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On Two Special Cases of the Optimum Decision Rule for the Radar Signal Processing

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In paper [1] the optimum decision rule for the radar signal processing was derived. That derivation was done under the assumptions of common covariance properties of the jamming random component of the radar signal. In the present paper we shall discuss the optimum decision rule in two technically important special cases of the jamming covariance properties. In the first case the jamming will consist only of the noise of the radar receiver (of the white Gaussian noise), in the second case the noise of the radar receiver will be negligible compared to clutter.

INTRODUCTION

Under the assumptions given in [1] the optimum decision rule is defined by the relation

$$Q \geqq \vartheta \tag{1}$$

where ϑ is the threshold of the decision rule (a previously chosen constant) and Q is the random variable that has arisen in the functional transformation of the tested section of the received signal (signal in the output of the radar receiver). This functional transformation is given by the expression:

$$Q = \sqrt{[\sum_{i=1}^{M}(A_i d_i - B_i e_i)]^2 + [\sum_{i=1}^{M}(A_i e_i + B_i d_i)]^2}\,. \tag{2}$$

In case (1) holds, we say that the radar signal contains echoes from an aircraft (hypothesis H_1), in the opposite case we say that the radar signal consists only of jamming (hypothesis H_0). The d_i, e_i in (2) are orthogonal components of the vector η_i of the received signal in the i-th moment. M is the number of elements of the finite random sequence of vectors $\{\eta_i\} = \eta_1, \eta_2, \ldots, \eta_M$, the information of which is used in decision.

The A_i, B_i are fixed constants (so-called weight coefficients), the values of which are given by the expressions:

(3)
$$A_i = \sum_{j=1}^{M} J_j c_{ji} \sin jF\,,$$
$$B_i = \sum_{j=1}^{M} J_j c_{ji} \cos jF$$

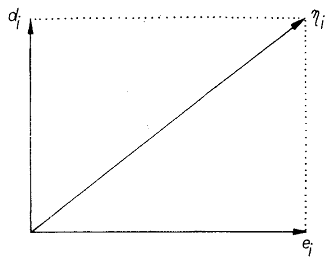

Fig. 1.

where $\{J_j\} = J_1, J_2, \ldots, J_M$ is the known sequence of modulus and F is the known increment of the phase of the useful signal vector (vector of echoes from the aircraft). Coefficients c_{ji} are elements of the inverse matrix **C** of the given $M \times M$ covariance

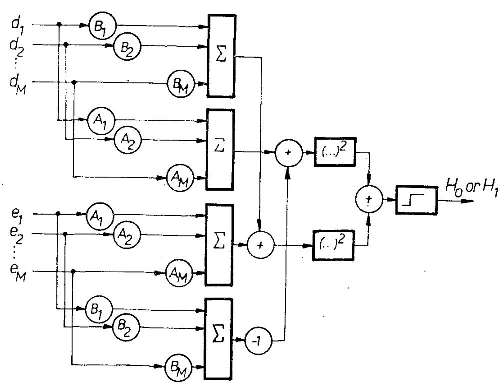

Fig. 2.

 matrix of jamming **R**, **C** = $\mathbf{R}^{-1}$. The block diagram of the optimum decision rule is shown in Fig. 2.

In [1] it is proved that the random variable Q has the Rayleigh-Rice probability density

$$w_Q(x \mid q) = \frac{x}{\sigma^2} \exp\left(\frac{-x^2 + qD^2}{2\sigma^2}\right) I_0\left(\frac{qDx}{\sigma^2}\right); \quad x > 0 \tag{4}$$

where q is some unknown quantity the value of which is 0 under H_0 (when the radar signal consists only of jamming) and 1 under H_1 (when the radar signal contains echoes from the aircraft). The following expressions are derived in [1] for parameters σ^2 and D of the probability density (4):

$$\sigma^2 = \sum_{j=1}^{M} [(\sum_{i=1}^{M} A_i K_{ij})^2 + (\sum_{i=1}^{M} B_i K_{ij})^2], \tag{5}$$

$$D = \sqrt{[\sum_{i=1}^{M} J_i(A_i \cos iF - B_i \sin iF)]^2 + [\sum_{i=1}^{M} J_i(A_i \sin iF + B_i \cos iF)]^2}; \tag{6}$$

K_{ij} in (5) are coordinate coefficients of the canonical expansion of the jamming covariance sequence.

