the actual situation, by $s' = s'(\omega)$ the situation following from s by applying n further steps (i.e. $lh(s') = lh(s) + n$). Then

$$P(\{\omega : Cn(H(S'(\omega), \omega)) = \mathscr{W}\}) \geqq$$

$$\geqq \sum_{O_p} \sum_i \sum_{X \in \mathscr{X}} \{p(Ob(a, j)\, p(X)\, p(Op(k, l))\, [g\ Op(k, l)\, (V(a, neg\ i, s''))]\, p(\bar{X}) \,.$$

$$\cdot \frac{1}{n!} \sum_{k=0}^{\infty} [1 - (1 - p(\xi_5)\, p(\xi_6))^k] \left[\frac{d^n}{dz^n} \tilde{F}^k(a, p, p_0, q, r, r_0, s)\right]_{z=0} \Bigg\},$$

where $\sum_{O_p}$ denotes summation over all operations not influencing i-th and $neg\ i$-th properties, $\sum_i$ denotes summation over all i such that $\mathscr{V}(a, i, s, \omega) = true$, s'' denotes the situation resulting from s when $Ob(a, j)$, X and $Op(k, l)$ applied, ξ_5 and ξ_6 are the same as in Theorem 3.1. The symbol a denotes the automaton position in the situation s.

Proof. Automaton is in a and $Ob(a, j)$ is applied, hence $V(a, j, s)$ enters to $H(Ob(a, j)\, s)$, as $\mathscr{V}(a, i, s, \omega) = true$. Then the automaton moves, e.g. to the north (i.e. $X = N$) and applies an operation $Op(k, l)$. This operation involves a random consequence consisting in fact that now $V(a, neg\ i, s'')$ holds. Then the automaton moves back to a. All this sequence of actions is performed with the probability

$$p(Ob(a, j))\, p(x)\, p(Op(k, l))\, p(\bar{X}) \,. \tag{5.1}$$

Multiplying this probability by $(g\ Op(k, l))\, (V(a, neg\ i, s''))$ we have the probability with which the automaton is again in a in situation s'''

$$s''' = \bar{X}\ Op(k, l)\, X\ Ob(a, j)\, s\,,$$

and in $H(s''')$ is $V(a, i, s)$, while $V(a, i, s''')$ does not hold. Following the same way of reasoning as in the proof of Theorem 3.1 we find that if, now, ξ_5 and ξ_6 applied,

$H(s''')$ becomes to be inconsistent. If this does not occur, the same possibility occurs in every step which brings the automaton to the point a again. Hence, $(1 - p(\xi_5) p(\xi_6))^k$ expresses the probability that $H(s''')$ does not become inconsistent when the automaton for the k-th time is in a and $1 - (1 - p(\xi_5) p(\xi_6))^k$ minorizes the probability that k enters of the automaton into a was sufficient for inconsistency of $H(s''')$.

Now, $\tilde{F}$ is the creative function of $\tilde{f}_n$, where $\tilde{f}_n$ is a lower bound for the probability of returning back to a in n steps. So $\tilde{F}^k$ defines a lower bound for probabilities that the automaton k-times comes back to a. In other words,

$$\left[\frac{1}{n!} \frac{d^n}{dz^n} \tilde{F}^k(a, p, p_0, q, r, r_0, s)\right]_{z=0}$$

minorizes the probability that in situation s' the automaton will be in a for at least k-th time counting from the situation s. Hence,

$$\frac{1}{n!} \sum_{k=0}^{\infty} [1 - (1 - p(\xi_5) p(\xi_6))^k] \left[\frac{d^n}{dz^n} \tilde{F}^k\right]_{z=0} \tag{5.2}$$

gives a lower bound for the probability that $H(s')$ is inconsistent supposing the automaton is in situation s'''. Combining (5.1) and (5.2) and taking into consideration that the special choose of i, $Op\,(k, l)$ and X, made when (5.1) derived is irrelevant we obtain just the assertion of the theorem. The theorem is proved.

