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The Computation of Characteristic Vectors of Logical–Probabilistic Expressions

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This paper is related to paper [2], knowledge of which is essential for a full understanding of the following text. Particular case of the computation of the characteristic vector of logical-probabilistic expressions are considered here.

The notion of logical-probabilistic expression was introduced in paper [2] for describing nets with logical and probabilistic elements. A logical-probabilistic expression Φ is a triple $[F, \Omega_F, \mathscr{P}_F]$, where F is a logical-probabilistic form (LP-form), Ω_F is a space of random events, and $\mathscr{P}_F$ is a system of probabilities on Ω_F.

An LP-form is a generalized logical form of the propositional calculus in which a new kind of unary connectives is introduced; let every such connective (probabilistic connective) be denoted by one of the symbols $\varphi_1, \varphi_2, \ldots$, (probabilistic connectives in a given LP-form must be denoted by different symbols). Let the connectives $\varphi_1, \ldots, \varphi_n$ occur in a given LP-form F. A space of random events (denoted $\Omega_i = \{\omega_0, \omega_1\}$ is associated with each connective φ_i. The value of the associated function of φ_i is 1 or 0, depending on the random events ω_1 and ω_0.

Further Ω_F is equal to $\mathsf{X}_{i=1}^n \Omega_i$. The system of probabilities $\mathscr{P}_F$ is

$$\{P(\gamma; \omega)\}_{\gamma \in \{0,1\}^n}, \ \gamma = (\gamma_1, \ldots, \gamma_n) .$$

Let $x = (x_1, \ldots, x_m)$ be variables occurring in F. Then for their given evaluation $\sigma \in \{0, 1\}^m$, γ_i is equal to the evaluation of a subform F' such that $\varphi_i(F')$ is a subform of F $(\varphi_i(F') \prec F)$, for $i = 1, \ldots, n$.

For a more precise description of the notion of LP-expression (and corresponding LP-net) see Definitions 8 and 5 in [2]. It is useful if $\mathscr{P}_F$ fulfills condition 4) from Definition 5 in [2].

As formulated and proved in Theorem 2 of [2], the probabilistic properties of an LP-expression Φ can be described by the characteristic vector $\boldsymbol{p}_\Phi$. $\boldsymbol{p}_\Phi$ is equal to the vector of probabilities $(P_\sigma(\Omega_1))_{\sigma \in \{0,1\}^m}$, where σ are possible evaluations of the variables

$x_1, \ldots, x_m$ and Ω_1 is a subset of Ω_F such that $func_F(\sigma, \omega) = 1$ for $\omega \in \Omega_1$, where $func_F$ is the function associated with F (see [2], Def. 8). Analogously for Ω_0.

Remark. The previous description of $\boldsymbol{p}_\Phi$ is valid only if every variable occurring in F occurs in the interior of some probabilistic connective φ_i. Otherwise if $x_{j_1}, \ldots, x_{j_s}$ are variables occurring in F, but not occurring in the interior of any φ_i, then $\Omega_1 \subseteq \subseteq \Sigma' \times \Omega_F$, where Σ' is the space of values of these variables.

The general method of computation of $\boldsymbol{p}_\Phi$ was described in the previous paper [2]. But this general method is too complicated to be used in many real cases. Theorem 3 from [2] made it possible for us to restrict the subject to the computation of the characteristic vectors of LP-expressions which are in probabilistic disjunctive normal form (PDNF). Roughly speaking in LP-expression $[F, \Omega_F, \mathscr{P}_F]$ is in PDNF, if for every φ_i occurring in F $(\varphi_i(F') \prec F)$ is its interior (F') of the form

$$F' \simeq (F_1 \mathbin{\&} \ldots \mathbin{\&} F_{n_1}) \vee \ldots \vee (F_{n_{l-1}+1} \mathbin{\&} \ldots \mathbin{\&} F_{n_l}),$$

where F_j are either variables or of the form $\varphi_j(F''_j)$.

The following considerations will be devoted to the computation of $\boldsymbol{p}_\Phi$ for several particular cases of LP-expressions in PDNF. These particular cases are given by restrictions on $\mathscr{P}_F$ (e.g. stochastic independence).

