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An Application of Logical-Probabilistic Expressions to the Realization of Stochastic Automata

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In this paper some methods are suggested for the realization of stochastic automata and probabilistic operators. These methods are based on the notion of logical-probabilistic expression. Knowledge of papers [4] and [5] is eesential for the understanding of the following considerations.

I. REALIZATION OF PROBABILISTIC OPERATORS

The notion of a probabilistic operator, as we shall use it in the following considerations, was introduced in paper [4] as follows: consider a triplet $\mathscr{A} = [A, \mathscr{P}_A, B]$, where $A = \{a_1, a_2, \ldots, a_k\}$ is the input alphabet, $B = \{b_1, b_2\}$ is the output alphabet and $\mathscr{P}_A = \{P_a\}_{a \in A}$ is a system of probabilities on B; we call this triplet the *probabilistic operator* with binary output. Analogously, the probabilistic operator with multiple valued output was defined ($B = \{b_1, \ldots, b_l\}$). The problem of the realization of these operators is understood here as a problem of how to find a probabilistic operator $\mathscr{A}' = [A', \mathscr{P}'_{A'}, B]$ with a previously given structure, that would be probabilistically equivalent to the original, i.e. such $\mathscr{A}'$, for which there exists a one-to-one mapping ψ of A on A' for which $P_a = P'_{\psi(a)}$, $a \in A$. Ke will now use logical-probabilistic expressions (LP-expressions). These expressions were introduced in the paper [4] as well. An LP-expression is a triplet $\Phi = [F, \mathscr{P}_F, \Omega_F]$, where: (1) F is a logical-probabilistic form (LP-form), i.e., a form of propositional calculus in which a new kind of unary logical connective is used ($\varphi_1, \varphi_2, \ldots$), (2) Ω_F is a finite space of random events; the values of associated functions of probabilistic connectives are determined by these random events, (3) $\mathscr{P}_F$ is a system of probabilities, $\mathscr{P}_F = \{P_\gamma\}_\gamma$, on Ω_F, where the values of parameter γ are given by the values of such subexpressions as F' for which $\varphi_i(F')$ is a subexpression of F (given a value on input variables). For further useful details see [4].

Probabilistic properties of an LP-expression Φ can be described by the characteristic vector $\boldsymbol{p}_\Phi$;

$$\boldsymbol{p}_\Phi = (P_\sigma(\Omega_1))_{\sigma \in \{0,1\}^m},$$

where $\boldsymbol{\sigma}$ are possible evaluations of the variables and Ω_1 is a subset of Ω_F such that $func_F(\sigma, \omega) = 1$ for $\omega \in \Omega_1$, where $func_F$ is the associated function of F (see [4]; Def. 8). Analogously the properties of $\mathscr{A}$ can be described by the vector $\boldsymbol{p}_{\mathscr{A}} = (P_a(b_2))_{a\in A}$. We shall now try to find a probabilistically equivalent operator in a form corresponding to a LP-expression to a given probabilistic operator. Further, this LP-expression should be in simple probabilistic disjunctive normal form (SPDNF, see [4]; Def. 12).

Theorem 1. Consider a probabilistic operator with binary output ($\mathscr{A} = [A, \mathscr{P}_A, B]$) There exists a natural number m, a mapping ψ of A to $\{0, 1\}^m$ and an LP-expression $\Phi = [F, \Omega_F, \mathscr{P}_F]$, being in SPDNF, such that for every $a \in A$ $P_a(b_2) = p_{\psi(a)}$, where $p_{\psi(a)}$ is a member of $\boldsymbol{p}_\Phi$.

The vector $\boldsymbol{p}_\Phi$, in which the members for which $\boldsymbol{\sigma} \notin \psi(A)$ are left out, we denote as $\boldsymbol{p}_{\Phi \restriction \psi(A)}$. Then we can write $\boldsymbol{p}_{\Phi \restriction \psi(A)} = \boldsymbol{p}_{\mathscr{A}}$.

Proof. Let $A = \{a_1, \ldots, a_k\}$, let m be the smallest natural number for which $k \leqq 2^m$. We construct ψ in the following way: let $\psi(a_i) = \boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is the binary form of the number $i - 1$. We put $\boldsymbol{p}_{\mathscr{A}} = (P_{a_1}(b_2), \ldots, P_{a_k}(b_2))^{\mathrm{T}}$ and $\boldsymbol{p}'_{\mathscr{A}} = (P_{a_1}(b_2), \ldots$ $\ldots, P_{a_k}(b_2), 0, \ldots, 0)^{\mathrm{T}}$ (dimension $\boldsymbol{p}'_{\mathscr{A}} = 2^m$). According to Theorem 4 from [4] we can construct an LP-expression $\Phi = [F, \Omega_F, \mathscr{P}_F]$ in SPDNF, such that $\boldsymbol{p}_\Phi = \boldsymbol{p}_{\mathscr{A}}$. There is

$$F \simeq \bigvee_{i=1}^{k} \varphi_i(x_1^{\varepsilon_1{}^i}) \,\&\, x_2^{\varepsilon_2{}^i} \,\&\, \ldots \,\&\, x_m^{\varepsilon_m{}^i}\,, \tag{1}$$

where

$$x_j^{\varepsilon_j{}^i} = \begin{cases} x_j & \text{if} \quad \varepsilon_j^i = 1\,, \\ \sim x_j & \text{if} \quad \varepsilon_j^i = 0\,, \quad \varepsilon^i = \psi(a_i) \end{cases}$$

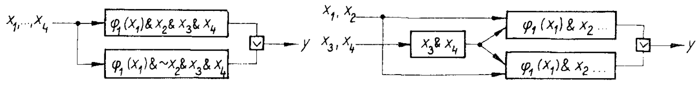
and the probabilistic parametres of φ_i are $(0, p_i)$, $p_i = P_{a_i}(b_2)$ $(P_i(0; 1) = 0, P_i(1; 1) = p_i)$; moreover, $\mathscr{P}_F = \{P(\gamma; \omega)\}_{\gamma\in\{0,1\}^k}$, where $P(\gamma; \omega) = \prod_{i=1}^{k} P_i(\gamma_i; \omega_i)$ for every $\gamma \in \{0, 1\}^k$ (the stochastical independence of Φ). □

For computation of the characteristic vector of Φ see also Example 3 from [5].

This LP-expression can be physically realized by a logical net with probabilistic elements (LP-net). The numbers of elements of the corresponding net (or connectives in the expression) will be denoted as $D(\&)$, $D(\vee)$, $D(\sim)$, $D(\varphi)$ respectively. For this realization according to the previous theorem we need $D(\&) = k(m - 1)$, $D(\vee) = k - 1$, $D(\sim) \leqq m2^{m-1}$ and $D(\varphi) = k$.