Theorem 5.2. Under the same conditions as in Theorem 5.1

$$P(\{\omega : Cn\,(H(s'(\omega), \omega)) = \mathscr{W}\}) \leqq$$
$$\leqq 1 - \sum_{X \in \mathfrak{X}} p(X)\,(1 - f_n(X0, 0, 0, p, p_0, q, r, r_0, S)) .$$

Proof. It is possible to suppose that the automaton is in $\langle 0, 0\rangle$ in the situation s because the distribution p restricted to $\mathfrak{X}$ is space-homogeneous. A necessary condition of inconsistency is that the automaton comes at least once back to 0. This probability is majorized by $p(x)$. $(1 - f_n(X0, 0, 0, p, p_0, q, r, r_0, s))$ supposing the move X is applied and is equal to $1 - \sum_{X \in \mathfrak{X}} p(X)$ supposing another step is applied. Hence,

$$P(\{\omega : Cn(H(s'(\omega), \omega)) = \mathscr{W}\}) \leqq 1 - p(X) + \sum_{X \in \mathfrak{X}} p(X)\,(1 - f_n) =$$
$$= 1 - \sum_{X \in \mathfrak{X}} p(X)\,(1 - f_n(X0, 0, 0, p, p_0, q, r, r_0, s))$$

which proves the theorem.

Theorem 5.3. Consider the same conditions as in Theorem 5.1. Let, moreover, $p = q > 0$ and $r = s > 0$. Then

$$P(\{\omega : Cn(H(s'(\omega), \omega)) = \mathscr{W}\}) \to 1$$

if $n = lh(s') - lh(s) \to \infty$.

In another words: if the random walk modeling the automaton behaviour is symmetric and non-degenerate, then the formal representation of the environment becomes, sooner or later, inconsistent.

Proof: The random event, consisting in the sequence $\langle Ob(a, j), X, Op(k, l); \bar{X} \rangle$ of steps together with the random change of $V(a, i, s)$ into $V(a, neg\ i, s'')$ (see the proof of Theorem 5.1) has a positive and constant probability, namely

$$p(Ob(a, j))\, p(x)\, p(Ob(k, l)\, p(\bar{X})\, [(g\ Op(k, l))\, (V(a, neg\ i\ s''))] .$$

Hence, with probability 1 sooner or later the situation, denoted in the proof of Theorem 5.1 by s''', occurs.

Now, as the random walk is symmetric and non-degenerate, the automaton returns infinitely many times to the point where it is in the situation s'''. As proved above, in every such a case there is a positive and constant probability (at least $p(\xi_5)\, p(\xi_6)$) that the set $H(s''')$ becomes inconsistent. This gives that with probability 1 this event, sooner or later, actually occurs, supposing the number of steps increases. Combining these results we obtain that, sooner or later, with probability 1 the set $H(s')$ is inconsistent, which proves the theorem.

Theorem 5.4. Consider the same conditions as in Theorem 5.1. Let, moreover, $p \neq q$ or $r \neq s$. Then, for any $a \in I^2$, $i \in N$, $s' \in S$

$$P\{\omega : Cn(H(s'(\omega), \omega) = \mathscr{W}\} \cap \{\omega : Cn(H(s'(\omega), \omega) - \{V(a, i, s')\}) \neq$$
$$\neq \mathscr{W}\}) \geqq |p - q|\, |r - s| .$$

Intuitively said, probability a lower bound of which is given in this theorem expresses the probability that the formula $V(a, i, s)$ will be the source of inconsistency of the set $H(s')$.

Proof. If $V(a, i, s) \in H(s)$, then the necessary condition for this formula being a source of inconsistency is that the automaton at least once returns to a. However, as given in Chapter 4, with a probability $|p - q|\, |r - s|$ the automaton never come back to a. So the assertion is valid.

Immediately from Theorems 5.1 – 5.4 follow the two assertions:

Theorem 5.5. Consider the same conditions as in Theorem 5.1. Then the probability that $H(s'(\omega), \omega)$ contains a formula which is not valid tends to 1 if $lh(s') - lh(s) \to \infty$.