Remark. It is very useful to bear in mind, that an LP-expression is a description of a net with logical and probabilistic elements (LP-net). For more particulars see Definitions 5 and 8 and Theorem 1 from [2].

Theorem 1. *Consider an* LP-*expression* $\Phi = [F, \Omega_F, \mathscr{P}_F]$, *where*

$$F \simeq \varphi_n((F_1 \mathbin{\&} \ldots \mathbin{\&} F_{n_1}) \vee \ldots \vee (F_{n_{l-1}+1} \mathbin{\&} \ldots \mathbin{\&} F_{n_l}))$$

and

$$\mathscr{P}_F = \{P(\gamma; \omega)\}_\gamma .$$

Let l be number of conjuctive members in *F*.

Then: 1) *For every conjuctive member*

$$K_i \simeq F_{n_{i-1}+1} \mathbin{\&} \ldots \mathbin{\&} F_{n_i}$$

(*more precisely, for the corresponding subexpression*) *we have* $\boldsymbol{p}_{K_i} = \boldsymbol{p}_{\boldsymbol{F}_i,\boldsymbol{1}}$, *where* $\boldsymbol{p}_{\boldsymbol{F}_i,\boldsymbol{1}}$ *is the last column of the matrix* $\boldsymbol{P}_{\boldsymbol{F}_i}$ $(\boldsymbol{F}_i = (F_{n_{i-1}+1}, \ldots, F_{n_i}))$.

2) *For the subform* $F' = int(\varphi_n, F)$ *we have* $\boldsymbol{p}_{F_,} = \boldsymbol{1}^T - \boldsymbol{p}_{\boldsymbol{K},0}$, *where* $\boldsymbol{p}_{\boldsymbol{K},0}$ *is the first column of the matrix* $\boldsymbol{P}_{\boldsymbol{K}}$ $(\boldsymbol{K} = (K_1, \ldots, K_l))$ *and* $\boldsymbol{1} = (1, \ldots, 1)$.

Moreover, let

$$P(\gamma; \omega) = P_1(\gamma_1; \omega_1) \,.\, P_2(\gamma_n; \omega_n)$$

(*for every* $\gamma \in \{0, 1\}^n$), *where*

$$\gamma_1 = (\gamma_1, \ldots, \gamma_{n-1}), \quad \omega_1 = (\omega_1, \ldots, \omega_{n-1}).$$

Then:

3) $$p_F = (I^T - p_{K,0})\, p_1 + p_{K,0} p_0\,,$$

where p_0, p_1 *are probabilistic parameters of* φ_n.

Remarks. 1) We suppose that the number of probabilistic connectives in F is n. For conventions concerning the enumeration of these connectives, see [2], convention 1).

2) An LP-subexpression $[F', \Omega_{F'}, \mathscr{P}_{F'}]$ of a given LP-expression $[F, \Omega_F, \mathscr{P}_F]$ is determined by a subform $F' \prec F$.

3) $Int\,(\varphi_i, F)$ is the F' for which $\varphi_i(F') \prec F$.

4) It is necessary to explain the meaning of symbols $\boldsymbol{P}_{F_i}$, $\boldsymbol{P}_K$. We have

$$\boldsymbol{P}_K = (p_{K;\sigma,\zeta})_{\sigma,\zeta}\,(\zeta \in \{0, 1\}^l)\,,$$

where

$$p_{K;\sigma,\zeta} = P(func_F\, F_1(\sigma,\omega) = \zeta_1, \ldots, func_F\, K_l(\sigma, \omega) = \zeta_l)\,.$$

The right-hand side of the preceding equation is equal to

$$P(\sigma;\, \Omega_{K_1}^{\zeta_1}, \ldots, \Omega_{K_l}^{\zeta_l})\,,$$

where (for $i = 1, \ldots, l$) $\Omega_{K_i}^{\zeta_i}$ is a set which plays the same role for K_i as the set Ω_1 (if $\zeta_i = 1$) or Ω_0 (if $\zeta_i = 0$) for F (see above; for more details see [2], the remark before Theorem 6 and the method of computation of p_Φ in Part II). Analogously for $\boldsymbol{P}_{F_i}$.

