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Discrete Twice Optimal Control Systems

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In this paper the properties of the closed-loop linear discrete control systems with feedforward and feedback controllers are discussed. First the stability problem is mentioned. Then the optimal transfer functions relating the system output to the reference input and to the disturbance are developed. The main result is a twice optimal control.

INTRODUCTION

In this paper we shall consider discrete, constant linear systems which are minimal-state realizations of given transfer functions. These transfer functions will be written in the complex variables z or $\zeta = z^{-1}$.

We say that the polynomial $\hat{a} = a_0 z^n + a_1 z^{n-1} + \ldots + a_n$ is stable if and only if all its zeros lie inside Γ, where Γ denotes the unit circle $|z| = 1$.

The polynomial $a = a_0 + a_1\zeta + \ldots + a_n\zeta^n$ is stable, $a = a^+$, if and only if all its zeros lie outside Γ. Any polynomial a can be factorized into $a^- a^+ = a$, where a^- has no zeros inside Γ and a^+ is stable.

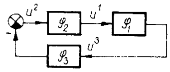
The transfer function a/α, where a, α are polynomials, is defined stable if and only if α is stable. Consequently, if any zero of a polynomial a lies on Γ then the polynomial a is unstable.

Let us consider polynomials a, b. We say that a divides b and write $a \mid b$, if and only if there exists a polynomial c such that $b = ac$.

If the greatest common divisor of a and b is a polynomial d with degree at least one, we shall write $(a, b) = d$, if the degree of d is zero we shall write $(a, b) = 1$.

STABILITY OF TWO-CONTROLLERS CLOSED-LOOP CONTROL SYSTEM

Consider the closed-loop control system (1),

(1) 

where $\mathscr{S}_1$, $\mathscr{S}_2$ and $\mathscr{S}_3$ are minimal realization of the transfer functions $\hat{S}_1$, $\hat{S}_2$ and $\hat{S}_3$ respectively. Let the $\mathscr{S}_1$, $\mathscr{S}_2$, $\mathscr{S}_3$ be given in the form

$$x_{n+1}^{(\alpha)} = \mathbf{A}_\alpha x_n^{(\alpha)} + \mathbf{B}_\alpha u_n^{(\alpha)}, \quad \alpha = 1, 2, 3,$$
$$y_n^{(\alpha)} = \mathbf{C}_\alpha x_n^{(\alpha)} + \mathbf{D}_\alpha u_n^{(\alpha)}, \quad \mathbf{D}_1 = 0,$$

then

(2) $$\hat{S}_\alpha = \mathbf{D}_\alpha + \mathbf{C}_\alpha(z\mathbf{I} - \mathbf{A}_\alpha)^{-1} \mathbf{B}_\alpha = \frac{\hat{s}_\alpha^1}{\hat{\sigma}_\alpha},$$

where $x_n^{(\alpha)}$ are state vectors, $u_n^{(\alpha)}$ are scalar inputs, $y_n^{(\alpha)}$ are scalar outputs and $\mathbf{A}_\alpha$, $\mathbf{B}_\alpha$, $\mathbf{C}_\alpha$, $\mathbf{D}_\alpha$ are matrices, $\hat{S}_\alpha$ are transfer functions and $\hat{s}_\alpha$, $\hat{\sigma}_\alpha$ are, for each α, relatively prime polynomials in complex variable z.

The state equation of the whole system is

$$\begin{bmatrix} x_{n+1}^{(1)} \\ x_{n+1}^{(2)} \\ x_{n+1}^{(3)} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 - \mathbf{B}_1\mathbf{D}_2\mathbf{D}_3\mathbf{C}_1, & \mathbf{B}_1\mathbf{C}_2, & -\mathbf{B}_1\mathbf{D}_2\mathbf{C}_3 \\ -\mathbf{B}_2\mathbf{D}_3\mathbf{C}_1, & \mathbf{A}_2, & -\mathbf{B}_2\mathbf{C}_3 \\ \mathbf{B}_3\mathbf{C}_1, & \mathbf{0}, & \mathbf{A}_3 \end{bmatrix} \begin{bmatrix} x_n^{(1)} \\ x_n^{(2)} \\ x_n^{(3)} \end{bmatrix},$$

in a shorthand notation

$$x_{n+1} = \mathbf{A} x_n .$$

The stability of the system (1) is given by the stability of the characteristic polynomial of $\mathbf{A}$, i.e. of

