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States of Affairs as Values for Formulas

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States of affairs are constructed as pairs consisting of a (relational) structure **S** and of a subset of the set of structures similar to **S**. This concept is adequate to the intuitively motivated relation of strong equivalence (identity of states of affairs). Some semantical applications are mentioned.

1. INTRODUCTION

It is generally accepted that the extension of a sentence is its truth-value. But it is not intuitively adequate that a truth-value is what a sentence speaks about (or possibly: its denotatum). The sentences, e.g.,

The Earth is round *John is human*

have the same value, but we feel that they speak about different things.

Many authors have required that there must be some entities as facts or states of affairs about which sentences speak (Wittgenstein [1], Russell [2; 3], Reichenbach [4], Baylis [5], etc.). But there is no exact formulation of facts or states of affairs.

In some earlier papers we have proposed an experimental construction of states of affairs as ontological correspondents of sentences (closed formulas). In the present paper a generalized theory for formulas of any kind is developed.

We will prefer that states of affairs are *possible* states constructed as entities of the *extensional* (set-theoretical) ontological domain. The states of affairs are assigned not only to true and simple (atomic) formulas but – in contrast to B. Russell – to formulas of *any form* (atomic, compound, general, etc.) as their *state-values*.

Our problem consists in answering two questions: 1) when two formulas have the same state of affairs as their value and 2) what entity is the state of affairs (the constructive definition of state of affairs).

2. THE LANGUAGE L AND ITS METALANGUAGE

The object of our interest is an arbitrary applied first-order language L of the predicate calculus with identity. The metalanguage ML of L contains the letters

$$A, B, C, D, A_1, B_1, C_1, D_1, A_2, \ldots$$

as variables for expressions of L. ML is essentially the language of set theory with the following constants:

'$\rightarrow$' (implication), '$\sim$' (negation), '$\vee$' (disjunction), '&' (conjunction), '$\equiv$' (equivalence), '$(\ldots)$', '$(E\ldots)$' (universal and existential quantifiers), '$=$' (identity), '$\in$' (membershiprelation), '$(\hat{\ldots})$', '$(\hat{\hat{\ldots}})$' (abstraction-operators), '$\{\ldots\}$' (sign of n-tuple), '$\langle\ldots\rangle$' (sign of ordered n-tuple), '$\times$' (cartesian product), '0' (empty set).

First we use the sign ' ' for construction of names of expressions (both of L and ML). When this sign is applied to metavariables (or to meta-metavariables) or to compound expressions containing metavariables, we mean, e.g. by 'A', '$A = B$', '$\sim A$' in fact the metavariables: "an arbitrary L-expression A", "an arbitrary L-formula of the form $A = B$", "an arbitrary L-formula of the form $\sim A$", etc.

The language L now contains the following primitive signs:

logical constants: '$\Rightarrow$', '$\neg$', '$+$', '$.$', '$\Leftrightarrow$', '$(\forall\ )$', '$(\exists\ )$', '$=$';
specific constants (individuals): 'a_1', 'a_2', ..., 'a_n';
(predicates): 'P_1', 'P_2', ..., 'P_m';

Variables: 'x_1', 'x_2', ..., 'x_n', ...

We suppose that a predicate constant 'P_i' is k_i-ary. The number of variables is unlimited.

The concepts of *L-formula*, *L-sentence* and other syntactical concepts are defined in the ordinary manner. The result of replacing 'B' by 'C' in 'A' will be denoted as follows '$A(B/C)$' (analogically in other cases).

3. ONTOLOGY, INTERPRETATION AND VALUATION

Now, we suppose the existence of a domain of objects called *ontology* constructed in agreement with principles of set theory. The variables for objects of ontology are the following: $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, \alpha_2, \ldots$

Two truth-values (truth and falsehood) will be denoted as follows:

$$t, f.$$

The metalanguage ML is a many-sorted language. We have seen two sorts of variables. Other sorts will appear later.

Definition 1. *The interpretation of L is a function I assigning*

1. *to every variable the same nonempty set D^I (domain of I) and*
2. *to every constant 'A' exactly one entity $I(A)$ (denotatum of 'A' in I) such that: if 'A' is an individual constant, then $I(A) \in D^I$; if 'A' is a k-ary predicate, then $I(A) \subset \underbrace{D^I \times \ldots \times D^I}_{k\text{-times}}$.*

We will use the following symbols as the variables for interpretations:

$$I, K, J, I_1, K_1, J_1, I_2, \ldots$$

Definition 2. *The valuation of variables in I is function V (or V^I) assigning to every variable 'A' exactly one entity $V(A)$ such that $V(A) \in D^I$. The entity $V(A)$ is called the value of 'A' in V^I.*

We will use the following symbols as the variables for valuations in I:

$$V, U, W, V_1, U_1, W_1, V_2, \ldots$$

(with index I, K etc., or without index, when the use in a given context is clear).

Now we will define the general concept of the *value in I and V* for every expression of L (i.e. constants, variables and formulas of L). We denote this value of 'A' in I and V by '$v_V^I(A)$'.

Definition 3.

(1a) *If 'A' is a constant, then $v_V^I(A) = I(A)$.*
(1b) *If 'A' is a variable, then $v_V^I(A) = V^I(A)$.*
(2) *If 'A' is a k-ary predicate and 'A_1', …, 'A_k' are individual terms, then*

$$v_V^I(A(A_1, \ldots, A_k)) = t \equiv \langle v_V^I(A_1), \ldots, v_V^I(A_k) \rangle \in v_V^I(A).$$

(3) *If 'A' and 'B' are individual terms, then*

$$v_V^I(A = B) = t \equiv v_V^I(A) = v_V^I(B).$$

(4) *If 'A' is a formula, then*

$$v_V^I(\neg A) = t \equiv v_V^I(A) \neq t.$$

(5) *If 'A' and 'B' are formulas, then*

$$v_V^I(A \Rightarrow B) = t \equiv v_V^I(A) \rightarrow v_V^I(B).$$

(6) *If 'A' is a formula and 'x_i' is a variable, then*

$$v_V^I((\forall x_i)A) = t \equiv v_U^I(A) = t,$$

for every valuation U^I differing from V at most in the value for 'x_i'.

(7) *If 'A' is a formula, then*

$$v_V^I(A) \neq t \equiv v_V^I(A) = f\,.$$

(For conjunction, disjunction, equivalence and existential quantifier the rules of values are as customary.)

In the known manner other semantical concepts are defined:

Definition 4. *If 'A' is a formula , then*

'A' is satisfied by $V^I \equiv v_V^I(A) = t$;

$VER^I(A)$	$\equiv (V)(v_V^I(A) = t)$	*('A' is true in I)*;
$FALS^I(A)$	$\equiv (V)(v_V^I(A) = f)$	*('A' is false in I)*;
$L\text{-}VER(A)$	$\equiv (I)(V)(v_V^I(A) = t)$	*('A' is L-true)*;
$L\text{-}FALS(A)$	$\equiv (I)(V)(v_V^I(A) = f)$	*('A' is L-false)*;
$EQ^I(A, B)$	$\equiv (V)(v_V^I(A \equiv B) = t)$	*('A', 'B' are equivalent in I)*;
$L\text{-}EQ(A, B)$	$\equiv (I)(V)(v_V^I(A \equiv B) = t)$	*('A', 'B' are L-equivalent)*.

4. THE TRANSLATION OF *L*-FORMULAS IN *ML*

The *ML*-terms '$v_V^I(a_1)$', ..., '$v_V^I(a_n)$', '$v_V^I(P_1)$', ..., '$v_V^I(P_m)$', '$v_V^I(x_1)$', ..., '$v_V^I(x_p)$', ..., etc. will be called *I-V-terms*.

