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Number of Alternatives in Reducing Finite Spaces and Vector Spaces

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In this paper the number of different partitions of finite spaces and of n-dimensional vector spaces is given, as well as the number of all partitions, if a non exhaustive method is used. Relations between the corresponding numbers of partitions of both methods are presented, too.

A. Perez fomulated the following problem: Let X_n be a finite space with $|X_n| = n$ or an n-dimensional vector space.

Let $\mathcal{Y}_m$, $m \leqq n$, be any partition of X_n in m disjoint sets, resp. any cylindric partition of the n-dimensional vector space X_n, which corresponds to the rejecting of $n - m$ coordinates of X_n.

Let $R(\mathcal{Y}_m)$ be a real valued function of $\mathcal{Y}_m$, $m = 1, 2, \ldots, n$, where $\mathcal{Y}_n = X_n$. Let

$$\max_{\substack{\mathcal{Y}_m \\ m=1,2,\ldots,n}} R(\mathcal{Y}_m) = R_0 = R(\overline{\mathcal{Y}}_{\bar{m}}) .$$

The original task is to determine a maximizing $\overline{\mathcal{Y}}_{\bar{m}}$ among all the (admissible, if it is required to respect some given constraints) $\mathcal{Y}_m$'s.

The exhaustive method requests to consider all the possible alternatives of $\mathcal{Y}_m$, to calculate the respective $R(\mathcal{Y}_m)$ and to compare them in order to find some $\overline{\mathcal{Y}}_{\bar{m}}$.

Since the number of all possible alternatives grows very quickly with n, the exhaustive method will be, in general, unpracticable. This situation leads to approximative non-exhaustive methods.

One such method is the following: Take in the first step, $m = m - 1$ and let $\mathcal{Y}^0_{n-1}$ be a maximizing (admissible) partition, i.e.

$$R(\mathcal{Y}^0_{n-1}) = \max_{\mathcal{Y}_{n-1}} R(\mathcal{Y}_{n-1}) .$$

In the second step, take $m = n - 2$ and let $\mathcal{Y}^0_{n-2}$ be a maximizing (admissible) subpartition of $\mathcal{Y}^0_{n-1}$, i.e.

$$R(\mathcal{Y}^0_{n-2}) = \max_{\mathcal{Y}_{n-2} \text{ subpartitin of } \mathcal{Y}^0_{n-1}} R(\mathcal{Y}_{n-2}) ,$$

etc.

In the $(n - m)$-th step, take $m = m$ and let $\mathscr{Y}^0_m$ be a maximizing (admissible) subpartition of $\mathscr{Y}^0_{m+1}$, i.e.

$$R(\mathscr{Y}^0_m) = \max_{\mathscr{Y}_m \text{ subpartition of } \mathscr{Y}^0_{m+1}} R(\mathscr{Y}_m)\,.$$

Finaly, let m_0 be such that

$$R(\mathscr{Y}^0_{m_0}) = \max_m R(\mathscr{Y}^0_m)\,.$$

In general, $\mathscr{Y}^0_{m_0} \neq \overline{\mathscr{Y}}_{\bar{m}}$ and $R(\mathscr{Y}^0_{m_0}) \leqq R(\overline{\mathscr{Y}}_{\bar{m}}) = R_0$.

However, there are cases where the equality is approximately attained in the inequality above (e.g. the case of minus α-entropy of P with respect to Q). The problem formulated by A. Perez is to compare the numbers of alternatives to be considered in the exhaustive and non-exhaustive methods above.

I. NUMBER OF REDUCTIONS OF FINITE SPACES

Definition 1. Let m, n be fixed, $m < n$. A *reduction* of a space X_n with $|X_n| = n$ is a partition

$$\mathscr{Y}_m = \{Y_1, \ldots, Y_m\}$$

of the space X_n, when the following is valid:

$$Y_i \subset X_n\,, \quad i = 1, \ldots, m\,,$$
$$Y_i \cap Y_j = 0 \quad \text{for} \quad i \neq j\,,$$
$$\bigcup_{i=1}^{m} Y_i = X_n\,.$$

Let $m < n$ be fixed. Let $V_{n,m}$ be the number of all different partitions $\mathscr{Y}_m$ of the space X_n.