(3) $$\det(z\mathbf{I} - \mathbf{A}) = \det \begin{bmatrix} z\mathbf{I} - \mathbf{A}_1 + \mathbf{B}_1\mathbf{D}_2\mathbf{D}_3\mathbf{C}_1, & -\mathbf{B}_1\mathbf{C}_2, & \mathbf{B}_1\mathbf{D}_2\mathbf{C}_3 \\ \mathbf{B}_2\mathbf{D}_3\mathbf{C}_1, & z\mathbf{I} - \mathbf{A}_2, & \mathbf{B}_2\mathbf{C}_3 \\ -\mathbf{B}_3\mathbf{C}_1, & \mathbf{0}, & z\mathbf{I} - \mathbf{A}_3 \end{bmatrix}.$$

It is known [6] that the determinant of a matrix

$$\mathbf{G} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix},$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are submatrices and $\mathbf{a}$, $\mathbf{d}$ are invertible square matrices, can be computed as

(i) $$\det \mathbf{G} = \det \mathbf{a} \,.\, \det(\mathbf{d} - \mathbf{c}\mathbf{a}^{-1}\mathbf{b}) = \det \mathbf{d} \,.\, \det(\mathbf{a} - \mathbf{b}\mathbf{d}^{-1}\mathbf{c}).$$

Hence

(ii) $$\det\left(\boldsymbol{a} - \boldsymbol{b}\boldsymbol{d}^{-1}\boldsymbol{c}\right) = \det \boldsymbol{a} \,.\, \det \boldsymbol{d}^{-1} \,.\, \det\left(\boldsymbol{d} - \boldsymbol{c}\boldsymbol{a}^{-1}\boldsymbol{b}\right).$$

If we use the equation (i) and (2) then the characteristic polynomial (3) is given as

$$\det\left(z\boldsymbol{I} - \boldsymbol{A}\right) = \det\left(z\boldsymbol{I} - \boldsymbol{A}_2\right) . \det\left(z\boldsymbol{I} - \boldsymbol{A}_3\right) . \det\left(z\boldsymbol{I} - \boldsymbol{A}_1 + \boldsymbol{B}_1 \hat{S}_2 \hat{S}_3 \boldsymbol{C}_1\right).$$

Taking the property (ii) into account then

$$\det\left(z\boldsymbol{I} - \boldsymbol{A}\right) = \det\left(z\boldsymbol{I} - \boldsymbol{A}_1\right) . \det\left(z\boldsymbol{I} - \boldsymbol{A}_2\right) . \det\left(z\boldsymbol{I} - \boldsymbol{A}_3\right) . \left(1 + \hat{S}_1 \hat{S}_2 \hat{S}_3\right).$$

We supposed that $\mathscr{S}_1$, $\mathscr{S}_2$ and $\mathscr{S}_3$ are minimal realization of the transfer functions $\hat{S}_1$, $\hat{S}_2$ and $\hat{S}_3$ respectively, and hence

$$\det\left(z\boldsymbol{I} - \boldsymbol{A}_\alpha\right) = \hat{\sigma}_\alpha \,, \quad \text{for} \quad \alpha = 1, 2, 3\,.$$

The characteristic polynomial is given as

$$\hat{\chi} = \det\left(z\boldsymbol{I} - \boldsymbol{A}\right) = \hat{\sigma}_1 \hat{\sigma}_2 \hat{\sigma}_3 + \hat{s}_1 \hat{s}_2 \hat{s}_3\,.$$

Theorem 1. *Let $\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3$ be minimal realizations of the transfer functions $\hat{S}_1, \hat{S}_2, \hat{S}_3$ respectively. Then the closed-loop control system (1) is stable if and only if the polynomial $\hat{\chi} = \hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3 + \hat{s}_1\hat{s}_2\hat{s}_3$ is stable.*

We can write any transfer function also in the complex variable $\zeta = z^{-1}$. For example $S = s/\sigma$, where s and σ are relatively prime polynomials in ζ.