The *ML*-sentences constructed from formulas of set theory by replacing variables by *I-V*-terms or the term 'D^I' will be called *I-V- sentences*.

So called *I-V-translations* of *L*-formulas will be defined in the following way:

Definition 5.

(1) *If 'A' is a k-ary predicate and 'A_1', ..., 'A_k' are individual terms, then the I-V- translation of '$A(A_1, \ldots, A_k)$' is the ML-sentence*

$$\text{`}\langle v_V^I(A_1), \ldots, v_V^I(A_k)\rangle \in v_V^I(A)\text{'}\,.$$

(2) *If 'A' and 'B' are individual terms, then the I-V-translation of 'A = B' is the ML-sentence '$v_V^I(A) = v_V^I(B)$'.*

(3) *If 'A' is a formula and '$\mathscr{A}$' is its I-V-translation, then the I-V-translation of '$\neg A$' is the ML-sentence '$\sim\mathscr{A}$'.*

(4) *If 'A' and 'B' are formulas a '$\mathscr{A}$' and '$\mathscr{B}$' are their I-V-translations, then the I-V-translation of '$A \Rightarrow B$' is the ML-sentence '$\mathscr{A} \rightarrow \mathscr{B}$'.*

(5) *If 'A' is a formula, '$\mathscr{A}$' is its I-V-translation and 'x_i' is a variable, then the I-V-translation of '$(\forall x_i)A$' is the ML-sentence*

$$`(\alpha_i)_{\alpha_i \in D^I} \ \mathscr{A}(v_V^I(x_i)/\alpha_i)'.$$

We use the following symbols as the variables for metaexpressions

$$\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{A}_1, \mathscr{B}_1, \mathscr{C}_1, \mathscr{D}_1, \mathscr{A}_2, \ldots$$

Obviously, to every L-formula there is exactly one I-V-translation of this formula and vice versa (to every bound L-variable 'x_i' is assigned the bounded ML-variable 'α_i'). The set of I-V-translations is a proper subset of the set of I-V-sentences.

A fundamental relation between the value of L-formula 'A' in I and V and the I-V-translation of 'A' is expressed by the following theorem:

Theorem 1. *If 'A' is a formula and '$\mathscr{A}$' is its I-V-translation, then*

$$v_V^I(A) = t \equiv \mathscr{A} \text{ and}$$
$$v_V^I(A) = f \equiv \sim\mathscr{A}.$$

It follows immediately from Definitions 3 and 5.

Theorem 1 gives the possibility of reformulation of Definition 4.

Theorem 2. *If 'A' is an L-formula and '$\mathscr{A}$' is its I-V-translation, then*

$$VER^I(A) \equiv (V^I)\,\mathscr{A},$$
$$FALS^I(A) \equiv (V^I) \sim\mathscr{A},$$
$$L\text{-}VER(A) \equiv (I)\,(V)\,\mathscr{A},$$
$$L\text{-}FALS(A) \equiv (I)\,(V) \sim\mathscr{A}.$$

Theorem 3. *If 'A' and 'B' are L-formulas and '$\mathscr{A}$' and '$\mathscr{B}$' are their I-V-translations, then*

$$EQ^I(A, B) \equiv (V^I)\,(\mathscr{A} \equiv \mathscr{B}),$$
$$L\text{-}EQ(A, B) \equiv (I)\,(V)\,(\mathscr{A} \equiv \mathscr{B}).$$

The following theorem has an important role in our considerations:

Theorem 4. *If '$\mathscr{A}$' is a K-U-sentence containing exactly K-U-terms '$v_U^K(A_1)$', … …, '$v_U^K(A_i)$' and possibly 'D^K', then*

$$(I)\,(V)\,[\mathscr{A}(D^K, v_U^K(A_1), \dots, v_U^K(A_i))\,(K/I, U/V)] \equiv$$
$$\underset{\beta \neq 0}{(\beta)} \quad \underset{\omega_\beta^{\alpha_1}}{(\alpha_1)} \dots \underset{\omega_\beta^{\alpha_i}}{(\alpha_i)}\, \mathscr{A}(D^K/\beta, v_U^K(A_1)/\alpha_1, \dots, v_U^K(A_i)/\alpha_i)$$

where '$\omega_\beta^{\alpha_g}$' is '$\alpha_g \in \beta$', if 'A_g' is an individual term and '$\omega_\beta^{\alpha_g}$' is $\alpha \subset \underbrace{\beta \times \dots \times \beta}_{k\text{-times}}$ if 'A_g' is a k-ary predicate.

(By '$\mathscr{A}(K/I, U/V)$' we mean the result of replacing K by I and U by V in the K-U-sentence '$\mathscr{A}$').

Proof. Every interpretation I and every valuation V define an infinite sequence of entities

$$D^I, v_V^I(P_1), \dots, v_V^I(P_m), v_V^I(a_1), \dots, v_V^I(a_n), v_V^I(x_1), \dots, v_V^I(x_p), \dots$$

For the I-V-sentence '$\mathscr{A}$' only the values of terms contained in it are relevant. Therefore, the validity of '$\mathscr{A}$'for every I and V means its validity for every $(i + 1)$-tuple:

$$D^I, v_V^I(A_1), \dots, v_V^I(A_i)\,.$$

All these $(i + 1)$ tuples have only one common property:

$D^I \neq 0$ and

$v_V^I(A_g) \in D^I$, if 'A_g' is an individual term and

$v_V^I(A_g) \in \underbrace{D^I \times \dots \times D^I}_{k\text{-times}}$, if '$A_g$' is a k-ary predicate.

Therefore, we can equivalently express the validity of '$\mathscr{A}$' for every of these $(i + 1)$-tuples by the statement on the right side of the proved equivalence.

It follows from Theorems 4 and 2 that we can formulate the concepts of logical truth and logical falsehood as follows:

Theorem 5. *If 'A' is an L-formula and '$\mathscr{A}$' is its I-V-translation containing exactly the terms '$v_V^I(A_1)$', …, '$v_V^I(A_i)$' and possibly 'D^I', then*

$$L\text{-}VER(A) \equiv \underset{\beta \neq 0}{(\beta)} \quad \underset{\omega_\beta^{\alpha_1}}{(\alpha_i)} \dots \underset{\omega_\beta^{\alpha_i}}{(\alpha_i)}\, \mathscr{A}(D^I/\beta, v_V^I(A_1)/\alpha_1, \dots, v_V^I(A_i)/\alpha_i)\,,$$

$$L\text{-}FALS(A) \equiv \underset{\beta \neq 0}{(\beta)} \quad \underset{\omega_\beta^{\alpha_1}}{(\alpha_i)} \dots \underset{\omega_\beta^{\alpha_i}}{(\alpha_i)}\, \sim\mathscr{A}(D^I/\beta, v_V^I(A_1)/\alpha_1, \dots, v_V^I(A_i)/\alpha_i)\,.$$

(The conditions '$\omega_\beta^{\alpha_g}$' are the same as in Theorem 4.)

Second part of Theorem 5 is obvious by the equivalence

$$L\text{-}FALS(A) \equiv L\text{-}VER(\neg A) \equiv (I)\,(V) \sim \mathscr{A}\,.$$

As a corrollary to Theorem 4 we obtain

Theorem 6. *If '$\mathscr{A}$' is a K-U-sentence containing, among its terms, the term '$v_U^K(A)$', then*

$$(I)\,(V)\,[\mathscr{A}(v_U^K(A))\,(K/I,\,U/V)] \equiv (I)\,(V)\,[(\underset{\omega_{D^k}^{\alpha}}{(\alpha)}\;(v_U^K(A)/\alpha))\,(K/I,\,U/V)]\,.$$

If we can replace every term in a sentence '$\mathscr{A}$' by a "generalized variable", then we can surely replace one term only. Of course, the quatifiers (I) and (V) cannot be dropped (the sentence contains at least the term for domain D^I).