If we consider the numbers in binary forms, we need 2^{m-1} more zeros for numbers from 0 to $2^{m-1} - 1$, than for numbers from 2^{m-1} to $2^m - 1$. By the coding ψ to $\{0, 1\}^m$ in the same way as in the theorem, we need the first k numbers for the forms $\varphi_i(x_1^{\varepsilon_1{}^i}) \,\&\, \ldots \,\&\, x_m^{\varepsilon_m{}^i}$. The occurence of 0 corresponds to occurence of an element of negation. It is much more useful to code from the top down, i.e., $\psi(a_1) = (1, \ldots, 1)$,

$\psi(a_2) = (1, \ldots, 1, 0)$, $\psi(a_3) = (1, \ldots, 1, 0, 1)$ etc. By logical subexpression of an LP-expression we mean an LP-subexpression without any probabilisti connective. If we substitute the subnets corresponding to equivalent logical subexpressions by one subnet of the net corresponding to the form F of LP-expression (1) with the help of elements of forkjunction (see Fig. 1), then the number of used elements



a) b)

Fig. 1.

is considerably lower. By the substitution we do not change the corresponding LP-expressions, nor the computation of probabilities. The LP-expressions remains stochastically independent. Then $D(\&) \leqq 2^{m-1} - 4$ and the other numbers are without change. The preceding inequality follows from the fact that the number of distinct logical expressions of the form $x_{m-1}^{\varepsilon^i_{m-1}} \& x_m^{\varepsilon_m{}^i}$ occuring in expression (1) is less than $2^2 + 1$, and the number of distinct logical expressions of the form $x_{m-2}^{\varepsilon^i_{m-2}} \& \ldots \& x_m^{\varepsilon_m{}^i}$ is less than $2^3 + 1$ etc.; if we have such a subexpression of a given lenght, we need only one new conjunction to construct two new longer and distinct subexpressions. Then

$$D(\&) \leqq 2^2 + 2^3 + \ldots + 2^m = 2^{m+1} - 4 .$$

The equality in preceding inequalities occurs if and only if all members of $(p_1, \ldots, p_k)$ are different and positive. If $p_i = 0$ for some $i \in \{1, \ldots, k\}$, then we leave out the form $\varphi_i(x_1^{\varepsilon_1{}^i}) \& \ldots \& x_m^{\varepsilon_m{}^i}$.

Further simplifications with the help of the element of forkjunction are posible if for some $i, j, i \neq j$, we have $p_i = p_j$

Lemma 1. Consider a characteristic vector $p_{\mathscr{A}}$ and let there exist $i, j \in \{1, \ldots, k\}$, $i \neq j$, such that $p_i = p_j$. If we replace in LP-expression (1) $\varphi_i(x_1^{\varepsilon_1{}^i}) \& \ldots \& x_m^{\varepsilon_m{}^i}$ and $\varphi_j(x_1^{\varepsilon_1{}^j}) \& \ldots \& x_m^{\varepsilon_m{}^j}$ by subexpressions with the forms

$$F_i \simeq x_1^{\varepsilon_1{}^i} \& \ldots \& \varphi_i'(x_l^{\varepsilon_l{}^i}) \& \ldots \& x_m^{\varepsilon_m{}^i}$$

and

$$F_j \simeq x_1^{\varepsilon_1{}^i} \& \ldots \& \varphi_i'(x_l^{\varepsilon_l{}^j}) \& \ldots \& x_m^{\varepsilon_m{}^i} ,$$

where l is the indice for which $\varepsilon_l^i = \varepsilon_l^j$ and φ_i', φ_j' are functionally equivalent, then the equality $p_{\mathscr{A}} = p_{\Phi \setminus \psi(A)}$ is preserved.

Proof. We have

$$(2)\qquad P(func\ F_i(\sigma) = 1, func\ F_j(\sigma) = 1) = \\ = P_i(\sigma_l; 1)\, p^{11}_{\sigma_1,\ldots,\sigma_{l-1},1,\sigma_{l+1},\ldots,\sigma_m} + \\ + P_i(\sigma_l; 0)\, p^{11}_{\sigma_1,\ldots,\sigma_{l-1},0,\sigma_{l+1},\ldots,\sigma_m},$$

where

$$p^{11}_{\sigma_1,\ldots,\sigma_{l-1},\zeta,\sigma_{l+1},\ldots,\sigma_m} = \begin{cases} 1 & \text{if} \quad x_1^{\varepsilon_1{}^i} \& \ldots \& \zeta \& \ldots \& x_m^{\varepsilon_m{}^i} = 1 \\ & \text{and} \quad x_1^{\varepsilon_1{}^j} \& \ldots \& \zeta \& \ldots \& x_m^{\varepsilon_m{}^j} = 1 \\ & \text{for} \quad \boldsymbol{x} = \boldsymbol{\sigma}, \\ 0 & \text{in other cases}, \end{cases}$$

because $P_{i,j}(\sigma_l, \sigma_l; 1, 0) = 0$.

Further the right hand side of (2) is equal to 0, because $p^{11}_{\sigma_1,\ldots,\zeta,\ldots,\sigma_m} = 0$ for every $\boldsymbol{\sigma}$ and ζ, if we consider the form of F_i and F_j. Thus

$$P(func\ F_i(\sigma) = 1 \cup func\ F_j(\sigma) = 1) = \\ = P(func\ F_i(\sigma) = 1) + P(func\ F_j(\sigma) = 1) = \begin{cases} p_i & \text{if} \quad \boldsymbol{\sigma} = \boldsymbol{\varepsilon}^i \quad \text{or} \quad \boldsymbol{\sigma} = \boldsymbol{\varepsilon}^j, \\ 0 & \text{in other cases.} \end{cases}$$

The stochastic independence and distinctness are preserved with respect to other conjunctive members from the form (1). □

In the corresponding net we can then replace the subnets corresponding to F_i and F_j by a single net with an element of forkjunction. In more complicated cases such as $p_i = p_{i_j}$, $i \notin \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, k\}$, we proceed in the analogous way.

For other method see [1], [2], [3], [6], [8], [11]. The methods from [8] and [11] for binary outputs are similar to the method suggested here.

We now give our attention to the realization of the probabilistic operators with multiple values output. Then we have to consider characteristic matrices of the form $(P_a(b))^{a\in A}_{b\in B}$, i.e., characteristic matrices of such an operator.