Definition 1. The *pseudocharacteristic polynomial* of the system (1) is

$$\chi = \sigma_1 \sigma_2 \sigma_3 + s_1 s_2 s_3\,.$$

It is evident that

$$\hat{\chi} = \det\left(z\boldsymbol{I} - \boldsymbol{A}\right) = \det\left(z\boldsymbol{I}(\boldsymbol{I} - z^{-1}\boldsymbol{A})\right) = z^n \chi\,,$$

where n is order of the matrix $\boldsymbol{A}$.

Hence it follows that for the test of stability of the system (1) we can test only the pseudocharacteristic polynomial χ.

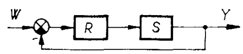
Theorem 2. *Let $\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3$ be minimal realizations of the transfer functions S_1, S_2, S_3 respectively. Then the closed-loop control system (1) is stable if and only if the polynomial $\chi = \sigma_1\sigma_2\sigma_3 + s_1s_2s_3$ is stable.*

It can happen that the least square control cannot be practically used for some systems. For example, let the transfer functions of the system be $S = \zeta/(1 - \zeta)$ and the reference signal $W = 1/(1 - 0{\cdot}5\zeta)$ then the optimal least square controller is

$$R = \frac{0{\cdot}5(1 - \zeta)}{1 - 0{\cdot}5\zeta}.$$

The configuration of the above system has the form (4):

(4)



The error signal is

$$E = \frac{1}{1 + RS} W = 1 .$$

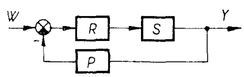
The pseudocharacteristic polynomial is (see Theorem 2)

$$\chi = (1 - \zeta)(1 - 0{\cdot}5\zeta) + 0{\cdot}5\zeta(1 - \zeta) = 1 - \zeta .$$

Hence, there is a mode in system (4) which does not depend on time and system is not suitable for applications. χ is not stable.

Consider the following control system (5).

(5)



where

$$S = \frac{\zeta}{1 - \zeta}, \quad W = \frac{1}{1 - 0{\cdot}5\zeta}, \quad R = 0{\cdot}5 , \quad P = 2{\cdot}0 .$$

The error signal $E = SR(P - 1)/(1 + SRP) = 1$ is equal to the above error signal but the pseudocharacteristic polynomial (see Theorem 2) is $\chi = 1 - \zeta + 0{\cdot}5 . 2{\cdot}0\zeta =$ $= 1$. This system can be applied because it is stable.

The sufficient and necessary stability conditions for the system (4) are known [3; 4] in the form

(a) $$K_{WY} = \frac{Y}{W} = sM ,$$

(b) $$1 - K_{WY} = \sigma N ,$$

where s/σ is the transfer function of the system $\mathscr{S}$ and M, N are any stable transfer functions for which

$$sM + \sigma N = 1\,,$$

but s, σ, M, N must be functions of complex variable $\zeta = z^{-1}$!

For the stable systems the above conditions are reduced to the condition (a).

CLOSED-LOOP SYSTEMS WITH TWO CONTROLLERS

The necessary stability condition for the system (5) is only one

$$K_{WY} = sM\,.$$

Theorem 3. *Let $\mathscr{S}$ be a minimal realization of the transfer function $S = s/\sigma$, then any transfer function $K_{WY} = sM$, where $M = m/\mu$ is arbitrary stable transfer function, can be realized by stable closed-loop system* (5).

More, there is always possible to find P and R such that the pseudocharacteristic polynomial is equal only to μ.

It is evident that χ can be chosen as any stable polynomial for which $\mu \mid \chi$.

Proof. Consider the control system (5) and denote $S = s/\sigma$, $R = r/\varrho$, $P = p/\pi$, $M = m/\mu$ and suppose, that $(r, \varrho) = 1$, $(s, \sigma) = 1$ and $(p, \pi) = 1$, $(m, \mu) = 1$.

It is evident that

$$K_{WY} = \frac{Y}{W} = \frac{SR}{1 + SRP} = \frac{sr\pi}{\sigma\varrho\pi + srp} = sM \tag{6}$$

and hence

$$M = \frac{m}{\mu} = \frac{r\pi}{\sigma\varrho\pi + srp}\,. \tag{7}$$

Choose

(i) $\quad m = r\pi \quad$ such that $\quad (sr, \sigma\pi) = 1\,,\quad$ it is always possible, and

(ii) $$\mu = \sigma\varrho\pi + srp\,.$$

The equation (ii) is a Diophantine equation in polynomials for the unknowns ϱ, p. This equation has a solution [1] if and only if $(\sigma\pi, sr) \mid \mu$.