5. STRONG EQUIVALENCE

In the present section an attempt is made to formulate the intuitively adequate conditions under which two formulas speak about the same state of affairs. In [6] and [7] the concept of *extensional isomorphism* of two sentences (in I an V) was formulated. Now, we must formulate these concepts for formulas of L (which may be open formulas).

We denote the relation of extensional isomorphism of two formulas 'A' and 'B' in I and V by '$EIS_V^I(A, B)$'.

Definition 6. *If 'A' and 'B' are L-formulas, the $EIS_V^I(A, B)$, iff there are constants or variables free in 'A': 'A_1', ..., 'A_i' and there are constants or variables free in 'B' : 'B_1', ..., 'B_i', such that*

$$v_V^I(A_1) = v_V^I(B_1) \textit{ and } \dots \textit{ and } v_V^I(A_i) = v_V^I(B_i)$$

and

$$`A' = `B(B_1/A_1, \dots, B_i/A_i)'\,.$$

In many cases it seems to agree with our intuition that L-formulas speak about the same state of affairs, when the first of them results from the second one by replacing terms with the same value in I and V. But not only such formulas speak about the same state of affairs. Obviously, the formulas

$$`P_1(x_1)' \quad \text{and} \quad `P_1(x_1)\,.\,P_1(x_1)' \quad \text{and} \quad `P_1(y_1)\,.\,P_1(y_1)'$$

speak about the same state of affairs in I and V, if

$$v_V^I(x_1) = v_V^I(y_1)$$

holds. The second formula is a simple "logical transformation" of the first one and the third formula is extensionally isomorphic (in I and V) with the second one.

Therefore, we formulate a generalized concept of identity of states of affairs, namely the concept of "+strong equivalence in I and V" (symbolically '$+STREQ^I_V$'), in the following manner:

Definition 7. *If 'A' and 'B' are L-formulas, then* $+STREQ^I_V(A, B)$, *iff there are L-formulas 'C' and 'D' such that*: $EIS^I_V(A, C)$ *and* $EIS^I_V(B, D)$ *and* $L\text{-}EQ(C, D)$.

But this concept is too broad. The formulas

$$`x_1 = x_2' \quad \text{and} \quad `x_1 = x_1'$$

are $+STREQ$ in I and V, when $v^I_V(x_1) = v^{I'}_V(x_2)$ holds. This seems not to agree with intuition. The second sentence is logically true, the first sentence is only satisfied in I and V. The first formula results from the second one (or vice versa) by replacing the variable 'x_2' by the variable 'x_1'. But the variable 'x_2' is "essentially contained" in the first formula, whereas 'x_1' is contained in the second formula unessentially ('x_1' can be replaced in the second formula — in all its occurrence — by any other variable "salva veritate").

Therefore, we define the relation of strong equivalence in a different manner; but first we must formulate the concept of essential and unessential occurence of a term in a formula.

Definition 8. *If 'A' is an L-formula and 'B' is a constant or variable free in 'A', then*

$$\text{'A' essentially contains 'B', iff } (EI)(EV)[\mathscr{A} \not\equiv_{\omega^\alpha_{DI}} (\alpha)\ \mathscr{A}(v^I_V(B)/\alpha)]$$

where 'ω^α_{DI}' is customary and '$\mathscr{A}$' is the I-V-translation of 'A'.

The new concept of strong equivalence ($STREQ$) is now defined as follows:

Definition 9. *If 'A' and 'B' are L-formulas, then* $STREQ^I_V(A, B)$ *iff two following conditions are satisfied*:

1. $+STREQ^I_V(A, B)$ *and*
2. *for every 'C' contained essentially in 'A' there is 'D' essentially contained in 'B' such that* $v^I_V(C) = v^I_V(D)$ *and vice versa (where 'C' is a constant or a variable free in 'A' and 'D' is a constant or a variable free in 'B').**

In contrast to $+STREQ$ the following holds (under condition $v^I_V(x_1) = v^I_V(x_2)$):

$$\sim STREQ^I_V(x_1 = x_2, x_1 = x_1).$$

* This concept of strong equivalence is one of three concepts proposed in [6; 7], where it was denoted by $STREQ_0$.

In general, the extensional isomorphism does not imply the strong equivalence of two formulas.

Theorem 7. *There are L-formulas 'A' and 'B' and there are I and V such that:*

$$EIS^I_V(A, B) \,\&\, \sim STREQ^I_V(A, B)\,.$$

Theorem 8. *If 'A' and 'B' are L-formulas, then*

$$L\text{-}EQ(A, B) \rightarrow STREQ^I_V(A, B)\,,$$

$$STREQ^I_V(A, B) \rightarrow EQ^I_V(A, B)\,, \quad \textit{for every I and V.}$$

Theorem 9. *The relation STREQ is reflexive, symmetric and transitive.*

6. ABSTRACTION

In the preceding section we have formulated the criterion of having the same state-values in I and V. We must now give a definition of state of affairs as a set-theoretical entity. The state of affairs assigned to a formula (in and I and V) is constructed as as a pair of entities designated by members '$\mathscr{A}$' and '$\mathscr{B}$' of "abstraction-form"

$$\mathscr{A} \in \mathscr{B}$$

of the given formula (more exactly: of "abstraction-form" of the I-V-translation of the given formula). The known abstraction principle

$$\mathscr{A}(\alpha_1, \ldots, \alpha_i) \equiv \langle \alpha_1, \ldots, \alpha_i \rangle \in (\hat{\beta}_1, \ldots, \hat{\beta}_i)\, \mathscr{A}(\alpha_1/\beta_1, \ldots, \alpha_i/\beta_i)$$

must be used in a different form. It is intuitively adequate that the sentences '$a_1 = a_2$' and '$a_2 = a_1$' have the same states of affairs as values. But, abstraction-forms of their I-V-translations have different components

$$\langle v^I_V(a_1), v^I_V(a_2) \rangle \in \underset{\alpha_1 \in D^I \,\&\, \alpha_2 \in D^I}{(\hat{\alpha}_1, \hat{\alpha}_2)} (\alpha_1 = \alpha_2)\,,$$

$$\langle v^I_V(a_2), v^I_V(a_1) \rangle \in \underset{\alpha_2 \in D^I \,\&\, \alpha_1 \in D^I}{(\hat{\alpha}_2, \hat{\alpha}_1)} (\alpha_2 = \alpha_1)\,.$$

Left-hand sides of these sentences designate different pairs and, therefore, the states of affairs are different too. But an equivalent form of the second sentence is

$$\langle v^I_V(a_1), v^I_V(a_2) \rangle \in \underset{\alpha_1 \in D^I \,\&\, \alpha_2 \in D^I}{(\hat{\alpha}_1, \hat{\alpha}_2)} (\alpha_2 = \alpha_1)$$

in which the left-hand and the right-hand sides both denote the same objects as the components of the first sentence. This equivalence is justified by a variant of abstraction-principle:

$$\mathscr{A}(\alpha_1, \ldots, \alpha_i) \equiv \langle \alpha_{p_1}, \ldots, \alpha_{p_i} \rangle \in (\hat{\beta}_{p_1}, \ldots, \hat{\beta}_{p_i})\, \mathscr{A}(\alpha_1/\beta_1, \ldots, \alpha_i/\beta_i)$$

where $\alpha_{p_1}, \ldots, \alpha_{p_i}$ is a permutation of $\alpha_1, \ldots, \alpha_i$.