Analogously, we can consider characteristic matrices of the vectors of LP-expressions, i.e., matrices of the form

$$(P(func_{F_1}(\sigma, \omega) = \zeta_1, \ldots, func_{F_k}(\sigma, \omega) = \zeta_k)^{\sigma\in\{0,1\}^m}_{\zeta\in\{0,1\}^k}$$

(see [4]; Def. 14).

Theorem 2. Consider a stochastic matrix $P = (p_{ij})^{i=1,\ldots,2^m}_{j=1,\ldots,2^n}$. Then a vector of LP-expressions $\boldsymbol{\Phi} = [\boldsymbol{F}, \Omega_{\boldsymbol{F}}; \mathscr{P}_{\boldsymbol{F}}]$, where $\boldsymbol{F} = (F_1(x_1, \ldots, x_m), \ldots, F_k(x_1, \ldots, x_m))$, exists such that $\boldsymbol{P}_{\boldsymbol{\Phi}} = \boldsymbol{P}$.

Proof. We define logical expressions

$$G_1(y_1, \ldots, y_{l-1}), \ldots, G_n(y_1, \ldots, y_{l-1}), \quad 1 = 2^m,$$

in such a way that their associated functions are given by the table:

$y = (y_1, \ldots, y_{l-1})$	$G_1, \ldots G_n$
$1, \ldots, 1$	$1, \ldots, 1 = \zeta_1$
γ_2	ζ_2
$\vdots$	$\vdots$
γ_{l-1}	ζ_{l-1}
$\left.\begin{matrix}\gamma_l \\ \vdots \\ \gamma_2^{l-1}\end{matrix}\right\}$	ζ_l

where each $\gamma_2, \ldots, \gamma_{l-1}$ contains only one member equal to 0, and $\gamma_2, \ldots, \gamma_{l-1}$ are different. The expressions are defined so that

$$P((G_1(y), \ldots, G_n(y)) = \zeta_1) \quad = P(y = (1, \ldots, 1)),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$P((G_1(y), \ldots, G_n(y)) = \zeta_{l-1}) = P(y = \gamma_{l-1}),$$

$$P((G_1(y), \ldots, G_n(y)) = \zeta_l) \quad = 1 - \sum_{i=1}^{l} P(y) = \gamma_i).$$

According to Theorem 1 we can find LP-expressions $\Phi_1, \ldots, \Phi_{l-1}$, stochastically independent, such that (if σ_j is the binary form of $j - 1$ and P'_r is the probability determined by $\mathscr{P}_r = \{P_{r,\gamma_r}\}$) $P'_r(\sigma_j; \Omega^r_1)$ is equal to p^r_j for $r = 1, \ldots, l - 1, j = 1, \ldots \ldots, 2^m$. We considered the LP-expressions $\Phi_r = [F_r, \Omega_r, \mathscr{P}_r]$ and $P(\sigma; \omega^*_1, \ldots, \omega^*_r) = \prod_{r=1}^{l-1} P'_r(\sigma; \omega^*_r)$. We now define a vector of LP-expressions given by a vector of LP-forms $F = (F'_1(x), \ldots, F'_n(x))$, where $F'_j = G_j(F^j_1, \ldots, F^j_{l-1})$ and F^j_r are functionally equivalent to F_r for $r = 1, \ldots, l - 1, j = 1, \ldots, n$.

If p^r_i, $i = 1, \ldots, 2^m$, are solutions of the equations

$$\text{(3)} \qquad \varepsilon_1\varepsilon_2 \ldots \varepsilon_{l-1} = p_{i1},$$

$$\varepsilon_{j_1}\varepsilon_{j_2} \ldots \varepsilon_{j_{l-2}} - p_{i1} = p_{i1}, \quad j = 2, \ldots, l - 1,$$

where $j_1, \ldots, j_{l-2}$ are those indices for which

$$\gamma_j^{j_r} = 1 \quad (\gamma_j = (\gamma_j^1, \ldots, \gamma_j^{l-1})),$$

then the above mentioned vector of LP-expressions fullfils the assertion of the theorem. This is because (considering the definition of LP-expressions Φ'_j by forms F'_j) we have

$$\text{(4)} \qquad p_{\sigma_i, \zeta_j} = \prod_{r=1}^{l-1} P'_r(\sigma_i; \Omega^r_{j,j_r})$$

$$(\zeta_j = (\zeta_j^1, \ldots, \zeta_j^n)),$$

where p_{σ_i, ζ_j} is a member of the characteristic matrix of $\boldsymbol{\Phi}$, and the right hand side of (4) is equal to $p_i^1 \dots p_i^{l-1} = p_{i1}$ for $j = 1$, to $p_i^1 \dots p_i^{l-2}(1 - p_i^{(l-1)}) = p_{i2}$ for $j = 2$, etc.

For $j = l$ we obtain the value of probability as a complement. □

If we have a probabilistic operator with matrix $P_{\mathscr{A}} = (p_{ij})$ of the type $k \times l$, we find the smallest natural number n for which $l \leqq 2^n$ and a one-to-one mapping ϑ of B to $\{0, 1\}^n$. Then we find a vector of LP-expressions such that $p_{ij} = p_{\psi(a_i)\vartheta(b_j)}$, where $p_{\psi(a_i)\vartheta(b_j)}$ is a member from $\boldsymbol{P}_{\Phi}$.

Example 1. Consider a probabilistic operator with the characteristic matrix

$$P_{\mathscr{A}} = \begin{pmatrix} p_{11}, \dots, p_{15} \\ \vdots \\ p_{31}, \dots, p_{35} \end{pmatrix}; \quad k = 3, l = 5 .$$

Then $n = 3$, $m = 2$, $l - 1 = 4$ and

y	G_1	G_2	G_3
1111	1	1	1
1110	1	0	0
1101	0	1	0
1011	0	0	1
0111	0	0	0

Then $G_1(y) = y_1 \,\&\, y_2 \,\&\, y_3$, $G_2(y) = y_1 \,\&\, y_3 \,\&\, y_4$ and $G_3(y) = y_1 \,\&\, y_2 \,\&\, y_4$. To obtain G_1, G_2 and G_3 in a simple form we construct ϑ in another way: $\vartheta(b_1) = (1, 1, 1)$, $\vartheta(b_2) = (1, 0, 0)$, etc.; we construct ψ in the usual way. For $x(a_i)$ we obtain equations

$$\begin{aligned} \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 &= p_{i1} , \\ \varepsilon_1\varepsilon_2\varepsilon_3(1 - \varepsilon_4) &= p_{i2} , \\ \varepsilon_1\varepsilon_2(1 - \varepsilon_3)\,\varepsilon_4 &= p_{i3} , \\ \varepsilon_1(1 - \varepsilon_2)\,\varepsilon_3\varepsilon_4 &= p_{i4} . \end{aligned}$$