Theorem 5.6. Consider the same conditions as in Theorem 5.1. Let there exist an $R \in N$ that operations, observations, deduced and actualizations are applicable only if the automaton is situated in $\mathcal{O}_R(\langle 0, 0 \rangle)$ (i.e. only moves are applicable outside $\mathcal{O}_R(\langle 0, 0 \rangle)$). Then there is a positive probability that $H(s')$ will never be inconsistent supposing $p \neq q$ or $r \neq s$.

Proof. Clearly, there is a positive probability that after a finite number of steps the automaton leaves $\mathcal{O}_R(\langle 0, 0 \rangle)$ and never more enters this set. From the foregoing theorems immediately follows that there is a positive probability, for any $n \in N$ that if $lh\,(s') - lh\,(s) = n$, then $H(s')$ is consistent. This proves the theorem.

Let us finish this paper with some remarks concerning the achieved results changing somehow the order in which they were proved. First, we proved that, sooner or later, a formula occurs in $H(s)$ which is not valid. If the random walk is symmetric, then something more is valid – sooner or later the set $H(s)$ becomes inconsistent. On the other hand, if the automaton can operate in an active form only in a bounded area and if the random walk is not symmetric, it is possible that the set $H(s)$, though containing a non-valid formula (or formulas), will be always consistent.

The possible inconsistency of $H(s)$ can be seen from two quite different points of view. First, inconsistent formal representation is considered (at least from the usual and classical point of view) to be useless as everything can be proved inside this formalization. However, realize that the automaton is able to derive that there is something wrong with its formal representation of the environment either by observation or by deducing a contradiction. If the automaton ability to observe the environment is limited, the possibility of deducing a contradiction may become a very important mean assuring at least some feedback between the environment and its formal representation in the automaton-environment system.

In any case, the inconsistency and inadequacy of the formal representation is an event requesting an appropriate intervention into the set $H(s)$, either by the user, or by the automaton itself. The aim of this paper was to show that this danger is great to be neglected. It is a matter of a further investigation to propose strategies or methodes eliminating or minimizing the danger mentioned above. The results of this paper can serve as a good justification for such an effort.

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REFERENCES

[1] D. Blackwell, M. A. Girschick: Theory of Games and Statistical Decisions. John Willey and Sons, New York 1954.

[2] W. Feller: An Introduction to Probability Theory and its Applications, vol. I. Second edition. John Willey and Sons, New York 1962.

[3] R. Fikes, N. Nilsson: STRIPS — New Approach to the Application of Theorem-Proving to Problem-Solving. Proceedings of the Second International Joint Conference on Artificial Intelligence, Imperial College, London 1971.

[4] P. J. Hayes, J. McCarthy: Some Philosophical Problems from the Standpoint of Artificial Intelligence. In: Machine Intelligence 4 (B. Meltzer and D. Michie, Editors). Edinburgh University Press, 1969.

[5] P. J. Hayes: Frame Problem and Related Problems in Artificial Intelligence. Research Report, Department of Computational Logic, Edinburgh University.

[6] H. W. Göttinger: Toward a Fuzzy Reasoning in the Behavioral Sciences. Ekonomicko-matematický obzor *9* (1973), 4, 404—422.

[7] S. C. Kleene: Introduction to Metamathematics. D. Van Nostrand Comp., New York 1952.

[8] I. Kramosil: Random Axiomatic Systems. Research Report, Institute of Information Theory and Automation, Prague 1973.

[9] I. Kramosil: Gentzen-like Random Axiomatic Systems. To appear in the Transactions of the European Meeting of Statisticians and Seventh Prague Conference, held in Prague, August 1974.

[10] R. T. C. Lee: Fuzzy Logic and the Resolution Principle. Journal of the Association for Computing Machinery *19* (1972), 1, 109—119.

[11] R. Parikh: Existence and Feasibility in Arithmetic. The Journal of Symbolic Logic *36* (1971), 3, 494—508.

[12] E. Sandewall: An Approach to the Frame Problem and its Implementation. In: Machine Intelligence 7 (B. Meltzer and D. Michie, Editors), Edinburgh University Press, 1972.

[13] F. Spitzer: Principles of Random Walk. Princeton University Press, Princeton 1964.

[14] L. A. Zadeh: Fuzzy sets. Information and Control *8* (1965), 338—353.

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