The derivation of the optimum decision rule in [1] was based on the common assumptions of the properties of jamming and of the useful radar signal component. The expressions (1), (2) and (3) therefore define the optimum decision rule for the processing of the wide class of possible input signals, while the radar signal is only its sub-class. From the point of view of the technical applications we shall be interested above all in the properties of the optimum decision rule when used for the radar signal. That si why we shall further assume, in accordance with [2], the covariance sequence of jamming in the form:

$$\begin{aligned} \varrho_{|i-j|+1} &= \Lambda^2 \exp[-\Omega(i-j)^2]; \quad i \neq j, \\ &= 1 + \Lambda^2; \quad i = j, \end{aligned} \tag{7}$$

where Λ^2 is the ratio of clutter to the receiver noise and Ω is a positive constant, characterizing the spectrum width of clutter (Ω depends e.g. on the wind velocity).

In the present paper we shall derive the structure and properties of the optimum decision rule in the two technically significant special cases of radar jamming. First we shall discuss the case when jamming consists only of the receiver noise (white normal noise) which corresponds to $\Lambda = 0$ in (7). Further, we shall deal with a case where the receiver noise is negligible comparing to the clutter. This corresponds to the case in (7) when $\Lambda \gg 1$.

Let the jamming consist only of white noise, i.e. we shall assume the case $A = 0$ in (7). Then the covariance matrix of jamming $\mathbf{R}$ and its inverse matrix $\mathbf{C} = \mathbf{R}^{-1}$ are two $M \times M$ unit matrices. Therefore

(8) $$c_{ij} = 1\,; \quad i = j\,, \qquad = 0\,; \quad i \neq j\,.$$

The assumptions of covariance properties of jamming in optimum decision rule are contained only in expressions (3) for the weight coefficients. Substituing (8) into (3) we have

(9) $$A_i = J_i \sin iF\,, \qquad B_i = J_i \cos iF\,.$$

Denote

(10) $$d_i = O_i \cos \Psi_i\,, \qquad e_i = O_i \sin \Psi_i$$

where O_i is the envelope and Ψ_i is the phase of the received signal η_i. Let us substitute (9) and (10) into (2). After an arrangement we obtain

(11) $$Q = \sqrt{[\sum_{i=1}^{M} O_i J_i \sin (iF - \Psi_i)]^2 + [\sum_{i=1}^{M} O_i J_i \cos (iF - \Psi_i)]^2} \geqq \vartheta\,.$$

The expression (11) defines the optimum decision rule for the processing of the radar signal that does not contain a correlated jamming.

The random variable Q has the Rayleigh-Rice probability density (4). Parameters σ^2 and D of this probability density can be obtained by substituting (9) into (5) and (6), making use of the fact that the coordinate coefficients of the canonical expansion create the unit matrix in our case, i.e.

(12) $$K_{ij} = 1\,; \quad i = j\,, \qquad = 0\,; \quad i \neq j\,.$$

Then

(13) $$\sigma^2 = \sum_{i=1}^{M} J_i^2\,,$$

(14) $$D = \sum_{i=1}^{M} J_i^2\,.$$

The vector of signal received in the i-th moment η_i is created by the sum of the useful signal vector $\varkappa_i$ and the jamming vector ξ_i, see Fig. 3.

Random variable Q in (11) is the length of the vector Z that is the vector sum of the vectors $\bar{\eta}_i$. Vector $\bar{\eta}_i$ arises from the vector η_i by its rotation by the angle $(iF - 2\Psi_i)$ and by multiplying it by the coefficient J_i. Owing to the rotation all the vectors of the

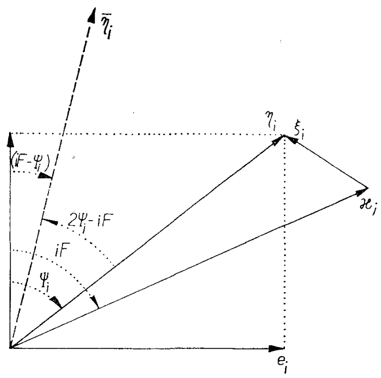

Fig. 3.

useful signal $\bar{\varkappa}_i$ are in phase. Thus we achieve that the power contributions of the useful signal $\varkappa_i$ are with respect to their weigths, i.e. D^2 in (14) is the weighted sum of powers of the signal received in the single moments i. In the special case when $J_i = \text{const.}$ for all $i \in \langle 1, M \rangle$, the decision rule (11) operates as the ideal coherent integrator.