Then

$$\begin{aligned} \varepsilon_1 &= (p_{i1} + p_{i2})(p_{i1} + p_{i3})(p_{i1} + p_{i4})/p_{i4}^2 , \\ \varepsilon_2 &= (p_{i1} + p_{i4})^{-1}\, p_{i1} , \\ \varepsilon_3 &= (p_{i1} + p_{i3})^{-1}\, p_{i1} , \\ \varepsilon_4 &= (p_{i1} + p_{i2})^{-1}\, p_{i1} . \end{aligned}$$

For $i = 1, \ldots, k$ we obtain $p_i^1, \ldots, p_i^{l-1}$ and therefore the characteristic vectors of $\Phi_1, \ldots, \Phi_l$ $p_{\Phi_j} = (p_1^j, \ldots, p_k^j)^T$. The desired vector of LP-expressions has the following vector of LP-forms:

$$F(x) = \begin{pmatrix} F_1(x) \,\&\, F_2(x) \,\&\, F_3(x) \\ F_1(x) \,\&\, F_3(x) \,\&\, F_4(x) \\ F_1(x) \,\&\, F_2(x) \,\&\, F_4(x) \end{pmatrix}.$$

For the construction of the corresponding net it is possible to use elements of fork-junction: (1) In the nets corresponding to $\Phi_1, \ldots, \Phi_{l-1}$ we can use the same method as in the case of a net with binary output (see above). (2) In the nets corresponding to $\Phi_1, \ldots, \Phi_{l-1}$ we can use common subnets corresponding to subexpressions $x_2^{\varepsilon_2{}^j} \,\&\, \ldots \,\&\, x_m^{\varepsilon_m{}^j}$ which in all $F_1, \ldots, F_{l-1}$ are identical. (3) The vector of LP-expressions was constructed so that the subexpressions substituted in distinct expressions $G_1, \ldots, G_n$ in the places of $y_1, \ldots, y_{l-1}$ are structurally equivalent (therefore functionally equivalent). Then we need only one $(l - 1)$-tuple of the nets corresponding to $\Phi_1, \ldots, \Phi_{l-1}$.

Example 2. Let $k = 6$, $l = 5$, then $m = 3$, $n = 3$ and

$$F_j = \bigvee_{i=1}^{6} \varphi_{ij}(x_1^{\varepsilon_1{}^i}) \,\&\, G_i'(x_2, x_3),$$

where

$$G_i'(x_2, x_3) = x_2^{\varepsilon_2{}^i} \,\&\, x_3^{\varepsilon_3{}^i}.$$

This example is continuation of Example 1, only the number of inputs is changed. The corresponding net is drawn in Fig. 2.

Let us now consider the numbers of used elements. Consider a probabilistic operator with l-valued output and k-valued input. The realizing net will consist: (1) $l - 1$ nets corresponding to $\Phi_1, \ldots, \Phi_{l-j}$, (2) n nets corresponding to $G_1, \ldots, G_n$.

(1) The common subnets for $\Phi_1, \ldots, \Phi_{l-1}$ have the number of elements corresponding to & equal to $(m - 2)\,k$ and every Φ_j has in addition k elements corresponding to &. Then $D_{(1)}(\&) \leqq (m - 2)\,k + (l - 1)\,k = k(m - 3)$. For every net the number of elements $\vee$ is equal to $k - 1$, then $D_{(1)}(\vee) \leqq (l - 1)(k - 1)$. We use the negationes only on the input variables, then $D_{(1)}(\sim) = m$.

(2) We know that it is possible to express every boolean function G in the form $\bigvee_{G(\sigma)=1} y_1^{\sigma_1} \,\&\, \ldots \,\&\, y_{l-1}^{\sigma_{l-1}}$. Therefore for the given table of values of the function we can construct a logical expression. With the help of minimalization (see, e.g., [7]) we obtain

$$G_1(y) = (y_1 \,\&\, y_2 \,\&\, y_3 \,\&\, y_4) \,\&\, ((y_5 \,\&\, y_6) \vee y_7),$$

$$G_2(y) = (y_1 \,\&\, y_2 \,\&\, y_5 \,\&\, y_6) \,\&\, ((y_3 \,\&\, y_4) \vee y_7),$$

$$G_3(y) = (y_1 \,\&\, y_3 \,\&\, y_5 \,\&\, y_6) \,\&\, ((y_2 \,\&\, y_4) \vee y_7),$$

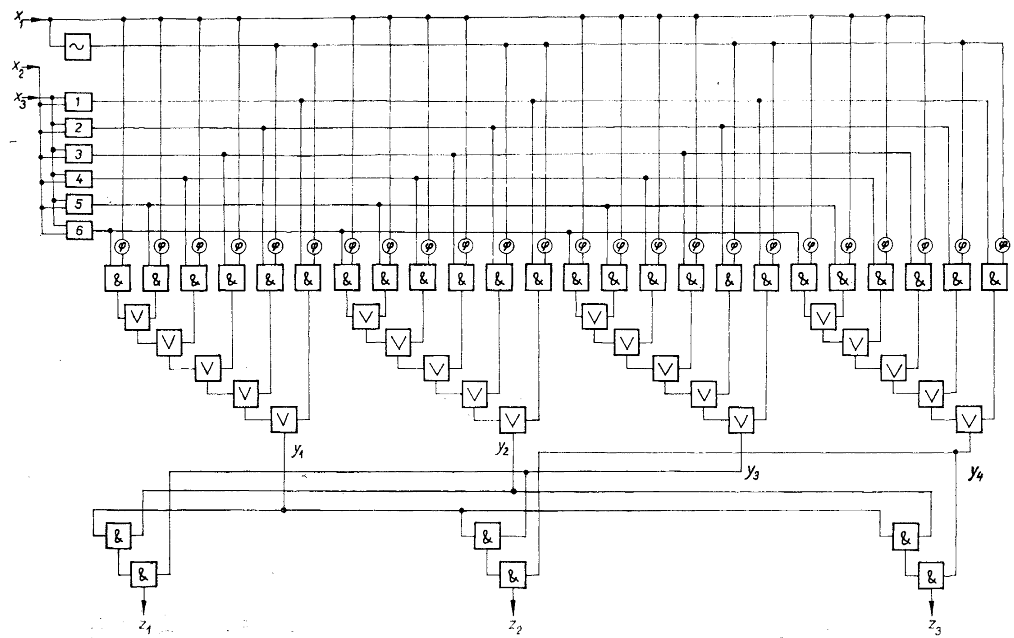


Fig. 2. (1, …, 6 — nets realizing expressions $G'_1, \ldots, G'_6$).

if the table is e.g.,

y	G_1	G_2	G_3	
1111111	1	1	1	$\vartheta(b_1)$
1111110	1	1	0	$\vartheta(b_2)$
1111101	1	0	1	$\vartheta(b_3)$
1111011	1	0	0	$\vartheta(b_4)$
1110111	0	1	1	$\vartheta(b_5)$
1101111	0	1	0	$\vartheta(b_6)$
1011111	0	0	1	$\vartheta(b_7)$
0111111	0	0	0	$\vartheta(b_8)$

(the usual coding for A, B, $n = 4$). Minimalization is possible because we choose the value of functions equal to 0 for y equals to (0111111) and any other vector with two or more zeros.