Theorem 1. *For $n \geqq m \geqq 1$ the fllowing formula holds:*

$$(1) \qquad V_{n,m} = \sum_{r=1}^{m} \sum_{(n_1,\ldots,n_r)\in N_r} \sum_{(k_1,\ldots,k_r)\in K_{N_r}} \frac{n!}{(k_1!)^{n_1}(k_2!)^{n_2}\ldots(k_r!)^{n_r}\, n_1!\ldots n_r!}$$

where

$$N_r = \{(n_1, \ldots, n_r) : n_i > 0,\ i = 1, \ldots, r,\ n_1 + \ldots + n_r = m\}\,,$$
$$K_{N_r} = \{(k_1, \ldots, k_r) : k_i > 0,\ k_{i-1} < k_i,\ i = 2, \ldots, r,$$
$$n_1 k_1 + \ldots + n_r k_r = n \ \forall (n_1, \ldots, n_r) \in N_r\}\,.$$

(For some r and N_r the K_{N_r} sets may be also empty.)

Proof. Formula (1) follows from the fact, that for fixed $k'_1, \ldots, k'_m$ such that $k'_1 + \ldots + k'_m = n$, where n_i of k'_j are the same, the number of all partitions is:

$$\frac{1}{n_1! \ldots n_r!} \binom{n}{k'_1} \binom{n - k'_1}{k'_2} \ldots \binom{n - k'_1 - \ldots - k'_{m-2}}{k'_{m-1}}.$$

If we denote the same k'_j by k_i, we may write the last expression as:

$$\frac{n!}{n_1! \ldots n_r! \, (k_1!)^{n_1} \ldots (k_r!)^{n_r}}.$$

Especialy we can deduce from formula (1):

$$V_{n,2} = \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{k} N,$$

$$k = \frac{n-1}{2}, \quad N = 1 \quad \text{for} \quad n \quad \text{odd},$$

$$k = \frac{n}{2}, \quad N = \tfrac{1}{2} \quad \text{for} \quad n \quad \text{even},$$

$$V_{n,n-1} = \binom{n}{2},$$

$$V_{n,n-2} = \binom{n}{3} + 3\binom{n}{n-4},$$

$$V_{n,n-3} = \binom{n}{4} + 10\binom{n}{n-5} + 15\binom{n}{n-6},$$

$$V_{n,n-4} = \binom{n}{5} + 25\binom{n}{n-6} + 105\left[\binom{n}{n-7} + \binom{n}{n-8}\right],$$

where we put $\binom{n}{n-j} = 0$ for $n - j < 0$.

Let $P_{n,m}$ be the number of all different partitions, if the mentioned non exhaustive procedure is used.

Theorem 2. *Let be $m < n$. Then*

$$P_{n,m} = \binom{n}{2} + \binom{n-1}{2} + \ldots + \binom{m+1}{2}. \tag{2}$$

Proof. By (1) the value of $V_{n,n-1}$ is equal to $\binom{n}{2}$ and $V_{n,n-1} = P_{n,n-1}$. As we form in the mentioned non exhaustive procedure the partition $\mathscr{Y}_{n,n-1}$ for $n = n, n-1, \ldots$ $\ldots, m+1$, the formula (2) is valid.

Some useful recurent formulas follow from (2):

(3)
$$P_{n,m} = P_{n-1,m} + \binom{n}{2},$$

$$P_{n,m} = P_{n,m+1} + \binom{m+1}{2},$$

$$P_{n,m} = P_{n,k} + P_{k,m} \quad \text{for} \quad m < k < n\,.$$

Values of $P_{n,m}$ and $V_{n,m}$ for some n and all $m < n$ are shown in Appendix. Of course, we are not justified to compare directly the value of $P_{n,m}$ and $V_{n,m}$, but we may do so for the value of $P_{n,m}$ and $R_{n,m}$, where $R_{n,m}$ is defined by:

$$R_{n,m} = \sum_{k=1}^{n-m} V_{n,n-k}\,.$$

Theorem 3. *The following relations are true*:

$$R_{3,1} = P_{3,1}$$

and

$$R_{n,m} > P_{n,m} \quad \textit{for} \quad n > m+1\,, \quad n > 3\,.$$

Proof. For $n = 3$ we may calculate it directly, and then we prove it by means of mathematical induction. Let be $n > 3$, fixed. For the first step we take: $m_{\max} = n - 2$. Then

$$R_{n,n-2} - P_{n,n-2} = V_{n,n-1} + V_{n,n-2} - P_{n,n-1} - P_{n-1,n-2} =$$
$$= \binom{n}{3} + 3\binom{n}{4} - \binom{n-1}{2} > 0\,.$$

So for the first step the assertion is valid. With next steps m diminishes. We suppose therefore the validity of the assertion for $m = k$, and we prove it for $m = k - 1$.