5) Probabilistic parameters of a probabilistic connective φ_j (in a given Φ) are defined by the following equalities:

$$p_0^j = \sum_{\{\omega;\omega_j = 1\}} P(\gamma_2, \ldots, \gamma_{j-1}, 0, \gamma_{j+2}, \ldots, \gamma_n;\, \omega)\,,$$

$$p_1^j = \sum_{\{\omega;\omega_j = 1\}} P(\gamma_2, \ldots, \gamma_{j-1}, 1, \gamma_{j+1}, \ldots, \gamma_n;\, \omega)\,.$$

In the preceding theorem, then,

$$p_0^n = P_2(0; 1) \quad \text{and} \quad p_1^n = P_2(1; 1)\,.$$

We will denote p_0^j, p_1^j by $\boldsymbol{p}_{\varphi_j}$.

Proof of Theorem 1: We can use Theorem 5 from [2].

1) According to this theorem we obtain $p_{K_i} = \boldsymbol{P}_{F_i} p_\&$; we know that $p_\& = 0, 0, \ldots \ldots, 0, 1)^T$ and the assertion is evident.

2) Again, according to Theorem 5, $p_{int(\varphi_i nF)} = \boldsymbol{P}_K p_V$. It is well known that $p_V = (0, 1. \ldots, 1)^T$ and thus

(1) $$p_{int(\varphi_n, F;)\sigma} = \sum_{\zeta \in \{0,1\}^l} p_{K;\sigma,\zeta} p_{V;\zeta} = \sum_{\zeta \neq 0} p_{K;\sigma,\gamma}\,.$$

Recall that the matrix P_K is stochastic; hence, the right hand-side of (1) equals $1 - p_{K;\sigma,0}$ and we obtain the second assertion.

3) We have $P_{int(\varphi,F)} = (p_{K,0}, 1 - p_{K,0})$ and so we obtain the third assertion (by Theorem 5). □

Examples. 1) Consider an LP-expression $\Phi = [F, \Omega_F, \mathscr{P}_F]$, where $F \simeq \simeq \varphi_1(x_1) \,\&\, \varphi_2(x_2)$; we assume stochastic idependence of Φ and we denote $q = 1 - p$. Let $p_{\varphi_1} = p_{\varphi_2} = (1 - p, p)$. Then:

$$P = \begin{pmatrix} p^2 & pq, & pq, & q^2 \\ pq, & p^2, & q^2, & pq \\ pq, & q^2, & p^2, & pq \\ q^2, & pq, & pq, & p^2 \end{pmatrix} \tag{2}$$

and $p_\& = (0, 0, 0, 1)^T$. Thus

$$p_F = P p_\& = (q^2, pq, pq, p^2)^T.$$

2) Analogously, for $F \simeq \varphi_1(x_1) \vee \varphi_2(x_2)$. There is $p_V = (0, 1, 1, 1)^T$ and thus

$$p_F = P p_V = \begin{pmatrix} q^2 + 2pq \\ p^2 + pq + q^2 \\ p^2 + pq + q^2 \\ p^2 + 2pq \end{pmatrix} = \begin{pmatrix} 1 - p^2 \\ 1 - pq \\ 1 - pq \\ 1 - q^2 \end{pmatrix}.$$

3) Let us now consider an LP-expression $\Phi = [F, \Omega_F, \mathscr{P}_F]$, where $F \simeq \simeq \varphi_3(\varphi_1(x_1) \vee \varphi_2(x_2))$. Then we have $K_1 \simeq \varphi_1(x_1)$ and $K_2 \simeq \varphi_2(x_2)$. We assume stochastic independence and $p_{\varphi 2} = p_{\varphi 2} = p_{\varphi_3} = (q, p)$ again. Now we obtain (P_K is equal to the matrix (2))

$$p_F = \begin{pmatrix} 1 - p^2 \\ 1 - pq \\ 1 - pq \\ 1 - q^2 \end{pmatrix} p + \begin{pmatrix} p^2 \\ pq \\ pq \\ q^2 \end{pmatrix} q = \begin{pmatrix} p + (q - p)\, p^2 \\ p + (q - p)\, pq \\ p + (q - p)\, pq \\ p + (q - p)\, q^2 \end{pmatrix}.$$

Let us have an LP-expression $\Phi = [F; \Omega_F, \mathscr{P}_F]$, where F is

$$\varphi_n((F_2 \,\&\, \ldots \,\&\, F_{n_1}) \vee \ldots \vee (F_{n_{l-1}+1} \,\&\, \ldots \,\&\, F_{n_l}))\,.$$

We will assume further that $\boldsymbol{x} = (x_1, \ldots, x_m)$ are variables occurring in F and $\boldsymbol{x}_i = = (x_{i_1}, \ldots, x_{i_{s_i}})$ are variables occurring in F_i ($i = 1. \ldots, n_l$). We can now formulate the following theorem.