We will always use a definite permutation called the *lexicographical permutation.* Let the *lexicographical order of I-V*-terms be given as follows:

(L) $\quad v_V^I(P_1), \ldots, v_V^I(P_m), v_V^I(a_1), \ldots, v_V^I(a_n), v_V^I(x_1), \ldots, v_V^I(x_p), \ldots$ etc .

Definition 10. *The lexicographical permutation of an i-tuple of I-V-terms*

$$v_V^I(A_1), \ldots, v_V^I(A_i)$$

is its permutation, in which each member on the left precedes in the lexicographical order (L) *each member on the right.*

We denote this lexicographical permutation as follows:

$$v_V^I(A_{L_1}), \ldots, v_V^I(A_{L_i})\ .$$

In the abstraction-form of each *I-V* translation the term 'D^I' occurs in the relation on the right-hand side (at least in conditions laid down on abstraction operator). Therefore, we must subject also the term 'D^I' to the abstraction procedure. The form of abstraction principle for our purposes will be as follows:

Theorem 10. *If '$\mathscr{A}$' is an I-V-sentence containing exactly the terms '$v_V^I(A_1)$', … …, '$v_V^I(A_i)$' and possibly the term 'D^I', then*

$$\mathscr{A}(D^I, v_V^I(A_1), \ldots, v_V^I(A_i)) \equiv$$
$$\equiv \langle D^I, v_V^I(A_{L_1}), \ldots, v_V^I(A_{L_i}) \rangle \in \underset{\beta \neq 0\ \&\ \omega_\beta^{\alpha_{L1}}\ \&\ \ldots\ \&\ \omega_\beta^{\alpha_{Li}}}{(\hat{\beta}\hat{\alpha}_{L_1} \ldots \hat{\alpha}_{L_i})} \mathscr{A}(D^I/\beta, v_V^I(A_1)/\alpha_1, \ldots$$
$$\ldots, v_V^I(A_i)/\alpha_i)$$

where the conditions '$\omega_\beta^{\alpha_{Lg}}$' are customary.

The result of this abstraction-transformation applied to an *I-V*-sentence 'A' will be denoted as 'ABS^L(A)'. It is obvious that

$$ABS^L(\mathscr{A}) \equiv \mathscr{A}\ , \quad \text{under every } I \text{ and } V.$$

From the abstraction principle mentioned above and from the principle of extensionality it follows:

Theorem 11. *If '$\mathscr{A}$' and '$\mathscr{B}$' are K-U-sentences containing the same K-U-terms, e.g.: '$v_U^K(A_1)$, ..., '$v_U^K(A_i)$', then*

$$\underset{\beta \neq 0}{(\beta)} \; \underset{\omega_\beta^{\alpha_1}}{(\alpha_i)} \ldots, \underset{\omega_\beta^{\alpha_i}}{(\alpha_i)} \left[\mathscr{A}(D^K/\beta, v_U^K(A_1)/\alpha_1, \ldots, v_U^K(A_i)/\alpha_i) \equiv \mathscr{B}(D^K/\beta, v_U^K(A_1)/\alpha_1, \ldots\right.$$

$$\ldots, v_U^K(A_1)/\alpha_i] \equiv [\underset{\beta \neq 0 \& \omega_\beta^{\alpha_1} \& \ldots, \& \omega_\beta^{\alpha_i}}{(\beta\hat{\alpha}_1 \ldots, \hat{\alpha}_i)} \mathscr{A}(D^K/\beta, v_U^K(A_1)/\alpha_i, \ldots, v_U^K(A_i)/\alpha_i) =$$

$$= \underset{\beta \neq 0 \& \omega_\beta^{\alpha_1} \& \ldots \& \omega_\beta^{\alpha_i}}{(\beta\hat{\alpha}_1 \ldots, \hat{\alpha}_i)} \mathscr{B}(D^K/\beta, v_U^K(A_1)/\alpha_1, \ldots, v_U^K(A_i)/\alpha_i)] \, .$$

Because of validity of Theorem 4 the left-hand side of Theorem 11 means a logical equivalence, i.e. an equivalence holding under every I and V. Therefore, we may assert (in an abbreviated form):

Theorem 12. *Under the same condition as in Theorem* 11:

$$(I)\,(V)\,[(A \equiv B)\,(K/I, U/V)] \equiv$$

$$\equiv [\underset{\beta \neq 0 \& \omega_\beta^{\alpha_1} \& \ldots \& \omega_\beta^{\alpha_i}}{(\hat{\beta}\hat{\alpha}_1 \ldots \hat{\alpha}_i)} \mathscr{A} = \underset{\beta \neq 0 \& \omega_\beta^{\alpha_1} \& \ldots \& \omega_\beta^{\alpha_i}}{(\hat{\beta}\hat{\alpha}_1 \ldots \hat{\alpha}_i)} \mathscr{B}] \, .$$

7. REDUCTION

It may happen that in I and V the following equality holds:

$$v_V^I(a_1) = v_V^I(a_3) \, .$$

The sentences '$a_1 = a_2$' and '$a_3 = a_2$' are equivalent (and *STREQ*) in I and V, but the abstraction-forms of their I-V-translations contain on the left-hand sides the triples:

$$\langle D^I, v_V^I(a_1), v^I(a_2)\rangle \quad \text{and} \quad \langle D^I, v^I(a_2), v^I(a_3)\rangle$$

(the I-V-terms are lexicographically ordered), which represent different triples of objects. We need a procedure with the result that equivalent I-V-terms (I-V-terms with the same denotatum in I and V) have the same place in the lexicographical order of I-V-terms. For these purposes we adopt the *operation of I-V- reduction* of I-V-terms and of I-V-sentences.

Definition 11. *If '$\mathscr{A}$' and '$\mathscr{B}$' are I-V-terms, then '$\mathscr{A}$' is I-V-reduced term of '$\mathscr{B}$', iff '$\mathscr{A}$' is lexicographically the first member of the set of I-V-terms having the same denotatum as '$\mathscr{B}$'.*

For instance, from the example mentioned above, '$v_V^I(a_1)$' is I-V-reduced term of '$v_V^I(a_3)$' and '$v_V^I(a_1)$' is I-V-reduced term of '$v_V^I(a_1)$'.

Definition 12. *If '$\mathscr{A}$' is a I-V-sentence containing exactly '$v_V^I(A_1)$', ..., '$v_V^I(A_i)$' and '$v_V^I(B_1)$', ..., '$v_V^I(B_i)$' are their I-V-reduced terms, then the I-V- reduced form of '$\mathscr{A}$' (abbreviated: '$R_V^I(\mathscr{A})$') is the sentence*

$$\text{'}\mathscr{A}(v_V^I(A_1)/v_V^I(B_1), \ldots, v_V^I(A_i)/v_V^I(B_i))\text{'}.$$

Clearly, for every I and V, $R_V^I(\mathscr{A}) \equiv \mathscr{A}$, where '$\mathscr{A}$' is an I-V-sentence.

An interesting case for our purposes is when an interpretation and valuation give the same reduction as another interpretation and valuation.

Definition 13. *I and V define an analogical reduction as K and U, iff for every two terms it holds that '$v_V^I(A)$' is I-V-reduced term of '$v_V^I(B)$', iff '$v_U^K(A)$' is K-U-reduced term of '$v_U^K(B)$' (abbreviated: '$R_V^I = R_U^K$').*

We may express the same fact by a ML-condition:

Theorem 13. *I and V define an analogical reduction as K and U, iff for every two L-terms 'A' and 'B'*

$$v_V^I(A) = v_V^I(B) \equiv v_U^K(A) = v_U^K(B)$$

holds.