In technical practice the opinion has been accepted that as for as the radar signal does not contain clutter, the phase information of the received signal is useless. Suitable decision rules are then looked for in the class of rules for the processing of the radar signal evelope.

The decision rule for the optimum processing of the radar signal envelope is given by the well-known expression [3]:

$$Q = \sum_{i=1}^{M} \ln I_0(J_i O_i) \geqq \vartheta \qquad (15)$$

where I_0 is the modified Bessel function of the first kind. The relations (11) and (15) are not equivalent. This implies that the ignoring of the phase information results in a worse performance of the decision process.

Further, we shall show that the rules (11) and (15) are equivalent when

$$J_i O_i \gg 1 \tag{16}$$

i.e. when the useful signal to noise ratio (S/N ratio) is high.

For $x \gg 1$ the $I_0(x)$ can be approximated by the term

$$I_0(x) \doteq \frac{\exp(x)}{\sqrt{(2\pi x)}}$$

and from here

$$\ln I_0(x) \doteq x - \tfrac{1}{2} \ln (2\pi x) \doteq x . \tag{17}$$

Substituting (17) into (16), we obtain

$$Q \doteq \sum_{i=1}^{M} J_i O_i \gtreqq \vartheta . \tag{18}$$

By this relation the optimum decision rule for the processing of the radar signal envelope is defined in case the S/N ratio is high.

When the S/N ratio is high, the influence of the noise on the received signal phase Ψ_i is evidently negligible. Therefore we can write

$$iF - \Psi_i = \text{const.} \tag{19}$$

Substituting (19) into (11), we obtain the expression for the common optimum decision rule that is valid when (16) holds and the jamming is not correlated

$$Q \doteq \sum_{i=1}^{M} J_i O_i \gtreqq \vartheta . \tag{20}$$

This relation is actually the same as (18).

2. THE CLUTTER PREDOMINATES IN JAMMING

Let the ratio of the clutter variance to the noise variance be so high that $\Lambda \gg 1$ in (7) is valid, i.e. the receiver noise is no practicaly use in the jamming. In this case (see Appendix) we can express the elements of matrix $\mathbf{C}$ approximately in the following manner:

$$\begin{aligned} c_{i1} &= k_i X , \\ c_{ij} &= c_{i1} k_j = k_i k_j X ; \quad X > 0 \end{aligned} \tag{21}$$

where X is a constant for the fixed Λ and Ω and k_i are coefficients given by the relation

(22) $$k_i = (-1)^{i-1} \binom{M-1}{i-1} = \frac{(-1)^{i-1}(M-1)!}{(i-1)!(M-i)!};$$

$\binom{M-1}{i-1}$ are binomial coefficients.

Let us substitute (22) into (3)

(23) $$A_i = k_i X \sum_{j=1}^{M} J_j k_j \sin jF = k_i \alpha ,$$

$$B_i = k_i X \sum_{j=1}^{M} J_j k_j \cos jF = k_i \beta .$$

α and β are evidently constants that are independent of i. Using (23) in (2) we obtain, after arrangement:

(24) $$Q = \sqrt{(\alpha^2 + \beta^2)} \cdot \sqrt{(\sum_{i=1}^{M} k_i d_i)^2 + (\sum_{i=1}^{M} k_i e_i)^2} .$$

The expression $\sqrt{(\alpha^2 + \beta^2)}$ does not contain components d_i, e_i of the received signal and therefore it is a constant. Let us denote

(25) $$G = \frac{Q}{\sqrt{(\alpha^2 + \beta^2)}} .$$

$$\bar{\vartheta} = \frac{\vartheta}{\sqrt{(\alpha^2 + \beta^2)}} .$$

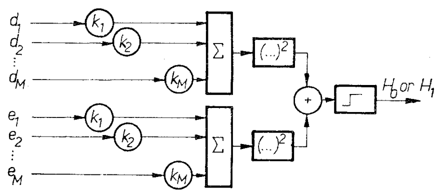

Fig. 4.

Then by the relation

(26) $$G = \sqrt{(\sum_{i=1}^{M} k_i d_i)^2 + (\sum_{i=1}^{M} k_i e_i)^2} \geqq \bar{\vartheta}$$

the optimum decision rule for the case $\Lambda \gg 1$ is defined. This rule will be referred to as the binomial decision rule because it requires the multiplying of the received signal values by the binomial coefficients. The block diagram of the binomial decision rule is shown in Fig. 4.