Theorem 2. *Consider an* LP-*expression* Φ *such as in Theorem* 1. *Let, moreover, all variables be different* (*i.e. each variable cannot occure more then once on* F)

 and let

$$P(\gamma; \omega) = \prod_{i=1}^{n_l} P_i(\gamma_i; \omega_i),$$

where, if $\varphi_{i_2}, \ldots, \varphi_{i_{k_i}}$ are probabilistic connectives occurring in F_i, γ_i is $(\gamma_{i_1}, \ldots, \gamma_{i_{k_i}})$ and ω_i is $(\omega_{i_1}, \ldots, \omega_{i_{k_i}})$.

Values of the variables $\mathbf{x}_i$ will be denoted by σ_i (if we have the evaluation σ on $\mathbf{x}$). Then

1) $$p_{int(\varphi_n, F);\sigma} = 1 - \prod_{j=1}^{l} (1 - \prod_{i=n_{j-1}+1}^{n_j} p_{i;\sigma_i}),$$

where $p_{i;\sigma_i} = P_i(\sigma_i; \Omega_i^1)$, and

2) $$p_{F;\sigma} = p_0^n + (p_1^n - p_0^n) \prod_{j=1}^{l} (1 - \prod_{i=n_{j-1}+1}^{n_j} p_{i;\sigma_i}).$$

Remark: It is helpful to note that

$$P_i(\sigma_i; \Omega_i^1) = P(func_F\, F_i(\sigma_i; \omega_i) = 1)$$

as we know from [2]. According to our assumptions

$$func_F\, F_i(\sigma_i; \omega_i) \quad \text{is equal to} \quad func_F\, F_i(\sigma; \omega).$$

Example. If we return to point 3) of the preceding example, we can see that Theorem 2 immediately gives

$$\boldsymbol{p}_\Phi = \begin{pmatrix} p + (q-p)(1-q)(1-q) \\ p + (q-p)(1-q)(1-p) \\ p + (q-p)(1-q)(1-p) \\ p + (q-p)(1-p)(1-p) \end{pmatrix} = \begin{pmatrix} p + (q-p)\,p^2 \\ p + (q-p)\,pq \\ p + (q-p)\,pq \\ p + (q-p)\,q^2 \end{pmatrix}.$$

Remark. The situation described in Theorem 2 is well known to everyone who deals with unreliable logical nets.

Proof of theorem 2. As a consequence of stochastical independence we have

$$p_{K_j;\sigma} = p_{F_j;\sigma,I} = \prod_{i=n_{j-1}+1}^{n_j} p_{i;\sigma_i}.$$

Furthemore

$$P_{K_j}(\Omega_j^0) = 1 - p_{K_j;\sigma} = 1 - \prod_{i=n_{j-1}+1}^{n_j} p_{i;\sigma_i}$$

and with respect to stochastic independence

$$p_{K;\sigma,0} = \prod_{j=1}^{l} (1 - \prod_{i=\,j-1+1}^{n_j} p_{i;\sigma_i})$$

($p_{K;\sigma,0}$ is a member of the first column of $\boldsymbol{P}_K$) and thus

$$p_{int(\varphi_n,F);\sigma} = 1 - \prod_{j=1}^{l} (1 - \prod_{i=n_{j-1}+1}^{n_j} p_{i;\sigma_i}) \, .$$

The second assertion is a consequence of point 3) from the previous theorem. □

Remarks. 1) The assumption that the variables must be different is not essential. But for practical computation it is useful to have different variables and to denote them in such a way that $\boldsymbol{x}_2 = (x_1, .., x_{m_1}), \ldots, \boldsymbol{x}_l = (x_{m_{l-1}+1}, \ldots, x_m)$. Then $\boldsymbol{\sigma} = (\sigma_1, \ldots \ldots, \sigma_{n_l})$ and the computation of matrices $\boldsymbol{P}_{F_i}$, $\boldsymbol{P}_K$ is simpler. We must consider this fact from the point of view of the inductive computation of $\boldsymbol{p}_\Phi$ for an LP-expression Φ which is in PDNF.