Proof. The reduction is determined uniquely by equivalences between I-V-terms. On the other hand, when stated what is a reduced term for every I-V-term, we can determine which of I-V-terms has the same denotatum (the lexicographical order is absolute; in no way it depends on I and V).

Therefore, we may consider the condition '$R_V^I = R_U^K$' as a ML-expression.

Furthermore, a few theorems about reduction will be proved. Evidently, under given I and V we replace I-V-terms by I-V-reduced terms, independently on the form of sentences; especially:

Theorem 14. *If '$\mathscr{A}$' and '$\mathscr{B}$' are I-V-sentences, then*

$$R_V^I(\mathscr{A} \equiv \mathscr{B}) \equiv [R_V^I(\mathscr{A}) \equiv R_V^I(\mathscr{B})].$$

Furthermore, we can state the following theorem:

Theorem 15. *If '$\mathscr{A}$' is a K-U-sentence, then*

$$(I)(V)[(R_U^K(\mathscr{A}))(K/I, U/V)] \equiv \underset{R_U^K = R_V^I}{(I)(V)} [\mathscr{A}(K/I, U/V)].$$

Proof. Let '$\mathscr{A}$' contain exactly K-U-terms '$v_U^K(A_1)$', ..., '$v_U^K(A_i)$' and let '$v_U^K(B_1)$', ..., '$v_U^K(B_i)$' be their K-U-reduced terms.

'$\mathscr{A}$' is therefore

(1) $$`\mathscr{A}(v_U^K(A_1), \ldots, v_U^K(A_i))\text{'} ,$$

'$R_U^K(\mathscr{A})$' is then

(2) $$`\mathscr{A}(v_U^K(B_1), \ldots, v_U^K(B_i))\text{'}$$

and '$(R_U^K(\mathscr{A}))(K/I, U/V)$' is

(3) $$`\mathscr{A}(v_V^I(B_1), \ldots, v_V^{I}(B_i))\text{'} .$$

(I) The validity of the left-hand side of the equivalence in Theorem 15 means

(a) $$(I)(V)\,\mathscr{A}(v_V^I(B_1), \ldots, v_V^I(B_i)) .$$

For every I and V the sentence (3) holds, which is the result of K-U-reduction of (1) and of replacing of K-U-reduced terms (in (2)) by corresponding I-V-terms.

The condition of validity of (3) for every I and V may be, therefore, the K-U-reduction. From this fact it follows that the sentence

(4) $$\mathscr{A}(v_V^I(A_1), \ldots, v_V^I(A_i))$$

holds in I and V satisfying the condition $R_U^K = R_V^I$, because exactly in these I and V the terms '$v_V^I(A_1)$', …, '$v_V^I(A_i)$' are equivalent with terms '$v_V^I(B_1)$', …, '$v_V^I(B_i)$'. Therefore, it holds

(b) $$\underset{R_U^K = R_V^I}{(I)(V)}\,[\mathscr{A}(K/I, U/V)] .$$

(II) On the other hand, if (b) holds and if we apply to the sentence '$\mathscr{A}(K/I, U/V)$' an I-V-reduction analogical with K-U-reduction, we obtain the sentence (3) holding for every I and V satisfying $R_V^I = R_U^K$. But this condition was in (3) already satisfied (all original terms were replaced by reduced terms) and, therefore, it can be dropped. The validity of (a) follows.

Theorem 16. *If 'A' and 'B' are L-formulas and '$\mathscr{A}$' and '$\mathscr{B}$' are their K-U-translations, then*

$$+STREQ_U^K(A, B) \equiv (I)(V)\,[(R_U^K(\mathscr{A}) \equiv R_U^K(\mathscr{B}))(K/I, U/V)] .$$

Proof. (I) The left-hand side of the equivalence means that there are L-formulas 'C' and 'D' such that $EIS_U^K(A, C)$ and $EIS_U^K(B, D)$ and L-$EQ(C, D)$. Let 'C^*' and 'D^*' be such L-formulas and let '$\mathscr{C}^*$' and '$\mathscr{D}^*$' be their K-U-translations. Therefore, it holds:

(1) $$EIS_U^K(A, C^*) \quad\text{and}\quad EIS_U^K(B, D^*) \quad\text{and}\quad L\text{-}EQ(C^*, D^*) .$$

It is obvious that K-U-reduced translations of L-formulas, which are EIS in K and U, must be identical:

(2) $$`R_U^K(\mathscr{A})' = `R_U^K(\mathscr{C}^*)'$$

and

(3) $$`R_U^K(\mathscr{B})' = `R_U^K(\mathscr{D}^*)' .$$

From (1) it follows also

(4) $$(I)\,(V)\,[(\mathscr{C}^* \equiv \mathscr{D}^*)\,(K/I,\, U/V)] .$$

Each reduction can – at most – identify some terms. Therefore, if (4) holds, it must hold also

(5) $$(I)\,(V)\,[(R_U^K(\mathscr{C}^*) \equiv R_U^K(\mathscr{D}^*))\,(K/I,\, U/V)]$$

(this can be seen also from Theorem 15 and from obvious specification of (44)). Because of (2) and (3), from (5) it follows:

(6) $$(I)\,(V)\,[(R_U^K(\mathscr{A}) \equiv R_U^K(\mathscr{B}))\,(K/I,\, U/V)] .$$

(II) Let us suppose the validity of the right-hand side of the equivalence (i.e. the validity of (6)). The K-U-reduced form of '$\mathscr{A}$' and '$\mathscr{B}$' are surely K-U-translations of some L-formulas. Let us designate the L-formula, the K-U-translation of which is '$R_U^K(\mathscr{A})$' as 'A^*' and the L-formula the K-U-translation of which is '$R_U^K(\mathscr{B})$' the K-U-translation as 'B^*'. Because of (6) we have

(1) $$L\text{-}EQ(A^*, B^*)$$

Furthermore, it is obvious that L-formulas 'A' and 'A^*' on the one hand and 'B' and 'B^*' on the other hand may differ by one aspect only: on the place where one formula contains a term, the other can at most contain an another but K-U-equivalent term (a term with the same value in K an U). Therefore, it holds in agreement with the definition of EIS:

(2) $$EIS_U^K(A, A^*) \quad \text{and} \quad EIS_U^K(B, B^*) .$$

If we now consider (1) with (2) together, we can say that
(3) there are L-formulas 'C' and 'D' (i.e. 'A^*' and 'B^*') such that (1) and (2) hold. The validity of (3) means that

(4) $$+STREQ_U^K(A, B) .$$

Theorem 17. *If 'A' and 'B' are L-formulas and '$\mathscr{A}$' and '$\mathscr{B}$' are their K-U-translations, then*

$$+STREQ_U^K(A, B) \equiv \underset{R_U^K = R_V^I}{(I)(V)} [(\mathscr{A} \equiv \mathscr{B})(K/I, U/V)] .$$

This theorem follows immediately from Theorems 16 and 14.

8. ELIMINATION

If we construct the states of affairs from denotata of two components of abstraction-form of the reduced *I-V*-translation, we obtain a result not corresponding to our intuition (such result represents the first variant of the definition of states of affairs in [7]). First: different states of affairs are assigned to the sentences '$a_1 = a_1$' and '$a_2 = a_2$', one containing the denotatum of 'a_1' the second the denotatum of 'a_2'. But both sentences are logically equivalent and we will say that they speak about the same thing. Furthermore, the sentence '$(P_1(a_1) . P_1(a_2)) + P_1(a_1)$' is logically equivalent to '$P_1(a_1)$', but we obtain different states of affairs: the state of affairs for the first sentence contains the denotatum of 'a_2' in contrast to the state of affairs assigned tho the second sentence. But what the first sentence says about the denotatum of 'a_2', can be said about everything.