Using the relation (26) we can see an interesting and important property of the binomial decision rule. Only coefficients k_i and the constant M (the length of the rule) are the factors in the functional transformation of the random sequence of the received signal vectors $\{\eta_i\}$ to the random variable G. Under the assumption $\Lambda \gg 1$, the constant M is the only parameter of the above mentioned functional transformation that contains some assumptions about the signal

$$G = G(M) \tag{27}$$

contrary to the optimum decision rule (2), where

$$Q = Q(M, \{J_i\}, F, \Lambda, \Omega). \tag{28}$$

This implies, for the given M, that the random variable G of the binomial decision rule is constructed in the same way for all the combinations of values of the signal parameters $(\{J_i\}, F, \Lambda, \Omega)$, $\Lambda \gg 1$. An example of the construction of G from the sequence of the received signal vectors $\eta_i = [d_i, e_i]$ is shown in Fig. 5 for $M = 5$.

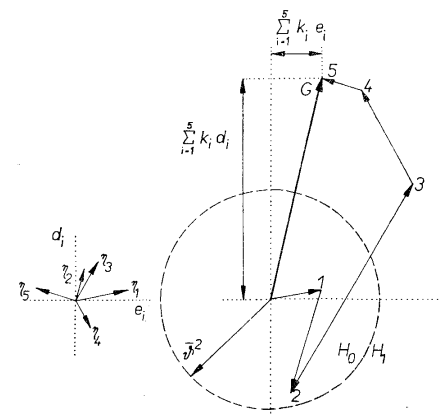

Fig. 5.

Now we determine the probability density $w_G(x \mid q)$ of the random variable G. Transformation (25) is of no influence on the type of the probability density function, thus G again has the Rayleigh-Rice probability density (4). We obtain parameters σ^2

and D of (4) substituting (23) into (5) and (6) and considering (25). After the arrangement we have

(29) $$\sigma^2 = \sum_{j=1}^{M} \left(\sum_{i=1}^{M} k_i K_{ij} \right)^2 ,$$

(30) $$D = \sqrt{\left(\sum_{i=1}^{M} k_i J_i \cos iF \right)^2 + \left(\sum_{i=1}^{M} k_i J_i \sin iF \right)^2} .$$

From the expression (30) it follows that for the given M and $\{J_i\}$

(31) $$D(F) = D(2\pi - F) .$$

Thus the function $D(F)$ is symmetric with respect to the value $F = \pi$.

CONCLUSION

The special cases of the optimum decision rule that have been discussed in this paper correspond to the two limit values of Λ in (7). The results that we have obtained for the first and the other special case have different significances.

The results, obtained for the case of the noise-jamming, are mainly of a theoretical importance. They show us the mechanism of common optimum processing and its relationship with the optimum processing of the envelope and with the ideal coherent integration. As the S/N ratio is usually high enough, the technically sufficient results can be reached already by using simpler methods, e.g. the optimum processing of the envelope and its approximations [5].

On the other hand, the results, obtained for the case when the clutter predominates in jamming, have also a considerable practical importance. Comparing the relations (2) and (26) or Fig. 2 and Fig. 4, it is obvious that the binomial decision rule has an essentially simpler structure than the common optimum decision rule and, moreover, the binomial decision rule has only the integer weight coefficients.

Thus, the necessary number of operations with the signal can be reduced by using the binomial decision rule. In [6] it is proved that decreasing the value of Λ, the performance of the decision process of the binomial decision rule (expressed e.g. by the type I and type II errors) does not get worse. This implies that if we choose the value of the threshold $\bar{\vartheta}$ so that under $\Lambda = \Lambda_1$ the necessary performance of decision process is reached then the performance does not get worse, if $\Lambda < \Lambda_1$. The results of the calculations of properties of the binomial decision rule in [6] for the typical range of radar signal behaviour show us that using the binomial decision rule technically very good results can be reached as far as the increment of the phase $F \in \in (\frac{2}{3}\pi + 2i\pi; \frac{4}{3}\pi + 2i\pi)$; $i = 0, \pm 1, \pm 2$, etc.

For illustration of the approximate binomial properties of the inverse matrix $\boldsymbol{C}$ under $\Lambda \to \infty$, the typical values of its elements c_{ij} are shown and the method of their calculation is described in this section.