For our purpose $(\sigma\pi, sr) = 1$ and the equation (ii) has always a solution.

All solutions has the form

$$\varrho = \varrho_0 + srt\,,$$

$$p = p_0 - \sigma\pi t\,,$$

where t is an arbitrary polynomial, ϱ_0, p_0 is a particular solution.

The effective method of solution is given in [1]. Now we must check validity of the condition

$$\text{(iii)} \qquad (r, \varrho) = 1\,, \quad (p, \pi) = 1\,.$$

Suppose that $(r, \varrho) = d_1$ and $(p, \pi) = d_2$, then from (i)

$$d_1 \mid m\,, \quad d_2 \mid m$$

and from (ii)

$$d_1 \mid \mu\,, \quad d_2 \mid \mu\,.$$

Hence $(m, \mu) \neq 1$ and since we supposed that $(m, \mu) = 1$. It can be seen that the above problem has infinitely many solutions.

Example 1. Let

$$S = \frac{\zeta(1 - 2\zeta)}{1 - \zeta} \quad \text{and} \quad K_W = \frac{\zeta(1 - 2\zeta)^2}{\zeta - 2}\,, \text{ find}$$

the controllers R, P such that the pseudocharacteristic polynomial $\chi = \mu = \zeta - 2$.

From (6) and (7)

$$M = \frac{K_{WY}}{s} = \frac{1 - 2\zeta}{\zeta - 2}\,.$$

Write

$$\mu = \sigma\varrho\pi + srp\,, \quad m = r\pi\,.$$

The condition $(sr, \sigma\pi) = 1$ gives

$$r = 1 - 2\zeta\,, \quad \pi = 1$$

and the pseudocharacteristic polynomial is in the form

$$(\zeta - 2) = (1 - \zeta)\,\varrho + \zeta(1 - 2\zeta)^2\,p$$

Hence, the general solution is

$$\varrho = -2 - 4\zeta^2 + \zeta(1 - 2\zeta)^2\,t\,,$$

$$p = -1 - (1 - \zeta)\,t\,.$$

Choose $t = 0$, then

$$R = \frac{1 - 2\zeta}{-2(1 + 2\zeta^2)}\,, \quad P = -1\,.$$

Let us be given $S = s/\sigma$ and a reference signal $W = w/v$, v has no zero inside Γ, synthesize such controllers R and P that the cost functional

$$I = \frac{1}{2\pi j} \int E(\zeta^{-1})\, E(\zeta) \frac{d\zeta}{\zeta} \tag{8}$$

is minimized and the psuedocharacteristic polynomial is stable with minimal degree.

It can be seen that

$$E = (1 - sM)\, W$$

and the classical approach [4] to the minimization of the above cost functional gives the optimal transfer function

$$M = \frac{v^*}{w^* s^*} \left[\frac{s^= w^*}{s^{*=} v^*} \right]_{\oplus}, \tag{9}$$

where $w^=$ denotes the substitution of ζ^{-1} for ζ, w^* is the spectral factor of $w^= w$ such that w^* has no zeros inside Γ and $w^{*=} w^* = w^= w$ and $[\cdot]_{\oplus}$ denotes the partial fraction expansion without unstable fractions.

The LSC problem has a solution if and only if the transfer function M from (9) is stable.

Further procedure is the same as the proof of Theorem 3.

Example 2. Consider

$$S = \frac{\zeta}{(1 - \zeta)^2}, \quad W = \frac{1}{2 - \zeta},$$

then the optimal M from (9) is

$$M = (2 - \zeta) \left[\frac{\zeta^{-1}}{2 - \zeta} \right]_{\oplus} = 0{\cdot}5,$$

the error signal

$$E = 0{\cdot}5,$$

and the cost functional

$$I = 0{\cdot}25.$$

From Theorem 3 the minimal degree of the pseudocharacteristic polynomial is given by the denominator of M.