Analogously if n is a higher number. Then: (a) all negationes are omitted, (b) every G_j contains 2^{n-1} & connectives and less then 2^{n-2} conjuctive members (each with $2^{n-1} - 3$) variables) and each with $2^{n-1} - 4$ connectives. Then for G_j we have

$$D_j(\&) \leqq 2^{n-1} + 2^{n-1} - 3 + (2^{n-1} - 4)(2^{n-2} - 2).$$

Without minimalization it would be $D_j(\&) \leqq {}^{2n-1} - 2^n$. Analogously we obtain $D_j(\vee) \leqq 2^{n-1} - 1$

Conclusion 1. The realizing net for a probabilistic operator with k-valued input and l-valued output needs the following numbers of elements:

$$D(\&) \leqq k(m + l - 3) + n(2^{n-1} - 2^n + 2^{2n-3} + 1),$$
$$D(\vee) \leqq (l - 1)(k - 1) + n(2^{n-2} - 1),$$
$$D(\sim) \leqq m,$$
$$D(\varphi) \leqq k(l - 1).$$

In particular cases, some simplifications are possible.

Theorem 3. Consider a stochastic matrix $\boldsymbol{P}$ of the type $k \times l$. Suppose that there exists vectors $p^1, \ldots, p^{l-1}$, $p^i = (p_1^i, \ldots, p_k^i)^{\mathrm{T}}$, such that

$$p_{ij} = \prod_{r, B(r,j) = 1} p_j^r \prod_{r, B(r,j) = 0} (1 - p_j^r),$$

where $B(r, j)$ is the r-th member of binary form of $j - 1$.

Then we can find a vector of LP-expressions $\boldsymbol{\Phi} = [\boldsymbol{F}, \Omega_{\boldsymbol{F}}, \mathscr{P}_{\boldsymbol{F}}]$ such that $\boldsymbol{P} = \boldsymbol{P}_{\boldsymbol{\Phi}}$

and if $\boldsymbol{F} = (F_1, \ldots, F_n)$ $(2^n = l)$, then

$$p_{ij} \prod_{r=1}^{n} P'_r(\sigma_i, \Omega^r_{\zeta_j r}) \quad \text{for every} \quad i = 1, \ldots, k,\ j = 1, \ldots, l$$

$(\zeta_j = (\zeta_j^1, \ldots, \zeta_j^n)$ is binary form of $j - 1$, $\boldsymbol{\sigma}_i$ is binary form of $i - 1)$.
In the realization using this theorem we have the following

$$D(\&) \leqq k(m + n - 2),$$
$$D(\vee) \leqq n(k - 1),$$
$$D(\sim) \leqq m,$$
$$D(\varphi) \leqq k(n - 1).$$

Consider now the realization of the stochastic automata. Given Mealy stochastic automaton, $A = \{a_1, \ldots, a_k\}$, $B = \{b_1, \ldots, b_l\}$ $Q = (q_i, \ldots, q_s)$, and transition matrices

$$P_1 = (p^1_{i,p,j}), \quad P_2 = (p^2_{i,p,q}),$$
$$i = 1, \ldots, k; \; p = 1, \ldots, s; \; j = 1, \ldots, l; \; q = 1, \ldots, s,$$

let m, n, r be the smallest natural numbers for which $k \leqq 2^m$, $l \leqq 2^n$, $s \leqq 2^r$; we can define three one-to-one mappings $\psi: A \to \{0, 1\}^m$, $\vartheta: B \to \{0, 1\}^n$ and $\tau: Q \to \{0, 1\}^r$ and find a vector of LP-expressions $\boldsymbol{\Phi} = [\boldsymbol{F}, \Omega_{\boldsymbol{F}}, \mathscr{P}_{\boldsymbol{F}}]$ such that

(i) $$\boldsymbol{F} = (\boldsymbol{F}_1(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{F}_2(\boldsymbol{x}, \boldsymbol{y})),$$

where

$$\boldsymbol{F}_1(\boldsymbol{x}, \boldsymbol{y}) = (F_1(\boldsymbol{x}, \boldsymbol{y}), \ldots, F_n(\boldsymbol{x}, \boldsymbol{y})), \quad \boldsymbol{F}_2(\boldsymbol{x}, \boldsymbol{y}) = (G_1(\boldsymbol{x}, \boldsymbol{y}), \ldots, G_r(\boldsymbol{x}, \boldsymbol{y})),$$

(ii) $$p^1_{i,p,j} = p^{\vartheta(b_j)}_{\psi(a_i)\tau(q_p)}, \quad p^2_{i,p,q} = p^{\tau(q_q)}_{\psi(a_i)\tau(q_p)},$$

where $p^{\vartheta}_{\psi,\tau}$ and $p^{\tau}_{\psi,\tau}$ are members of the characteristic matrices of $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$, and

(iii) $$P_{\boldsymbol{\Phi}}(\gamma; \omega) = P_{\boldsymbol{\Phi}1}(\gamma_1; \omega_1) \,.\, P_{\boldsymbol{\Phi}2}(\gamma_2; \omega_2).$$

For the nets realizing $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ we can use common subnets realizing subexpressions like

(5) $$x_2^{\varepsilon_2{}^i} \,\&\, \ldots \,\&\, x_m^{\varepsilon_m{}^i} \,\&\, y_2^{\varepsilon^i{}_{m+2}} \,\&\, \ldots \,\&\, y_r^{\varepsilon^i{}_{m+r}}$$

without violating the condition of stochastical independence.