When we take into account only "essential" occurrences of *L*-terms, our difficulties disappear. The sentences '$a_1 = a_1$' and '$a_2 = a_2$' contain 'a_1' and 'a_2' unessentially. The first of the last couple of sentences mentioned contains 'P_1' and a_1 essentially, but it contains 'a_2' unessentially. Therefore, we must *eliminate* the terms contained unessentially.

We define the meaning of "to contain essentially" and "to contain unessentially" for *I-V*-sentences:

Definition 14. *If '$\mathscr{A}$' is a K-U-sentence and '$\mathscr{B}$' is a K-U-term, then '$\mathscr{A}$' essentially contains '$\mathscr{B}$', iff*

$$(EI)(EV)[(\mathscr{A} \not\equiv \underset{\omega_{DK}^{\alpha}}{(\alpha)} \mathscr{A}(\mathscr{B}/\alpha))(K/I, U/V)]$$

and '$\mathscr{A}$' does not contain essentially '$\mathscr{B}$', iff

$$(I)(V)[(\mathscr{A} \equiv \underset{\omega_{DK}^{\alpha}}{(\alpha)} \mathscr{A}(\mathscr{B}/\alpha))(K/I, U/V)]$$

where 'ω_{DK}^{α}' is customary.

Definition 15. *If '$\mathscr{A}$' is a K-U-sentence and '$\mathscr{B}$' is a K-U-term, then '$\mathscr{A}$' contains unessentially '$\mathscr{B}$', iff '$\mathscr{A}$' contains '$\mathscr{B}$' but not essentially.*

Definition 16. *If '$\mathscr{A}$' is a K-U-sentence containing unessentially exactly the K-U-terms '$v_U^K(A_1)$', ..., '$v_U^K(A_i)$', then the first eliminated form of '$\mathscr{A}$' (abbreviated: '$ELIM_1(\mathscr{A})$') is*

$$\underset{\omega_{D^K}^{\alpha_1}}{(\alpha_1)} \ldots \underset{\omega_{D^K}^{\alpha_i}}{(\alpha_i)} \mathscr{A}(v_U^K(A_1)/\alpha_1, \ldots, v_U^K(A_i)/\alpha_i) .$$

We must separately define the essential (unessential) occurrence of the term 'D^K' and its possible elimination, because it can appear in the K-U-sentence first after the mentioned elimination (we can see this from the conditions '$\omega_{D^K}^{\alpha_g}$' in the first eliminated form).

Definition 17. *If '$\mathscr{B}$' is '$ELIM_1(\mathscr{A})$', where '$\mathscr{A}$' is a K-U-sentence and '$\mathscr{B}$' contains 'D^K', then '$\mathscr{B}$' contains essentially 'D^K', iff*

$$(EI)(EV)[(\mathscr{B} \not\equiv \underset{\beta \neq 0}{(\beta)} \mathscr{B}(D^K/\beta))(K/I, U/V)]$$

and '$\mathscr{B}$' contains unessentially 'D^K', iff

$$(I)(V)[(\mathscr{B} \equiv \underset{\beta \neq 0}{(\beta)} \mathscr{B}(D^K/\beta))(K/I, U/V)] .$$

Definition 18. *If '$\mathscr{B}$' is '$ELIM_1(\mathscr{A})$', where '$\mathscr{A}$' is a K-U-sentence, then the second eliminated form of '$\mathscr{B}$' (abbreviated: '$ELIM_2(\mathscr{B})$') is*

1. '$\underset{\beta \neq 0}{(\beta)} \mathscr{B}(D^K/\beta)$', *when '$\mathscr{B}$' contains '$D^K$' unessentially and*
2. '$\mathscr{B}$', *when '$\mathscr{B}$' does not contain 'D^K' or contains it essentially.*

Instead of '$ELIM_2(ELIM_1(\mathscr{A}))$' we will write '$ELIM(\mathscr{A})$'.

It is obvious that

Theorem 18. *If '$\mathscr{A}$' is a K-U-sentence, then* $(I)(V)[(ELIM(\mathscr{A}) \equiv \mathscr{A})(K/I, U/V)]$.

An auxiliary concept will be usefull:

Definition 19. *If '$\mathscr{A}$' is a K-U-sentence containing exactly K-U-terms '$v_U^K(A_1)$', ..., '$v_U^K(A_1)$' and possibly 'D^K', then the totally generalized form of '$\mathscr{A}$' (abbreviated: '$TG(\mathscr{A})$') is*

$$\text{'}\underset{\beta \neq 0}{(\beta)} \quad \underset{\omega_\beta^{\alpha_1}}{(\alpha_1)} \ldots \underset{\omega_\beta^{\alpha_i}}{(\alpha_i)} \mathscr{A}(D^K/\beta, v_U^K(A_1)/\alpha_1, \ldots, v_U^K(A_i)/\alpha_i)\text{'} .$$

It is clear that

Theorem 19. *If '$\mathscr{A}$' is a K-U-sentence and '$ELIM(\mathscr{A})$' contains no K-U-term and does not contain 'D^K', it holds*:

1. $\mathscr{A}(K/I, U/V) \equiv TG(\mathscr{A})$, *for every I and V*,
2. '$ELIM(\mathscr{A})$' = '$TG(\mathscr{A})$'.

We can now reformulate Theorems 4 and 5 briefly as follows:

Theorem 20. *If '$\mathscr{A}$' is a K-U-sentence, then*

1. $(I)(V)[\mathscr{A}(K/I, U/V) \equiv TG(\mathscr{A})]$,
2. $(I)(V)[\sim\mathscr{A}(K/I, U/V) \equiv TG(\sim\mathscr{A})]$.

Theorem 21. *If '$\mathscr{A}$' is the K-U-translation of an L-formula '$\mathscr{A}$', then*

1. $L\text{-}VER(A) \equiv TG(\mathscr{A})$,
2. $L\text{-}FALS(A) \equiv TG(\sim\mathscr{A})$.

We now define the analytic and the synthetic formulas as follows:

Definition 20. *If '$\mathscr{A}$' is the K-U-translation of a formula 'A', then*

1. $ANAL(A) \equiv TG(\mathscr{A}) \vee TG(\sim\mathscr{A})$,
2. $SYNT(A) \equiv \sim ANAL(A)$.

9. STATES OF AFFAIRS

We now define the states of affairs in the following manner (the abbreviation '$\mathscr{S}^I_V(A)$' means: the state of affairs assigned to an L-formula 'A' under given interpretation I and valuation V):

Definition 21. *If 'A' is an L-formula and SYNT(A) and '$\mathscr{A}$' is its I-V-translation, then*

$$\mathscr{S}^I_V(A) = \langle \alpha, \beta \rangle, \quad \textit{if} \quad `\alpha \in \beta\text{'} = `ABS^L(R^I_V(ELIM(\mathscr{A})))\text{'}.$$

From Theorem 10, exhibiting the operation ABS^L in detail, we see:

Theorem 22. *If 'A' is an L-formula and SYNT(A) and '$\mathscr{A}$' is its I-V-translation* then $\mathscr{S}^I_V(A)$ is the ordered pair construed of

$$\langle D^I, v^I_V(A_{L_1}), \ldots, v^I_V(A_{L_i}) \rangle$$

and of

$$\underset{\beta \neq 0 \,\&\, \omega_\beta^{\alpha_{L1}} \,\&\, \ldots \,\&\, \omega_\beta^{\alpha_{Li}}}{(\hat{\beta}\hat{\alpha}_{L_1} \ldots \hat{\alpha}_{L_i})} \quad [(R^I_V(ELIM(\mathscr{A})))\,(D^I/\beta, v^I_V(A_1)/\alpha_1, \ldots, v^I_V(A_i)/\alpha_i)]$$

where '$\omega_\beta^{\alpha_g}$' is a customary condition and the terms '$v_V^I(A_1)$', ..., '$v_V^I(A_i)$' are all different I-V-terms, which remain in '$R_V^I(ELIM(\mathscr{A}))$'.