For our purpose $\mu = 1$,

$$r\pi = 0{\cdot}5, \quad r = 0{\cdot}5, \quad \pi = 1$$

and

$$(1 - \zeta)^2 \varrho + 0{\cdot}5\zeta p = 1.$$

$$\varrho = 1\,, \quad p = 4 - 2\zeta \quad \text{and} \quad R = 0{\cdot}5\,, \quad P = 4 - 2\zeta\,.$$

This optimal control system has the very interesting property, its free motion reaches the steady state in a finite time.

For the control system (4) the least square control gives the same error signal but the pseudocharacteristic polynomial $\chi = (1 - \zeta)^2$ and the closed-loop system is not stable. Moreover, other solutions which minimize the above quadratic cost functional do not exist.

OTOC — CLOSED-LOOP OUTPUT TIME OPTIMAL CONTROL

Let us have $S = s/\sigma$ and a reference signal $W = w/v$ such that $(s^-, v) = 1$, synthesize such controllers R, P that the error signal is a polynomial with minimal degree and the pseudocharacteristic polynomial is stable and of minimal degree.

By Theorem 3 $K_{WY} = sM$, where M is any stable transfer function.

The error signal

$$E = (1 - sM)\, W$$

must be a polynomial and

$$e = \frac{\mu w - smw}{\mu v}\,.$$

After rearranging of this equation we can write

$$\mu w = \mu v e + swm\,.$$

It is evident that $\mu \mid sw$. We choose

(10) $$\mu = s^+ w^+$$

as stable polynomial, then

$$w = ve + s^- w^- m.$$

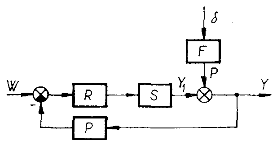
Hence $e = w^- x$, where x is a polynomial, and we obtain

(11) $$w^+ = vx + s^- m\,.$$

All solutions x, m of the equation (11) give the polynomial error signal $e = w^- x$. The time-optimal error signal is given as $e = w^- x$, where x is the minimal degree solution of the equation (11). The problem has a solution if and only if $(s^-, v) = 1$.

Consider the configuration (12),

(12)



where δ denotes white noise with zero mean and variance equal one and F is a stable transfer function.

It is known that then the spectral density of the signal D is given as $F^{=}F$.

The variance of the signal Y for $W = 0$ and stable closed-loop system is

$$I = \frac{1}{2\pi j} \int \frac{F^{=}F}{(1 + S^{=}R^{=}P^{=})(1 + SRP)} \frac{d\zeta}{\zeta}. \tag{13}$$

The basic problem in our task are stability conditions.

Theorem 4. *The necessary and sufficient conditions for the stability of the control system* (12) *are*

(i) $$K_{DY} = \frac{Y}{D} = \sigma N,$$

(ii) $$1 - K_{DY} = sM,$$

where M, N are any stable transfer functions for which

$$\sigma N + sM = 1,$$

(iii) $$(\varrho^{-}\pi^{-}, r^{-}p^{-}) = 1,$$

where ϱ^{-}, π^{-}, r^{-}, p^{-} are unstable factors of the polynomials ϱ, π, r, p respectively.

Proof. The pseudocharacteristic polynomial has the form

$$\chi = \sigma\varrho\pi + srp$$

and the transfer functions

$$K_{DY} = \frac{\sigma\varrho\pi}{\sigma\varrho\pi + srp} = \sigma N,$$

$$1 - K_{DY} = \frac{srp}{\sigma\varrho\pi + srp} = sM.$$

Hence

$$N = \frac{\varrho\pi}{\chi}, \quad M = \frac{rp}{\chi}.$$

Necessity.

(i), (ii) Let χ is a stable polynomial, then it is evident that N, M are stable.

(iii) Suppose that

$$(r^- p^-, \varrho^- \pi^-) = d^-,$$

then the polynomial $\chi = \sigma\varrho\pi + srp$ has the factor d^-, a contradiction.

Sufficiency. Suppose the conditions (i), (ii) and (iii) hold and in addition consider that χ has an unstable factor β^-. Then from the conditions (i) and (ii) it follows that

$$\beta^- \mid \varrho^- \pi^-, \quad \beta^- \mid r^- p^-.$$

From this the polynomial $(r^- p^-, \varrho^- \pi^-)$ contains the factor β^- which is a contradiction with condition (iii).