On the other hand it is possible to transform every LP-form F to the LP-form F' in which all variables are different. Then we obtain the result for the original LP-expression by omitting some members in $\boldsymbol{p}_{\Phi'}$.

2) For Theorem 2 it is sufficient that

$$P(\boldsymbol{\sigma}; \omega_1^*, \ldots, \omega_{n_l}^*) = \prod_{i=1}^{n_l} P_i(\sigma_i; \omega_i^*) \, ,$$

where $P(\boldsymbol{\sigma}; \omega_1^*, \ldots, \omega_{n_l}^*)$ is the joint probability of evaluations of $F_1, \ldots, F_{n_l}$ if the variables are evaluated by $\boldsymbol{\sigma}$ (ω_i^* is a variable with two possible values $\Omega_{F_i}^0$, $\Omega_{F_i}^1$; see point 4) of the remark after Theorem 1).

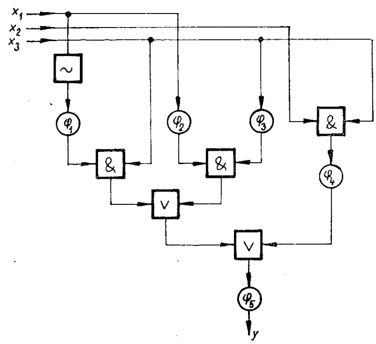

Fig. 1.

Examples. 1) Let us consider an LP-net $[N, \Omega_N, \mathscr{P}_N]$ (fig. 1). Assume stochastic independence and let the probabilistic parametres of all probabilistic elements (connectives) be (0·4, 0·6). The corresponding LP-expression is $[F(N), \Omega_N, \mathscr{P}_N]$, where

$$F(N) \simeq \varphi_5(((\varphi_1(\sim x_1) \mathbin{\&} x_2) \vee (\varphi_2(x_1) \mathbin{\&} \varphi_3(x_3)) \vee (\varphi_4(x_2 \mathbin{\&} x_3))) .$$

Then

$$F_1 \simeq \varphi_1(\sim x_1)\,, \quad F_2 \simeq x_2\,, \quad F_3 \simeq \varphi_2(x_1)\,, \quad F_4 \simeq \varphi_3(x_3)\,,$$
$$F_5 \simeq \varphi_4(x_2) \mathbin{\&} x_3)$$

and

σ	p_{F_1}	p_{F_2}	p_{F_3}	p_{F_4}	p_{F_5}	p_F	ζ
000	.6	0	.4	.4	.4	.550	0
001	.6	0	.4	.6	.4	.510	0
010	.6	1	.4	.4	.4	.560	0
011	.6	1	.4	.6	.6	.576	1
100	.4	0	.6	.4	.4	.510	0
101	.4	0	.6	.6	.4	.538	1
110	.4	1	.6	.4	.4	.546	1
111	.4	1	.6	.6	.6	.588	1

where ζ are, for comparison, the values of expression $F(N')$ without probabilistic elements (connectives).

2) Consider an LP-net $[N, \Omega_N, \mathscr{P}_N)$ (fig. 2). Assume stochastic independence. The

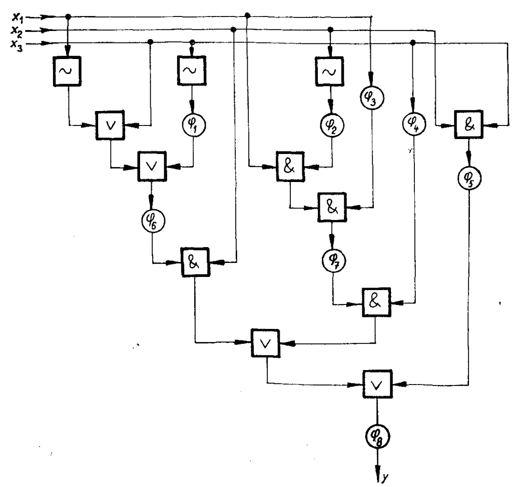

Fig. 2.