It is $R_{n,k-1} = R_{n,k} + V_{n,k-1}$ and from (3):

$$R_{n,k-1} - P_{n,k-1} = R_{n,k} + V_{n,k-1} - P_{n,k} - \binom{k}{2}.$$

By assumption, it is $R_{n,k} - P_{n,k} > 0$, so that we must prove:

$$V_{n,k-1} - \binom{k}{2} \geqq 0\,.$$

It must be $m \geqq 1$, i.e. $k - 1 \geqq 1$, therefore $k \geqq 2$; for $k = 2$ is $V_{n,k-1} - \binom{k}{2} = 0$.
For $k > 2$ $V_{n,k-1}$ contains the member with $k_1 = 1$, $n_1 = k - 2$, $k_2 = n - (k - 2)$, $n_2 = 1$, which is:

$$\frac{n!}{(k-2)!\,[n-(k-2)]!} = \binom{n}{k-2}.$$

Since $n > m + 1$, so that $n > k - 1 + 1 = k$, hence $\binom{n}{k-2} \geqq \binom{k}{2}$ and $V_{n,k-1} \geqq \binom{k}{2}$, q.e.d.

II. NUMBERS OF DIFFERENT REDUCTIONS OF n-DIMENSIONAL VECTOR SPACES

We denote a vector space of n dimensions by X_n, so that:

$$X_n = Z_1 \times Z_2 \times \ldots \times Z_n\,,$$

i.e.

$$X_n = \bigcup_{\substack{z_1 \in Z_1,\\ \ldots\\ z_n \in Z_n}} \{(z_1, z_2, \ldots, z_n)\}\,.$$

Definition 2. Let be $m < n$. A *reduction* of an n-dimensional vector space X_n is a cylindric partition

$$\mathcal{Y}_m = \mathcal{Y}_m^{k_1,\ldots,k_{n-m}} = \{Y_1, \ldots, Y_r, \ldots\}$$

of the space X_n, when the following is valid:

$$Y_r = \mathop{\times}_{i=1}^{n} A_i\,,$$

where

$$A_i = \{z_i\} \quad \text{for} \quad i \neq k_j\,, \quad j = 1, 2, \ldots, n - m\,,$$

$$A_{k_j} = Z_{k_j} \quad \text{for} \quad j = 1, 2, \ldots, n - m\,.$$

Let $m < n$ be fixed. Let $W_{n,m}$ be the number of all cylindric partitions $\mathcal{Y}_m$ of the n-dimensional vector space X_n.

Theorem 4. *Let be* $m < n$. *Then*

$$W_{n,m} = \binom{n}{m}.$$

The value of $W_{n,m}$ is obviously equal to the number of all possible groups of $n - m$ coordinates which we reject from n coordinates, i.e. $\binom{n}{n - m} = \binom{n}{m}$.

Let $m < n$ be fixed. Let $Q_{n,m}$ be the number of all cylindric partitions, resulting from n-dimensional space X_n, when the non exhaustive procedure, mentioned above, is used.

Theorem 5. *Let be* $m < n$. *Then*

$$Q_{n,m} = \frac{n + m + 1}{2} (n - m). \tag{4}$$

The value of $W_{n,n-1}$ is equal to n and as we form in the mentioned non exhaustive procedure the partition $\mathscr{Y}_{n-1}$ for $n = n, n - 1, \ldots, m + 1$, the following equation holds:

$$Q_{n,m} = n + (n - 1) + \ldots + (m + 1),$$

it means, the formula (4) is valid.

Analogous recurent formulas, as for $P_{n,m}$, are valid also for $Q_{n,m}$. We mention the most useful one:

$$Q_{n,m} = Q_{n,m+1} + m + 1. \tag{5}$$

Let $m < n$ and let $S_{n,m}$ be defined by:

$$S_{n,m} = \sum_{k=1}^{n-m} W_{n,n-k}.$$

Then

$$S_{n,m-1} = S_{n,m} + \binom{n}{m - 1} \tag{6}$$

and

$$Q_{n,n-1} = W_{n,n-1} = S_{n,n-1}$$

immediately follow.