Task formulation MVC. Let us have $S = s/\sigma$ and a stable filter $F = f/\varphi$ for the configuration (12). Synthesize such controllers that the closed-loop system is stable and the functional (13) is minimized.

Four our purpose it is suffices to find only $R_1 = RP$ see (13), and if the condition (iii) from Theorem 4 is satisfied then this problem is reduced to the problem with one controller, which is treated in [5] for both stable and unstable S.

Those results are:

The controller

$$R_1 = \frac{M_1}{N_1}, \tag{14}$$

where

$$M_1 = \frac{x}{s^* f^* \sigma^{-\sim}}, \quad N_1 = \frac{\varphi y}{s^{-\sim} f^* \sigma^*} \tag{15}$$

and x, y is such a solution of the equation

$$s^- x + \sigma^- \varphi y = s^{-\sim} f^* \sigma^{-\sim} \tag{16}$$

that the polynomial y has the minimal degree, is the solution of the MVC problem if and only if M_1, N_1 are stable.

Then the variance of the signal Y is

$$I = \frac{1}{2\pi \mathrm{j}} \int \frac{y^= y}{b^{-=} b^-} \frac{\mathrm{d}\zeta}{\zeta}. \tag{17}$$

s^*, f^*, σ^* denotes the spectral factors of $ss^=, ff^=, \sigma\sigma^=$ respectively, $s^{-\sim}, \sigma^{-\sim}$ denotes the reciprocal polynomial, $s^{-\sim} = s^-(\zeta^{-1})\,\zeta^{\partial s^-}$, where ∂s^- denotes the degree of the polynomial s^-.

Considering the two-controller control systems (12). We can require stability of the closed-loop system and optimal transfer functions $K_{DY} = Y/D$ and $K_{WY} = Y/W$ simultaneously.

From the practical point of view it is possible to consider the following problem.

What transfer functions K_{WY} can be achieved if the transfer functions S and K_{DY} in a stable control system (12) are given?

Theorem 5. *Let S and K_{DY} be given such that $K_{DY} = \sigma N_1$, $1 - K_{DY} = sM_1$ where M_1, N_1 are stable transfer functions (see Theorem 4). Then the necessary and sufficient conditions for stability of the system (12) are*

(i) $$K_{WY} = sM ,$$

where $M = m/\mu$ is a stable transfer function,

(18) $$m^- \mid r_1 \varrho_1 ,$$

where $R_1 = M_1/N_1 = r_1/\varrho_1$ and $(r_1, \varrho_1) = 1$.

Proof. Necessity.

(i) Let the pseudocharacteristic polynomial χ is stable, then

$$M = \frac{r\pi}{\chi}$$

is stable.

(18) Let χ is stable.

The transfer function K_{WY} can be written as

$$K_{WY} = sM$$

and

$$M = \frac{r_1 \pi}{(\sigma \varrho_1 + s r_1)\, p} = \frac{\varrho_1 r}{(\sigma \varrho_1 + s r_1)\, \varrho} .$$

Consider that m^- includes a unstable factor d^- which does not divide $r_1 \varrho_1$, then $d^- \mid r$, $d^- \mid \pi$. It contradicts to stability condition (iii) from Theorem 4.

Sufficiency. Let (i) and (18) hold. For given M and $r_1 \varrho_1$ we can write

(19) $$R = \frac{M}{N_1} = \frac{r}{\varrho} = \frac{m(\sigma \varrho_1 + s r_1)}{\mu \varrho_1} ,$$

$$P = \frac{M_1}{M} = \frac{p}{\pi} = \frac{\mu r_1}{m(\sigma \varrho_1 + s r_1)} ,$$

where

$$(r, \varrho) = 1\,, \quad (p, \pi) = 1\,.$$

Factorize

$$r_1 = r_{10}^- r_{11}^- r_1^+\,, \quad \varrho_1 = \varrho_{10}^- \varrho_{11}^- \varrho_1^+$$

such that

$$m^- = r_{10}^- \varrho_{10}^-\,,$$

then from (19) it follows that

$$r^- = r_{10}^-\,, \quad \varrho^- = \varrho_{11}^-\,, \quad p^- = r_{11}^-\,, \quad \pi^- = \varrho_{10}^-$$

and the stability condition (iii) from Theorem 4 is satisfied.