corresponding LP-expression is $[F(N), \Omega_N, \mathscr{P}_N]$, where $F(N)$ is

$$\varphi_8((\varphi_5(\sim x_1 \vee x_3 \vee \varphi_1(\sim x_3)) \& x_2) \vee (\varphi_7(x_1 \& \varphi_2(\sim x_2) \& \varphi_3(x_1) \& \varphi_4(x_3))) \vee \\ \vee \varphi_5(x_2 \& x_3)\,.$$

For all probabilistic connectives let the probabilistic parametres be (0·1, 0·9). Then

$$F_1 \simeq \varphi_6(\sim x_1 \vee x_3 \vee \varphi_1(\sim x_3))\,, \quad F_2 \simeq x_2\,, \quad F_3 \simeq \varphi_7(x_1 \& \varphi_2(\sim x_2) \& \varphi_3(x_1))$$
$$F_4 \simeq \varphi_4(x_3)\,, \quad F_5 \simeq \varphi_5(x_2 \& x_3)$$

(for the second step of induction) and $\boldsymbol{p}_F$ is

$$(.2514,\ .2514,\ .7744,\ .8872,\ .2000,\ .3117,\ .3304,\ .8872)^T\,;$$

for comparison, the characteristic vector of this net without probabilistic elements is $(0, 0, 1, 1, 0, 0, 0, 1)^T$.

3) As an example, we can compute the characteristic vector of LP-expression Φ' from the proof of Theorem 4 of [2]. There we had $\Phi' = [F', \Omega_{F'}, \mathscr{P}_{F'}]$, where

$$F' \leqq \bigvee_{i=1}^{2^m} \varphi_i(x_1^{\varepsilon_1{}^i}) \& x_2^{\varepsilon_2{}^i} \& \ldots \& x_m^{\varepsilon_m{}^i}$$

(where

$$x_j^{\varepsilon_j{}^i} = \begin{cases} x_j & \text{if} \quad \varepsilon_j^i = 1\,, \\ \sim x_j & \text{if} \quad \varepsilon_j^i = 0\,). \end{cases}$$

The LP-expression Φ' was assumed to be stochastically independent, i.e.

$$P(\gamma; \omega) = \prod_{i=1}^{2^m} P_i(\gamma_i; \omega_i)\,,$$

and $P_i(0; 1) = 0$, $P_i(1; 1) = p_i$ $(i = 1, \ldots, 2^m)$. According to Theorem 2 we have

$$p_{F;\sigma} = 1 - \prod_{i=1}^{2^m} (1 - \prod_{j=n_{i-1}+1}^{n_i} p_{j;\sigma_j})\,,$$

where $\sigma_j = \sigma_l$, if $j = (i-1)\,m + l$ $(l = 1, \ldots, m)$. For

$$j = (i-1)\,m + l \quad (l \geqq 2) \quad p_{j;\sigma_j} = \begin{cases} 0, & \text{if} \quad \sigma_l \neq \varepsilon_i^l\,, \\ 1, & \text{if} \quad \sigma_l = \varepsilon_i^l\,, \end{cases}$$

and for

$$j = (i-1)\,m + 1 \qquad p_{j;\sigma_j} = \begin{cases} 0, & \text{if} \quad \sigma_1 \neq \varepsilon_i^1\,, \\ p_i, & \text{if} \quad \sigma_1 = \varepsilon_i^1\,. \end{cases}$$

Then

$$1 - \prod_{j=(i-1)m+1}^{im} p_{j;\sigma_j} = \begin{cases} 1, & \text{if} \quad \sigma \neq \varepsilon^i\,, \\ 1 - p_i, & \text{if} \quad \sigma = \varepsilon^i\,, \end{cases}$$

and thus

$$p_{F;\sigma} = p_i \quad (\sigma = \varepsilon^i).$$

As we mentioned in [2], the class of stochastically independent LP-expressions is not closed with respect to the transformation to PDNF. During this transformation, stochastically independent groups of functionally equivalent probabilistic connectives are formed. The same problem can arise if we consider logical nets with

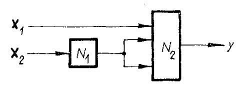

Fig. 3.