The $M \times M$ standardized covariance matrix of jamming $\boldsymbol{R}$ with elements

$$r_{ij} = \frac{\varrho_{|i-j|+1}}{1 + \Lambda^2}$$

is ill conditioned matrix for high values of Λ. Using the usual methods for matrix inversion or using standard software programs, the calculation of the inverse matrix $\boldsymbol{C} = \boldsymbol{R}^{-1}$ is numerically unstable even when the order of the matrix is small, e.g. $M = 4$. For this reason the program INVER has been created and debugged. This program enables us to carry out the inversion of the standardized covariance matrix of jamming even when $\Lambda \to \infty$. Besides, this program enables us to invert the ill conditioned matrices even in the cases when the other methods fail.

The program INVER utilizes a combination of two methods, the method of inversion by succesive bordering of the matrix and the method of successive approximation of the inverse matrix elements. Both these methods are described in [4]. In order to limit the influence of the round off errors on the stability of the calculation, the program INVER utilizes some simulated arithmetic. This arithmetic (created as subprograms) operates on the basis of number recording to 144 significant decimal places (72 decimal places preceeding the decimal point and 72 decimal places following the decimal point) and enables us to carry out all arithmetic operations with an absolute error less than 10^{-72}.

Successive approximation of elements of the obtained inverse matrix is made by iteration procedure after each step of bordering. The deviation of the matrix $\boldsymbol{R}_n\boldsymbol{C}_n$ from the unit matrix $\boldsymbol{E}_n$

$$\Delta = \boldsymbol{E}_n - \boldsymbol{R}_n\boldsymbol{C}_n$$

is measured by the norm

$$\|\Delta\| = \max_j \sum_{i=1}^{n} |\delta_{ij}| ,$$

where δ_{ij} are elements of the matrix Δ and $n = 2, 3, \ldots, M$ is the order of the matrix corresponding to the $(n - 1)$-th step of the bordering.

The values of coefficients

$$b_{ij} = \frac{c_{ij}}{Y}$$

under $\Lambda \to \infty$, are given in Tables 1 and 2 (Y is a constant).

Table 1 is valid for $\Omega = 5{\cdot}27 \,.\, 10^{-4}$ which corresponds to the wind velocity $v = 32$ km/h. Table 2 is valid for $\Omega = 3{\cdot}2 \,.\, 10^{-6}$ which corresponds to $v = 0$ km/h. Table 3 gives the values of the coefficients $b_{ij} = \binom{M-1}{i-1}\binom{M-1}{j-1}$.

Table 1. ($Y = 1{\cdot}62 \,.\, 10^{23}$, $\|\Delta\| = 4{\cdot}1 \,.\, 10^{-71}$).

1·02					
−10·14	100·92				
45·45	−452·54	2027·09			
−120·83	1202·38	−5389·66	14331·56		
211·01	−2099·96	9414·01	−25035·31	43737·95	
−252·92	2517·61	−11287·28	30020·20	−52452·34	62909·65
210·79	−2098·18	9408·05	−25024·74	43728·72	
−120·58	1200·35	−5382·83	14319·46		
45·31	−451·13	2023·25			
−10·10	100·58				
1·01					

Table 2. ($Y = 2{\cdot}4324 \,.\, 10^{45}$, $\|\Delta\| = 2{\cdot}84 \,.\, 10^{-56}$).

1·0001					
−10·0007	100·003				
45·0018	−450·005	2024·97			
−120·0026	1199·990	−5399·83	14399·29		
210·0019	−2099·957	9449·59	−25198·49	44096·91	
−252·0007	2519·934	−11339·45	30238·05	−52916·08	63499·10
210·0006	−2099·947	9449·55	−25198·43	44096·85	
−120·0011	1199·978	−5399·79	14399·22		
45·0010	−449·998	2024·95			
−10·0004	100·001				
1·0001					

Table 3.

1					
10	100				
45	450	2025			
120	1200	5400	14400		
210	2100	9450	25200	44100	
252	2520	11340	30240	52920	63504
210	2100	9450	25200	44100	
120	1200	5400	14400		
45	450	2025			
10	100				
1					

For a better survey only those elements b_{ij} are given the indices of which $j \in \in \langle 1, \frac{1}{2}(M + 1)\rangle$; $i \in \langle j, M + 1 - j\rangle$. The other elements can be obtained using the symmetry of the inverse matrix with respect to both diagonals, i.e.

$$b_{ij} = b_{ji} = b_{M+1-j,M+1-i} = b_{M+1-i,M+1-j}\,.$$

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