Example 3 (MVC and LSC together). Let

$$S = \frac{\zeta(1 + 2\zeta)}{3 - \zeta}\,, \quad F = \frac{1}{4 - \zeta}\,, \quad W = \frac{1}{1 - \zeta}$$

be given in the configuration (12). Synthesize such controllers R, P that the system (12) is optimal in both MVC and LSC sense.

MVC:

$$s^- = \zeta(1 + 2\zeta)\,, \quad s^{-\sim} = 2 + \zeta\,, \quad \sigma^- = 1$$

$$\varphi = 4 - \zeta\,, \quad f = f^* = 1\,.$$

The equation (16) has the form

$$\zeta(1 + 2\zeta)\, x + (4 - \zeta)\, y = 2 + \zeta\,.$$

The solution with the minimal degree of y is

$$x = \tfrac{1}{6}\,, \quad y = \tfrac{1}{2} + \tfrac{1}{3}\zeta\,.$$

From (15) and (14)

$$M_1 = \frac{1}{6(z + \zeta)}\,, \quad N_1 = \frac{(4 - \zeta)(3 + 2\zeta)}{6(2 + \zeta)(3 - \zeta)}\,,$$

$$R_1 = \frac{M_1}{N_1} = \frac{3 - \zeta}{(4 - \zeta)(3 + 2\zeta)}\,.$$

It can be seen that M_1, N_1 are stable.

LSC:

Suppose that all transfer functions of the form $K_{WY} = sM$ can be achieved, then the optimal M for LSC is given by (19) as

$$M = \frac{1 - \zeta}{2 + \zeta}\left[\frac{\zeta^{-1}(1 + 2\zeta^{-1})}{(2 + \zeta^{-1})(1 - \zeta)}\right]_{\oplus} = \frac{1}{2 + \zeta}\,.$$

In this case the stability condition (18) is satisfied and the system (12) will be stable.

From (19)

$$R = \frac{M}{N_1} = \frac{(3 - \zeta) \cdot 6}{(4 - \zeta)(3 + 2\zeta)},$$

$$P = \frac{M_1}{M} = \frac{1}{6}$$

and the solution is complete.

The pseudocharacteristic polynomial is

$$\chi = 36(3 - \zeta)(\zeta + 2).$$

Example 4 (MVC and OTOC together). Consider S, F, W as in Example 3. Synthesize controllers R, P such that the system (12) is MVC and OTOC optimal simultaneously.

MVC:

See Example 3.

OTOC:

The equation (11) has the form

$$(1 - \zeta)x + \zeta(1 + 2\zeta)m = 1.$$

The solution with minimal degree of x is

$$x = 1 + \tfrac{2}{3}\zeta, \quad m = \tfrac{1}{3}, \quad E = 1 + \tfrac{2}{3}\zeta.$$

From (10)

$$\mu = 1, \quad M = \tfrac{1}{3}.$$

For this M the stability condition (18) is satisfied. Analogously to Example 3

$$R = \frac{(2 + \zeta)(3 - \zeta)2}{(4 - \zeta)(2\zeta + 3)},$$

$$P = \frac{1}{2(2 + \zeta)}.$$

The pseudocharacteristic polynomial is

$$\chi = 12(3 - \zeta)(2 + \zeta)^2.$$

Example 5 (MVC and OTOC together). Let

$$S = \frac{\zeta}{3 - \zeta}, \quad F = \frac{2 + \zeta}{4 - 2\zeta^2}, \quad W = \frac{4}{(1 - \zeta)(4 + 3\zeta)}$$

be given in the configuration (12).

MVC:

$$s^{-=} = \zeta\,, \quad s^{-\sim} = 1\,, \quad \sigma^{-} = 1\,,$$

$$\varphi = 4 - 2\zeta^2\,, \quad f = f^* = 2 + \zeta\,.$$

The equation (16) has the form

$$\zeta x + (4 - 2\zeta^2)\, y = 2 + \zeta\,.$$

The solution with the minimal degree of y is

$$x = 1 + \zeta\,, \quad y = \tfrac{1}{2}\,.$$

From (15) and (14)

$$M_1 = \frac{1 + \zeta}{2 + \zeta}\,, \quad N_1 = \frac{2 - \zeta^2}{(2 + \zeta)(3 - \zeta)}\,.$$

It can be seen that M_1, N_1 are stable.