stochastically independent probabilistic elements, but, in addition, with elements of forkjunction. If we have such a net, e.g. having the structure of fig. 3, we can transform this net to that net which will be an LP-net according to our concept (see [2], Def. 2). Let us consider the net from fig. 3 and assume that N_1, N_2, N_3 are nets without elements of forkjunction. Let N_1' be a new net such that $\mathcal{N}_1 = [N_1, \Omega_{N_1}, \mathcal{P}_{N_1}]$ and $\mathcal{N}_4' = [N_1', \Omega_{N_1'}, \mathcal{P}_{N_1'}]$ are functionally equivalent, i.e. $P(func_{N_1}(\sigma, \omega) = func_{N_1'}(\sigma, \omega)) = 1$ (for every σ) (see [2], Def. 10). Then we trans-

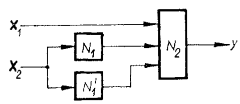

Fig. 4.

form the net from fig. 3 to the net having the structure of fig 4. The new net is a net in our sense of word.

Generally we proceed in the same way as in the process of transforming an LP-expression to PDNF. If we need the new subnet, we take new functionally equivalent probabilistic elements and substitute them in the given structure. Transforming the net, we proceed by induction on degrees of probabilistic elements. For more particulars see [2], Lemma 2, Theorems 3 and 7 and their proofs.

Now, we must consider the computation of characteristic vectors for these cases, i.e., when we have an LP-expression for which

$$P(\gamma'; \omega') = \begin{cases} P(\gamma; \omega), & \text{if for } \gamma_i = \gamma_{i1} = \ldots = \gamma_{in_i} \\ & \text{is } \quad \omega_i = \omega_{i1} = \ldots = \omega_{in_i} \quad (i = 1, \ldots, n)\,. \\ 0 & \text{in other cases} \end{cases}$$

($\gamma' = (\gamma_1, \ldots, \gamma_n, \gamma_{11}, \ldots, \gamma_{nn_n})$, $\gamma = (\gamma_1, \ldots, \gamma_n)$, analogously for ω', ω).

Then

$$\varphi_i = (\varphi_i, \varphi_{i1}, \ldots, \varphi_{in_i})$$

is such a group of functionally equivalent probabilistic connectives.

We can proceed according to Theorem 1; for computation of probabilities $p_{F_i;\sigma,I}$ and $p_{K;\sigma,0}$ we must consider the occurrence of functionally equivalent subexpressions. First, we can consider some particular cases:

1) Assume $F_1(x_1) \equiv_f F_2(x_2)$. Then (for $F_i = (F_1, F_2)$)

$$p_{F_i;\sigma,I} = p_{F;\sigma} = p_{F_2;\sigma}$$

($p_{F_1;\sigma}$, $p_{F_2;\sigma}$ are members of p_{F_1} and p_{F_2}).

2) Assume $F_1(x_1) \equiv_f F_2(x_2)$, $F_i = (F_1, \varphi_j(F_2))$, and let probabilistic parametres of φ_j be p_0, p_1. Then, using the general method of computation (see [2], Part II), we obtain

$$p_{F_i;\sigma,I} = \sum_{\gamma_j \in \{0,1\}} P'(\gamma_j;\, \Omega^1_{F_1}, \Omega^{\gamma_j}_{F_2}, 1) = \tag{3}$$

$$= P'(1;\, \Omega^1_{F_1}, \Omega^1_{F_2}, 1) = P(1;\, \Omega^1_{F_1}, 1)\,,$$

because

$$P'(0;\, \Omega^1_{F_1}, \Omega^0_{F_2}, 1) = 0\,.$$

$P'(\gamma_j; \omega_1^*, \omega_2^*, \omega_j)$ is the probability for subexpression corresponding to F_1, F_2 and the probabilistic connective φ_j. Using stochastic independence the right-hand side of (3) is then equal to $p_{F_1;\sigma}p_1$.

Now we can formulate an auxiliary theorem which solves a more general case.