For $\boldsymbol{\Phi}_1$ we then obtain the following numbers of used elements

$$D_1(\&) \leqq ks(m + r + l - 3) + n(2^{n-1} - 2^n + 2^{2n-3} + 1),$$
$$D_1(\vee) \leqq (ks - 1)(l - 1) + n(2^{n-2} - 1),$$
$$D_1(\sim) \leqq m + r, \quad D_1(\varphi) \leqq ks(l - 1),$$

for Φ_2 we obtain

$$D_2(\&) \leqq ks(m + r + s - 3) + r(2^{r-1} - 2^r + 2^{r-3} + 1),$$
$$D_2(\vee) \leqq (ks - 1)(s - 1) + r(2^{r-2} - 1),$$
$$D_2(\sim) \leqq m + r, \quad D_2(\varphi) \leqq ks(s - 1).$$

It is not possible to make the number of probabilistic connectives needed for realization lower without further conditions concerning some probabilistic characteristic of the automaton; the number corresponds to the number of independent stochastic parametres. This fact is also true for other estimates of the number of probabilistic connectives in Part I of this paper.

Conclusion 2. If we consider the common subnets for the nets corresponding to (5) then we have:

$$D(\&) \leqq ks((\max(m, r) + l + s - 3)$$
$$+ n(2^{n-1} - 2^n + 2^{2n-3} + 1)$$
$$+ r(2^{r-1} - 2^r + 2^{2r-3} + 1),$$
$$D(\vee) \leqq (ks - 1)(l + s - 2) + n(2^{n-2} - 1) + r(2^{r-2} - 1),$$
$$D(\sim) \leqq m + r, \quad D(\varphi) \leqq (ks - 1)(l + s - 2).$$

For a general stochastic automaton we cannot separate nets for outputs and states. We have the four-dimensional array

$$\boldsymbol{P} = (p_{i,p,j,q}),$$
$$i = 1, \ldots, k; \; p = 1, \ldots, s; \; j = 1, \ldots, l; \; q = 1, \ldots, s.$$

We must define one-to-one mappings ψ, ϑ, ω as in the preceding case and we can find a vector of LP-expressions $\boldsymbol{\Phi}$, where $\boldsymbol{F} = (\boldsymbol{F}_1, \boldsymbol{F}_2)$, $\boldsymbol{F}_1 = (F_1(\boldsymbol{x}, \boldsymbol{y}), \ldots, F_n(\boldsymbol{x}, \boldsymbol{y}))$,

$$\boldsymbol{F}_2 = (G_1(\boldsymbol{x}, \boldsymbol{y}), \ldots, G_r(\boldsymbol{x}, \boldsymbol{y}),$$

such that

$$p_{i,p,j,q} = p_{\psi(a_i),\tau(q_p)}^{\vartheta(b_j),\tau(q_q)}.$$

We can find this vector in the following way: we construct new alphabets

$$A' = \psi(A) \times \tau(Q), \quad A' \subseteq \{0, 1\}^{m+r}, \quad a'_{i,p} = (\psi(a_i), \tau(q_p)),$$

and

$$B' = \vartheta(B) \times \tau(Q), \quad B' \subseteq \{0, 1\}^{n+r}, \quad b'_{j,q} = (\vartheta(b_j), \tau(q_q)),$$

and we obtain then a matrix $\boldsymbol{P}' = (p_{i,p}^{j,q})$ where $p_{i,p}^{j,q} = p_{i,p,j,q}$. According to Theorem 2 we can find a vector of LP-expressions $\boldsymbol{\Phi}$ such that

$$\boldsymbol{P}' = \boldsymbol{P}_{\boldsymbol{\Phi} \upharpoonright A',B'} .$$

Conclusion 3. We obtain the following estimates of the numbers of used elements:

$$D(\&) \leqq ks(m + r - ls - 3) + (n + r)(2^{n+r-2} - 2^{n+r} + 2^{2(n+r)-3} + 1),$$

$$D(\vee) \leqq (sl - 1)(sk - 1) + (n + r)(2^{n+r-2} + 1),$$

$$D(\sim) \leqq m + r \quad \text{and} \quad D(\varphi) < sk(sl - 1).$$

We have proved that, if we have the partition of the set of all probabilistic operators (or stochastic automata) according to probabilistic equivalence, then it is possible to find a representative of every class from the set of vectors of LP-expressions formed with help of &, $\vee$, $\sim$ and stochastically independent probabilistic connectives (in the case of stochastic automata moreover, we need to use delay elements).

II. REALIZATION OF PROBABILISTIC CONNECTIVES

In the previous part of this paper we assumed the probabilistic connectives (realizing elements) with all possible probabilistic parametres. We will now turn our attention to the case when we have only independent probabilistic connectives with given and equal probabilistic parametres.

The expression of the form $\varphi_i(x_j)$ will now be called the *elementary LP-expression.*

Theorem 4. Consider an elementary LP-expression $\varphi_i(x)$, $\boldsymbol{p}_{\varphi_i} = (0, p_i)$.

We can find (in a finite number of steps) a stochastically independent LP-expression Φ (in SPDNF) containing only probabilistic connectives φ with probabilistic parametres (0, 0·1) such that $P_0(\Omega_1) = 0$ and $|P_1(\Omega_1) - p| \leqq \varepsilon_0$, where ε_0 is a given real number, $\varepsilon_0 > 0$.

Proof. Denote

$$\varphi^{(b)}(x) = \varphi_1(x) \,\&\, \dots \,\&\, \varphi_b(x),$$

$$\bigvee^b \varphi(x) = \varphi_1(x) \vee \dots \vee \varphi_b(x),$$

where $\varphi_1, \dots, \varphi_b$ have equal probabilistic parametres and are stochastically independent. We try to find Φ of the form

$$F(x) \simeq \bigvee^{b_1} \varphi(x) \vee \bigvee^{b_2} \varphi^{(2)}(x) \vee \bigvee^{b_3} \varphi^{(3)}(x) \vee \dots$$

We proceed in the following way:

1) b_1: we compute succesivelly $P(\bigvee^b \varphi(1) = 1)$, $b = 0, 1, \dots$ until we find b_1 such that $P(\bigvee^{b_1} \varphi(1) = 1) \leqq p < P(\bigvee^{b_1+1} \varphi(1) = 1)$,

2) b_2: we compute succesively $P(\bigvee^{b_1} \varphi(1) \vee \bigvee^{b} \varphi^{(2)}(1) = 1)$ for $b = 0, 1, \ldots$ until we find b_2 such that

$$P(\bigvee^{b_1} \varphi(1) \vee \bigvee^{b_2} \varphi^{(2)}(1) = 1) \leqq p < P(\bigvee^{b_1} \varphi(1) \vee \bigvee^{b_2+1} \varphi^{(2)}(1) = 1).$$

Analogously we find $b_3, b_4, \ldots$ until we find k such that $|P(\bigvee^{b_1} \varphi(1) \vee \ldots \vee \bigvee^{b_k} . \varphi^{(k)}(1) = 1) - p| \leqq \varepsilon_0$.