OTOC:

The equation (11) has the form

$$\text{(DE)} \qquad (1 - \zeta)(4 + 3\zeta)\, x + \zeta m = 4\,.$$

The solution with minimal degree of x is

$$x = 1\,, \quad m = 1 + 3\zeta$$

and the error signal is $E = 1$.

From (10)

$$\mu = 4\,, \quad m = 1 + 3\zeta\,.$$

For this m the stability condition (18) is not satisfied.

In spite of the fact we shall find a solution of the OTOC problem.

Consider the general solution of (DE)

$$x = 1 + \zeta t\,,$$

$$m = 1 + 3\zeta - (1 - \zeta)(4 + 3\zeta)\, t\,,$$

where t is an arbitrary polynomial with degree ∂t. We shall try to find a solution x, m such that $m^- \mid r_1 \varrho_1$ holds while ∂x is minimal. This solution can be obtained by setting $\partial t = 0, 1, 2, \ldots$ successively until the required solution is found.

There are two posibilities:

(i) to find a t with minimal degree such that m is a stable polynomial,

(ii) to find a t with minimal degree such that

$$m^- = r_1^- \varrho_1^- = 1 + \zeta\,.$$

Sub (i)

Suppose that the degree of t is chosen zero, $t = \tau_0 \neq 0$. Then

$$m = 1 - 4\tau_0 + (3 + \tau_0)\zeta + 3\tau_0\zeta^2 . \tag{a}$$

The stability check [4] gives the following condition

$$\left| \frac{1 - 8\tau_0 + 7\tau_0^2}{-3 + 20\tau_0 + 7\tau_0^2} \right| > 1 .$$

By rearranging of this inequality we obtain

$$\left| 1 - \frac{4}{3 + \tau_0} \right| > 1$$

and hence follows

$$\tau_0 < -1 . \tag{b}$$

The solution of our problem is any τ_0 which satisfies (b).

Sub (ii)

Suppose that the degree of t is zero and $(1 + \zeta) \mid m$, then $m(-1) = 0$ and from (a)

$$0 = -2 - 2\tau_0 .$$

Hence

$$\tau_0 = -1 .$$

Now we have to check if $m^- \mid \varrho_1 r_1$.

For this τ_0

$$m = 5 + 2\zeta - 3\zeta^2 .$$

Making use of $(1 + \zeta) \mid m$ we can compute

$$m = (1 + \zeta)(5 - 3\zeta) .$$

Because $m^- = 1 + \zeta$ the problem is solved.

The error signal is given as

$$E = 1 + \zeta\tau_0 ,$$

where

$$\tau_0 < -1$$

or as

$$E = 1 - \zeta .$$

In our case both error signals are polynomials with degree one. Let us choose for example $\tau_0 = -1$, then

$$R = \frac{M}{N_1} = \frac{(1 + \zeta)(5 - 3\zeta)(2 + \zeta)(3 - \zeta)}{2 - \zeta^2} ,$$

$$P = \frac{M_1}{M} = \frac{1}{(2 + \zeta)(5 - 3\zeta)} .$$

CONCLUSION

This paper shows properties of two-controllers closed-loop control systems and methods of synthesis. It seems that the twice optimal control, where OTOC problem is solved under the condition that system is MVC optimal, has always a solution white MVC and LSC problem together cannot always be solved.

Both controllers are realized by a D.D.C. algorithm and it makes no difference to realise two algorithms instead of one.

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REFERENCES

[1] V. Kučera: Algebraic theory of discrete optimal control for single-variable systems I. Kybernetika *9* (1973), 2, 94–107.

[2] V. Strejc a kol.: Syntéza regulačních obvodů s číslicovým počítačem. NČSAV, Praha 1965.

[3] Я. З. Цыпкин: Теория линейных импульсных систем. Москва 1963.

[4] V. Strejc: Teorie lineární regulace. Lecture notes. Technical university, Prague 1970.

[5] V. Kučera: Algebraic approach to discrete stochastic control. Kybernetika. *11*, (1975) 2, 114–147.

[6] D. K. Fadějev: Numerické metody lineární algebry. SNTL, Praha 1964.

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