The formal justification of the definition shows the following

Theorem 23. *If 'A' is an L-formula and SYNT(A), then there exists the unique α such that* $\alpha = \mathscr{S}_V^I(A)$

Proof. If 'A' is synthetic, then neither '$TG(\mathscr{A})$' nor '$TG(\sim\mathscr{A})$' holds, i.e. '$ELIM(\mathscr{A})$' is different from '$TG(\mathscr{A})$'. The sentence '$ELIM(\mathscr{A})$' contains, therefore, at least one I-V-term or 'D^I'. This term must remain also after application of reduction (a reduction can only diminish the number of terms contained, but cannot remove them) and after application of abstraction. Therefore, there exist the first member and the second member of the pair $\langle \alpha, \beta \rangle$. Furthermore, the members α and β are unique, because of univocality of I-V-translation, elimination, reduction and lexicographical abstraction.

For analytic formulas there are no states of affairs (they "speak" about the same thing, namely, "about nothing"). Analytic sentences contain all terms unessentially (their I-V-translation contains unessentially also 'D^I'). All terms and 'D^I', therefore, must be eliminated and we cannot apply the operation of abstraction.

We have seen that a state of affairs is a set-theoretical entity, namely a construct of relational structures. We call two relational structures

$$\langle \beta, \alpha_1, \ldots, \alpha_i \rangle \quad \text{and} \quad \langle \beta', \alpha_1', \ldots, \alpha_i' \rangle$$

similar, if $i = j$ and if for every α_g and α_g' it holds $\omega_\beta^{\alpha_g} \equiv \omega_\beta^{\alpha'_g}$ (these conditions are customary).

We can state that every state of affairs is a pair consisting of one structure $\mathbf{S}^A$ and of the set of structures $\mathbf{T}^A$:

$$\langle \mathbf{S}^A, \mathbf{T}^A \rangle$$

where each structure of $\mathbf{T}^A$ is similar to $\mathbf{S}^A$ and satisfying the formula

$$(R_V^I(ELIM(\mathscr{A})))\,(D^I/\beta, v_V^I(A_1)/\alpha_1, \ldots, v_V^I(A_i)/\alpha_i)$$

(the structure $\mathbf{S}^A$ is determined by a formula 'A', interpretation I and valuation V).

The great disadvantage of our construction of states of affairs is their relative existence. We can construct a state of affairs only for a given formula under given I and V. We need "absolute" states of affairs existing independently of formulas. This will be an object of our further investigations. A possibility of such a formulation gives us the foregoing note on structures.

10. ADEQUACY

The construction of states of affairs is too complicated for an examination from the point of view of intuitive adequacy. But we have defined the relation of strong equivalence which represents a proposal of formulation of what we mean when we say that two sentences speak about the same thing (about the same state of affairs).*

Therefore, we can exhibit some relative intuitive adequacy of our construction of states of affairs with respect to the relation of strong equivalence. Such an adequacy states the following theorem:

Theorem 24. (Adequacy theorem). *If 'A' and 'B' are synthetic L-formulas and '$\mathscr{A}$' and '$\mathscr{B}$' are their K-U-translations, then*

$$STREQ_U^K(A, B) \equiv \mathscr{S}_U^K(A) = \mathscr{S}_U^K(B)\,.$$

Proof. (I) First we suppose

$$(1) \qquad STREQ_U^K(A, B)\,.$$

This means the conjunction of

$$(2) \qquad +STREQ_U^K(A, B)$$

and

(3) for every 'C' essentially contained in 'A' there is a 'D' essentially contained in 'B' such that $v_U^K(C) = v_U^K(D)$ and vice versa (where: 'C' is a constant or variable free in 'A' and 'D' is a constant or variable free in 'B').

The assumption (2) is by Theorem 17 first equivalent to

$$(4) \qquad \underset{R_V^I = R_U^K}{(I)\,(V)}\ [(\mathscr{A} \equiv \mathscr{B})\,(K/I, U/V)]$$

and by Theorem 18 to

$$(5) \qquad \underset{R_V^I = R_U^K}{(I)\,(V)}\ [(ELIM(\mathscr{A}) \equiv ELIM(\mathscr{B}))\,(K/I, U/V)]\,.$$

This can be writen equivalently (by Theorems 14 and 15) as:

$$(6) \qquad (I)\,(V)\,[(R_U^K(ELIM(\mathscr{A})) \equiv R_U^K(ELIM(\mathscr{B})))\,(K/I, U/V)]\,.$$

The second part of assumption (3) holds also for translations:

(7) For every term '$v_U^K(C)$' contained essentially in '$\mathscr{A}$' there is a term '$v_U^K(D)$' essentially contained in '$\mathscr{B}$' such that $v_U^K(C) = v_U^K(D)$ and vice versa.

The operation *ELIM* now eliminates the terms contained unessentially and, therefore, we obtain a result equivalent to (7):

(8) For every term '$v_U^K(C)$' contained in '$ELIM(\mathscr{A})$' there is a term '$v_U^K(D)$' contained in '$ELIM(\mathscr{B})$' such that $v_U^K(C) = v_U^K(D)$ and vice versa.

* More detailed discussion of these intuitive ideas is in [6] and [7].

The K-U-reduction now identifies these terms denoting the same thing and, therefore, we have equivalently:

(9) '$R_U^K(ELIM(\mathscr{A}))$' and '$R_U^K(ELIM(\mathscr{B}))$' contain the same K-U-terms.

Let now the terms mentioned in (9) be '$v_U^K(A_1)$', ..., '$v_U^K(A_i)$' and possibly 'D^K'. From the fact (9) and from the equivalence (6) it follows by Theorem 12 that the relations obtained by the operation ABS^L on the right-hand sides of abstraction-forms are identical:

(10)
$$\underset{\beta \neq 0 \,\&\, \omega_\beta^{\alpha_{L1}} \,\&\, \ldots \,\&\, \omega_\beta^{\alpha_{Li}}}{(\hat{\beta}\hat{\alpha}_{L_1} \ldots \hat{\alpha}_{L_i})} \left[(R_U^K(ELIM(\mathscr{A})))\,(D^K/\beta, v_U^K(A_1)/\alpha_1, \ldots, v_U^K(A_i)/\alpha_i)\right] =$$
$$= \underset{\beta \neq 0 \,\&\, \omega_\beta^{\alpha_{Li}} \,\&\, \ldots \,\&\, \omega_\beta^{\alpha_{Li}}}{(\hat{\beta}\hat{\alpha}_{L_1} \ldots \hat{\alpha}_{L_i})} \left[(R_U^K(ELIM(\mathscr{B})))\,(D^K/\beta, v_U^K(A_1)/\alpha_1, \ldots, v_U^K(A_i)/\alpha_i)\right].$$

Furthermore, it follows from (9) that the $(i + 1)$-tuples on both left-hand sides in abstraction-forms of sentences mentioned in (9) must be identical and must be expressed by identical expressions

(11) $$\langle D^K, v_U^K(A_{L_1}), \ldots, v_U^K(A_{L_i})\rangle$$

(they must always exist because the starting sentences are synthetic; in the case $i = 0$ they must contain only 'D^K').