Theorem 6. *Let be* $n > 2$. *Then*

$$S_{n,m} > Q_{n,m} \quad \textit{for} \quad n > m + 1.$$

We prove it analogously to Theorem 3:

$$S_{n,n-2} - Q_{n,n-2} = \binom{n}{n-1} + \binom{n}{n-2} - n - (n-1) > 0 \quad \text{for} \quad n > 2\,.$$

From (6) and (5)

$$S_{n,k-1} - Q_{n,k-1} = S_{n,k} + \binom{n}{k-1} - Q_{n,k} - k =$$

$$= S_{n,k} - Q_{n,k} + \frac{n(n-1)\dots[n-(k-2)] - k(k-1)\dots 2.1}{(k-1)!} > 0$$

follows, because $n > m + 1$, i.e. $n > k + 1$.

APPENDIX

m	$P_{3,m}$	$V_{3,m}$	R_{3m}
2	3	3	3
1	4	1	4

m	$P_{4,m}$	$V_{4,m}$	$R_{4,m}$
3	6	6	6
2	9	7	13
1	10	1	14

m	$P_{5,m}$	$V_{5,m}$	$R_{5,m}$
4	10	10	10
3	16	25	35
2	19	15	50
1	20	1	51

m	$P_{6,m}$	$V_{6,m}$	$R_{6,m}$
5	15	15	15
4	25	65	80
3	31	90	170
2	34	31	201
1	35	1	202

m	$P_{7,m}$	$V_{7,m}$	$R_{7,m}$
6	21	21	21
5	36	140	161
4	46	350	511
3	52	301	812
2	55	63	875
1	56	1	876

m	$P_{8,m}$	$V_{8,m}$	$R_{8,m}$
7	28	28	28
6	49	266	294
5	64	1050	1344
4	74	1701	3045
3	80	966	4011
2	83	127	4138
1	84	1	4139

m	$P_{9,m}$	$V_{9,m}$	$R_{9,m}$
8	36	36	36
7	64	462	498
6	85	2646	3144
5	100	6951	10095
4	110	7770	17865
3	116	3025	20890
2	119	255	21145
1	120	1	21146

m	$P_{10,m}$	$V_{10,m}$	$R_{10,m}$
9	45	45	45
8	81	750	795
7	109	5880	6675
6	130	22827	29502
5	145	29925	59427
4	155	34105	93532
3	161	9330	102862
2	164	511	103373
1	165	1	103374

m	$Q_{3,m}$	$W_{3,m}$	$S_{3,m}$
2	3	3	3
1	5	3	6

m	$Q_{4,m}$	$W_{4,m}$	$S_{4,m}$
3	4	4	4
2	7	6	10
1	9	4	14

m	$Q_{5,m}$	$W_{5,m}$	$S_{5,m}$
4	5	5	5
3	9	10	15
2	12	10	25
1	14	5	30

m	$Q_{6,m}$	$W_{6,m}$	$S_{6,m}$
5	6	6	6
4	11	15	21
3	15	20	41
2	18	15	56
1	20	6	62

m	$Q_{7,m}$	$W_{7,m}$	$S_{7,m}$
6	7	7	7
5	13	21	28
4	18	35	63
3	22	35	98
2	25	21	119
1	27	7	126

m	$Q_{8,m}$	$W_{8,m}$	$S_{8,m}$
7	8	8	8
6	15	28	36
5	21	56	92
4	26	70	162
3	30	56	218
2	33	28	246
1	35	8	254

m	$Q_{9,m}$	$W_{9,m}$	$S_{9,m}$
8	9	9	9
7	17	36	45
6	24	84	129
5	30	126	255
4	35	126	381
3	39	84	465
2	42	36	501
1	44	9	510

m	$Q_{10,m}$	$W_{10,m}$	$S_{10,m}$
9	10	10	10
8	19	45	55
7	27	120	175
6	34	210	385
5	40	252	637
4	45	210	847
3	49	120	967
2	52	45	1012
1	54	10	1022

m	$Q_{20,m}$	$W_{20,m}$	$S_{20,m}$
19	20	20	20
18	39	190	210
17	57	1140	1350
16	74	4845	6195
15	90	15504	21699
14	105	38760	60459
13	119	77520	137979
12	132	125970	263949
11	144	167960	431909
10	155	184756	616665
9	165	167960	784625
8	174	125970	910595
7	182	77520	988115
6	189	38760	1026875
5	195	15504	1042379
4	200	4845	1047224
3	204	1140	1048364
2	207	190	1048554
1	209	20	1048574

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