We must prove that such a k exist (that there is a finite number of steps). We have

$$P(\bigvee^{b_1} \varphi(1) \vee \ldots \vee \bigvee^{b_l} \varphi^{(l)}(1) = 1) < P(\bigvee^{b_1} \varphi(1) \vee \ldots \vee \bigvee^{b_l+1} \varphi^{(l)}(1) = 1),$$

because $P(\varphi^{(l)}(1) = 1) = 0{\cdot}1 > 0$ and $\varphi^{(l)}_{b_l+1}(1) = 1$ do not imply $\bigvee^{b_1} \varphi(1) \vee \ldots \vee \bigvee^{b_l} \varphi^{(l)}(1) = 1$.

1) For the first step we have

$$P(\bigvee^{a+1} \varphi(1) = 1) = P(\bigvee^{a} \varphi(1) = 1) + 10^{-1} - 10^{-1} P(\bigvee^{a} \varphi(1) = 1). \tag{6}$$

Remark that $P(\bigvee^{a} \varphi(1) = 1) \leqq p$ (if b_1 is not found yet, i.e., $a \leqq b_1$) and thus the right hand side of (6) is greather or equal to $P(\bigvee^{a} \varphi(1) = 1) + 10^{-1} + 10^{-1}p$. Then $P(\bigvee^{a+1} \varphi(1) = 1) - P(\bigvee^{a} \varphi(1) = 1) \geqq 10^{-1}(1 - p)$, and after the finite number of steps p must be surpassed.

2) We have, analogously, for the l-th step

(7)

$$P(\bigvee^{b_1} \varphi(1) \vee \ldots \vee \bigvee^{a+1} \varphi^{(l)}(1) = 1) - P(\bigvee^{b_1} \varphi(1) \vee \ldots \vee \bigvee^{a} \varphi^{(l)}(1) = 1) \geqq$$
$$\geqq 10^{-l}(1 - p).$$

3) We have $P(\bigvee^{b_1} \varphi(1) \vee \ldots \vee \bigvee^{b_l+1} \varphi^{(l)}(1) = 1) - P(\bigvee^{b_1} \varphi(1) \vee \ldots \vee \bigvee^{b_l} \varphi^{(l)}(1) = 1) \leqq 10^{-l}$ and then it is sufficient to make k number of steps, where k is such a number that $10^{-k} < \varepsilon_0$ □

In the last step is better to choose $b_1 = b_1'$ or $b_1 = b_1' + 1$ according to, such a case for which $|P(\bigvee^{b_1} \varphi(1) \ldots = 1| - p)$ is smaller. The error is then less or equal to $\frac{1}{2}10^{-k}$.

The estimation of the number of used elements:

We have

$$b_1 \leqq 10p(1 - p)^{-1}, \quad b_j \leqq 10(1 - p)^{-1}$$

for $j = 2, \ldots, k - 1$ and $b_k \leqq 10(1 - p)^{-1} + 1$. Then

$$\begin{aligned} D(\&) &\leqq k - 1 + \tfrac{1}{2}10k(k - 1)(1 - p)^{-1}, \\ D(\vee) &\leqq 1 + 10(p + (k - 1))(1 - p)^{-1}, \\ D(\varphi) &\leqq k + 10(p + k(k - 1))(1 - p)^{-1}. \end{aligned} \tag{8}$$

We have to find k such that $\frac{1}{2}10^{-k} \leqq \varepsilon$, then $k = [-\log 2\varepsilon + 1]$ (the integer part of the number). For $p > 0{\cdot}5$ it is better to construct $F = \sim F'$, where F' is the expression for $1 - p$. We assume then $p \leqq 0{\cdot}5$.

Conclusion 5. The we obtain the following estimates:

$$D(\&) \leqq [-\log 2\varepsilon](1 + 10[-\log 2\varepsilon + 1]),$$
$$D(\vee) \leqq 1 + 20(\tfrac{1}{2} + [-\log 2\varepsilon]), \quad D(\sim) \leqq 1$$

and

$$D(\varphi) \leqq [-\log 2\varepsilon + 1] + 10(1 + [-\log 2\varepsilon][-\log 2\varepsilon + 1]).$$

The proof of estimate (8): Denote the increase of probability in the case of transition from b to $b + 1$ by Δ_b. For the first step: $10^{-1} \geqq \Delta_b \geqq 10^{-1}(1 - p)$. Let us denote the number of members in the disjunction needed for the minimal increase as b_1', for the maximal increase as b_1''. We have then $(b_1' + 1)\,10^{-1}(1 - p) \geqq p \geqq$ $\geqq b_1'\,10^{-1}(1 - p)$, and thus $b_1' \leqq 10p(1 - p)^{-1}$, and $b_1'' \geqq 10p - 1$. Therefore $10p(1 - p)^{-1} \geqq b_1 \geqq 10p - 1$. We obtain $10^{-l} \geqq \Delta_b \geqq 10^{-l}(1 - p)$ for the l-th step and it implies

$$b_l \leqq b_l' \leqq 10^{-l}(p - P(\bigvee^{b_1} \varphi(1) \vee \ldots$$
$$\ldots \vee \bigvee^{b_{l-1}} \varphi^{(l)}(1) = 1)(1 - p)^{-1} \leqq 10(1 - p)^{-1}.$$

For the k-th step $b_k \leqq 1 + 10(1 - p)^{-1}$. Then

$$D(\&) \leqq (k - 1) + \sum_{j=1}^{k-1} j\,10(1 - p)^{-1},$$
$$D(\vee) = k + b_1 - 1 + b_2 - 1 + \ldots + b_k - 1 \leqq$$
$$\leqq k + 10p(1 - p)^{-1} - 1 + (10(1 - p)^{-1} - 1)(k - 2) + 10(1 - p)^{-1}$$

and

$$D(\varphi) = b_1 + 2b_2 + \ldots + kb_k \leqq$$
$$\leqq 10p(1 - p)^{-1} + \sum_{j=1}^{k-1} j\,10(1 - p)^{-1} + k. \qquad \square$$

Some other methods of the synthesis are given by R. L. Schirtladze in [9] and [10], and by J. Wartfield in [11]. All these methods are based on "white sources", i.e., on probabilistic connectives with parametres (p_0, p_0).

Remark 1. The method suggested in Part II of this paper can be modified in the following way (using the notions from [10]): we substitute $\varphi_i(\varphi_{i-1}(\ldots(\varphi_1(x))\ldots))$ for $\varphi^{(i)}(x)$ (note that $P(\varphi_i(\varphi_{i-1}(\ldots(\varphi_1(1))\ldots)) = 1 = 10^{-i})$. Then $D(\&) = 0$, but the LP-expression is not in SPDNF.