From this fact and from (10) it follows by means of definition of states of affairs that

(12) $$\mathscr{S}_U^K(A) = \mathscr{S}_U^K(B).$$

(II) We must now prove that (12) implies (1). We suppose (12). We denote the mentioned states of affairs as $\langle\alpha, \beta\rangle$ and $\langle\alpha', \beta'\rangle$. Therefore, it holds

(13) $$\langle\alpha, \beta\rangle = \langle\alpha', \beta'\rangle,$$

i.e.

(14) $$\alpha = \alpha' \quad \text{and} \quad \beta = \beta'.$$

From the definition of states of affairs $\mathscr{S}_U^K(A)$ and $\mathscr{S}_U^K(B)$ we can see that the identity '$\beta = \beta'$' must have the form (10) above and '$\alpha = \alpha'$' must have the form of identity between two identical $(i + 1)$-tuples, both denoted by the expression (11) above (in agreement with construction of states of affairs the terms in (11) are reduced and, therefore, the terms denoting the same objects are the same terms). From (10) and from the fact that α and α' are signed by the same expression (11) the validity of (6) follows immediately by Theorem 12 above and (6) is equivalent — as we have seen — to

(15) $$+STREQ_U^K(A, B).$$

The statement (9) also holds (it follows from identity of expressions for α and α'), and is equivalent to (3). These two facts give together the validity of

(16) $$STREQ_U^K(A, B).$$

The theorem of adequacy shows that the concept of state of affairs is adequate with respect to the relation *STREQ*. If two formulas are *STREQ* in *I* and *V*, then the states of affairs assigned to these formulas in *I* and *V* are the same and vice versa. In [6] and [7] two other variants of the concept of *STREQ* and of the concept of state of affairs are exhibited.

11. STATE FUNCTIONS AND FACTS

It is customary to say that each given *L*-formula '*A*' represents a truth-value function, i.e. a function assigning to every pair *I* and *V* a truth value. Let '*A*' contain two constants or variables 'B_1' and 'B_2'. We often express the function mentioned by the table:

interpretation and valuation	values of 'B_1', 'B_2'	value of '$A(B_1, B_2)$'
$I \quad V$	$v_V^I(B_1) \quad v_V^I(B_2)$	$v_V^I(A(B_1, B_2))$
$K \quad U$	$v_U^K(B_1) \quad v_U^K(B_2)$	$v_U^K(A(B_1, B_2))$
$J \quad W$	$v_W^J(B_1) \quad v_W^J(B_2)$	$v_W^J(A(B_1, B_2))$

Now, we have a new sort of such semantic functions. Each given *synthetic* formula represents a function assigning to each pair *I* and *V* a state of affairs. This expresses the table (where $A(B_1, B_2)$ is a synthetic formula):

interpretation and valuation	values of 'B_1', 'B_2'	value of '$A(B_1, B_2)$'
$I \quad V$	$v_V^I(B_1) \quad v_V^I(B_2)$	$\mathcal{S}_V^I(A(B_1, B_2))$
$K \quad U$	$v_U^K(B_1) \quad v_U^K(B_2)$	$\mathcal{S}_U^K(A(B_1, B_2))$
$J \quad W$	$v_W^J(B_1) \quad v_W^J(B_2)$	$\mathcal{S}_W^J(A(B_1, B_2))$
etc.		

We call such functions the *state-functions*. Each synthetic formula thus represents a state-function.

When we will study the relation between truth-value functions and state functions, we must introduce a new concept. Intuitively, not all states of affairs are "real states".

Only those are real which correspond to true synthetic sentences (or: which are assigned to a synthetic formula, when its value is truth). We will call them *facts*. We define the class *FACT* which will have as members these facts.

Definition 22.

$$\alpha \in FACT \equiv (EI)(EV)(EA)[\alpha = \mathscr{S}^I_V(A) \,\&\, \alpha = \langle \beta, \gamma \rangle \,\&\, \beta \in \gamma]$$

where 'A' is a synthetic formula.

Therefore, facts are such states of affairs $\langle \beta, \gamma \rangle$, for which $\beta \in \gamma$. Other states of affairs are *non-facts* (i.e. when $\beta \notin \gamma$).

From our constructions it is easy to see that

Theorem 25. *If 'A' is a synthetic L-formula and '$\mathscr{A}$' is its I-V-translation, then*

$$\mathscr{S}^I_V(A) \in FACT \equiv \mathscr{A} \equiv v^I_V(A) = t,$$
$$\mathscr{S}^I_V(A) \notin FACT \equiv \sim\mathscr{A} \equiv v^I_V(A) = f.$$

(When we take '$ABS^L(R^I_V(ELIM(\mathscr{A})))$' as '$\beta \in \gamma$', it is clear that $\beta \in \gamma \equiv \mathscr{A}$, because evidently $ABS^L(R^I_V(ELIM(\mathscr{A}))) \equiv \mathscr{A}$; and $\langle \beta, \gamma \rangle = \mathscr{S}^I_V(A)$.)

This gives us a new expression for truth-value function represented by synthetic formulas and we can reformulate semantic definitions as follows:

Theorem 26. *If 'A' is a synthetic L-formula, then*

$$\text{'}A\text{' is satisfied in } I \text{ and } V \equiv \mathscr{S}^I_V(A) \in FACT,$$
$$VER^I(A) \equiv (V)\, \mathscr{S}^I_V(A) \in FACT,$$
$$FALS^I(A) \equiv (V)\, \mathscr{S}^I_V(A) \notin FACT.$$

The truth-functions of propositional logic can be redefined in the case of synthetic formulas:

Theorem 27. *If 'A' and 'B' are synthetic L-formulas, then*

$$\mathscr{S}^I_V(A \Rightarrow B) \in FACT \equiv \mathscr{S}^I_V(A) \in FACT \rightarrow \mathscr{S}^I_V(B) \in FACT,$$
$$\mathscr{S}^I_V(A \,.\, B) \in FACT \equiv \mathscr{S}^I_V(A) \in FACT \,\&\, \mathscr{S}^I_V(B) \in FACT,$$
$$\mathscr{S}^I_V(A + B) \in FACT \equiv \mathscr{S}^I_V(A) \in FACT \vee \mathscr{S}^I_V(B) \in FACT,$$
$$\mathscr{S}^I_V(A \Leftrightarrow B) \in FACT \equiv \mathscr{S}^I_V(A) \in FACT \equiv \mathscr{S}^I_V(B) \in FACT,$$
$$\mathscr{S}^I_V(\neg A) \in FACT \equiv \mathscr{S}^I_V(A) \notin FACT.$$

It is possible that the mentioned procedure enables us to solve certain questions about so called non-extensional contexts. It is known that the following statements do not hold:

$$A \Leftrightarrow B \equiv \textit{It is necessary that } A \Leftrightarrow \textit{It is necessary that } B\,,$$

$$A \Leftrightarrow B \equiv \textit{It is believed that } A \Leftrightarrow \textit{It is believed that } B\,.$$

But what is "necessary" or what is "believed"? Is it a truth value? In my opinion, we assert that a state of affairs is necessary a fact and it is also an entity which is believed to be a fact. For synthetic formulas it holds:

$$STREQ_V^I(A, B) \equiv \textit{It is necessary that } \mathscr{S}_V^I(A) \in FACT \equiv$$
$$\equiv \textit{It is necessary that } \mathscr{S}_V^I(B) \in FACT\,,$$

$$STREQ_V^I(A, B) \equiv \textit{It is believed that } \mathscr{S}_V^I(A) \in FACT \equiv$$
$$\equiv \textit{It is believed that } \mathscr{S}_V^I(B) \in FACT\,.$$

The main direction of our further investigations will be a construction of states of affairs independently of formulas. This will make possible a reconstruction of the foundations of logical semantics and many interesting applications.

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