In [11] $D(\&) \leqq {}_{k_0}2^{k_0}$, $D(\vee) \leqq 2^{k_0}$, $D(\sim) \leqq \frac{1}{2}(2^{k_0})$ and $D(\varphi) \leqq k_0 + 1$ is obtained for the number of used elements, if $k_0 = [\log \varepsilon/\log 2]$.

Some more general considerations are contained in paper [9]. This paper deals with connectives with parametres (p_0, p_0) where $p_0 \in (0, 1)$, and solves the problem of how for every $p \in (0, 1)$ to find a logical expression $f(x)$ such that $|p - P(f(\varphi_1, \dots, \varphi_n) = 1| \leqq \varepsilon$, where $\varphi_1, \dots, \varphi_n$ are stochastically independent and have characteristic vectors (p_0, p_0). The estimate of used elements is then (in our form):

$$D(\&) \leqq \tfrac{1}{2}(n-1)(n-2),$$

$$D(\vee) \leqq n - 1,$$

$$D(\sim) \leqq \tfrac{1}{2}(n-1)(n-2),$$

and

$$D(\varphi) \leqq n,$$

where $n \leqq n(\varepsilon, p_0)$, if $n(\varepsilon, p_0) = \max(n(\varepsilon), n(p_0))$, where $n(\varepsilon) = [1 + \log(2\varepsilon/p_0) : : \log(1-p_0)]$, $n(p_0) = [|\log_2(1-p_0)|^{-1}]$ for $p_0 \in (0, \frac{1}{2}\rangle$, and $n(\varepsilon) = [1 + \log . (2\varepsilon/(1-p_0))/\log p_0]$, $n(p_0) = [|\log_2 p_0|^{-1}]$ for $p_0 \in \langle\frac{1}{2}, 1)$.

Remark 2. It is possible to generalize Theorem 4 for connectives $\varphi_1, \dots, \varphi_n$ with $p_{\varphi_j} = (0, p_0)$ (in this theorem we used $p_0 = 0{\cdot}1$). Then we have

$$P(\bigvee^{a+1} \varphi(1) = 1) - P(\bigvee^{a} \varphi(1) = 1) \geqq p_0(1-p)$$

and

$$P(\bigvee^{b_1} \varphi(1) \vee \dots \vee \bigvee^{a+1} \varphi^{(l)}(1) = 1) - P(\bigvee^{b_1} \varphi(1) \vee \dots \vee \bigvee^{a} \varphi^{(l)}(1) = 1) \geqq$$
$$\geqq p_0^l(1-p)$$

in the proof.

It is necessary to find k for which $\frac{1}{2}p_0^k \leqq \varepsilon$. The estimate of the number of used elements is then

$$D(\&) \leqq k_0 + \frac{1}{2p_0} k_0(k_0 +),$$

$$D(\vee) \leqq 1 + 1p_0(1 + k_0)$$

and

$$D(\varphi) \leqq k_0 + \frac{1}{p_0}(1 + \tfrac{1}{2}k_0(k_0 + 1)),$$

where

$$k_0 = [\log 2\varepsilon/\log p_0].$$

For the method modified as in Remark 1 we have $D(\&) = 0$.

For illustration, we can now make a comparison of the method mentioned above. We denote Wartfield's method W, Schirtladze's method S, our method from Theorem 4 (or modified by Remark 2) as 1, and this method modified by Remark 1 (or modified by Remarks 1 and 2) as 2. We obtain the following tables:

For $p_0 = 0{\cdot}1$:

	$\varepsilon = 0{\cdot}001$			$\varepsilon = 0{\cdot}0001$		
	1	2	S	1	2	S
$D(\&) \leqq$	32	0	1553	43	0	3081
$D(\vee) \leqq$	51	51	58	71	71	79
$D(\sim) \leqq$	1	1	1553	1	1	3081
$D(\varphi) \leqq$	73	73	59	134	134	80
	$k_0 = 2$		$n \leqq 59$	$k_0 = 3$		$n \leqq 80$

for $p_0 = 0{\cdot}5$ and $\varepsilon = 0{\cdot}0001$:

	1	2	S	W
$D(\&) \leqq$	276	0	45	106 496
$D(\vee) \leqq$	51	51	11	8 192
$D(\sim) \leqq$	1	1	45	4 096
$D(\varphi) \leqq$	279	279	12	14
	$k_0 = 2$		$n \leqq 12$	$k_0 = 13$

The last problem is this: we realize probabilistic connectives with an error. We ask how big this error can be provided the error of the whole net realizing a probabilistic operator should be lower then ε_0 given in advance.

For the operator with binary output we could require the satisfaction of the following condition:

$$\sum_{\sigma \in \psi(A)} P(x = \sigma) \left|P'(\sigma, \Omega_1) - p_\sigma\right| \leqq \varepsilon_0 .$$

We will require a stronger condition:

$$\left|P'(\sigma, \Omega_1) - p_\sigma\right| \leqq \varepsilon_0 \quad \text{for every} \quad \sigma \in \psi(A) . \tag{9}$$

Our realizing LP-expression (1) is such that to satisfy (9) it is sufficient that $|p_i - p_\sigma| \leqq \varepsilon$ (σ is binary form of $i - 1$). For the probabilistic operator we will require

$$\left|p_{\psi(q_i)}^{\vartheta(b_j)} - p_{ij}\right| \leqq \varepsilon_0 \quad \text{for} \quad i = 1, \ldots, k, \; j = 1, \ldots, l .$$

If $|P'_r(\psi(a_i); \Omega_1^r) - p_i^r| \leqq \varepsilon$ $(r = 1, \ldots, l - 1)$, where p_i^r are solutions of equations (3), then

$$\left|\prod_{r=1}^{l-1} P_r(\psi(a_i); \Omega_1^r) - p_{i1}\right| \leqq$$

$$\leqq (l - 1)\,\varepsilon + (l - 1)\,\varepsilon^2 + \ldots + (l - 1)\,\varepsilon^{l-2} + \varepsilon^{l-1} \doteq \frac{(l - 1)(1 - \varepsilon^{l-2})\,\varepsilon}{1 - \varepsilon}$$

if we neglect ε^{l-1}. Then we need $(l - 1)\,\varepsilon(1 - \varepsilon^{l-2})(1 - \varepsilon)^{-1} \leqq \varepsilon_0$. A sufficient condition is $(l - 1)\,\varepsilon(1 - \varepsilon)^{-1} \leqq \varepsilon_0$ (thus we eliminate the influence of having neglected ε^{l-1}, and we obtain $\varepsilon \leqq \varepsilon_0/l$; for stochastic automaton we obtain following inequality: $\varepsilon \leqq \varepsilon_0/ls$.

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