Theorem 3. *Consider two subexpressions with subforms*

$$F_1 \simeq G_1(F_1'(x_0), x_1) \quad \text{and} \quad F_2 \simeq G_2(F_2'(x_0), x_2)\,,$$

where $F_1' \equiv_f F_2'$ *and for every* γ *let*

$$P(\gamma; \omega) = P_1(\gamma_1; \omega_1)\, P_2(\gamma_2; \omega_2)\, P_3(\gamma_3; \omega_3)\,,$$

where (γ_1, ω_1) *corresponds to probabilistic connectives from* F_1, (γ_2, ω_2) *to probabilistic connectives from* G_1, *and* (γ_3, ω_3) *to probabilistic connectives from* G_2.
Then for every $\sigma \in \{0, 1\}^m$ $(F_i = (F_1, F_2))$, *we have following*:

$$p_{F_i;\sigma} = p_{F'_1;\sigma_0} p_{G_1;1,\sigma_1} p_{G_2;1,\sigma_2} + (1 - p_{F'_1;\sigma_0})\, p_{G_1;0,\sigma_1} p_{G_2;0,\sigma_2}\,.$$

Proof. If we apply the disjointness of random events, we have

$$p_{F_i;\sigma} = \sum_{l,k \in \{0,1\}^2} P'(\sigma_0, l, \sigma_1, k, \sigma_2;\, \Omega^1_l, \Omega^2_k, \Omega^3_1, \Omega^4_1)$$

(where $P'(\sigma_0, l, \sigma_1, k, \sigma_2; \omega_1^*, \omega_2^*, \omega_3^*, \omega_4^*)$ is the joint probability for the subexpressions corresponding to F_1', F_2', $G_1(y, x_1)$ and $G_2(y, x_2)$.

Furthermore,

$$p_{F_i;\sigma} = \sum_{l \in \{0,1\}} P''(\sigma_0, l, \sigma_1, l, \sigma_2; \Omega_l^1, \Omega_1^3, \Omega_1^4)$$

and by applying stochastic independence we complete the proof. □

In the same way we can proceed in some more complicated cases, e.g., if F_1 is $G_1F_1'(x_1), F_1''(x_2), x_3)$ and F_2 is $G_2(F_2'(x_1), F_2''(x_2), x_3)$, where $F_1' \equiv_f F_2'$ and $F_1'' \equiv_f F_2''$. Moreover, we can proceed in the same way in the computation of probabilities $p_{K;\sigma,0}$.

Conclusion. In all cases we compute the characteristic vector of an LP-expression (LP-net) with the help of its PDNF (more precisely, with the help of a functionally equivalent LP-expression in PDNF). We can apply points 1) and 2) from Theorem 1. In real cases there can be some simplifications:

1) If a probabilistic connective is stochastically independent on the probabilistic connectives from its *int* (φ, F), we can apply point 3) from Theorem 1, or its modification for joint probabilities, if the connective is not the last one.

2) If the PDNF is stochastically independent, we can apply Theorem 2 inductively.

3) If a given PDNF contains stochastically independent groups of functionally equivalent probabilistic connectives $\boldsymbol{\varphi}^1, \boldsymbol{\varphi}^2, \ldots, \boldsymbol{\varphi}^n$ such that for every $k \leqq n$ and every $i, j \leqq n_n$ if $\varphi_1 \in \boldsymbol{\varphi}^k$ and $\varphi_j \in \boldsymbol{\varphi}^k$, then $d(\varphi_1, F) = d(\varphi_j, F)$ ($d(\varphi, F)$ means the degree of probabilistic connective in F; see [2]). we can use all three points of Theorem 2, because the independence of φ_i on *int* (φ_i, F) is presenved, and the probabilities $p_{F_i;\sigma,1}$ and $p_{K;\sigma,0}$ can be computed analogously as in Theorem 3. It is possible to formulate a general theorem for this case, but it would be too complicated and incomprehensible. If, for $F_1, \ldots, F_k$ for which we compute $p_{F;1}$ or $p_{F;0}$, it holds that no pair F_i, F_j $(i \neq j)$ contains a pair of functionally equivalent subexpressions $F_1' \prec F_i$ and $F_2' \prec F_j$, we can apply Theorem 2.

Remarks. 1) The condition of stochastic independence of φ_i on *int* (φ_i, F) is preserved if we transform a net with stochastically independent probabilistic elements and elements of forkjunction to our LP-net (F describes the structure of our new net). In the original net notion of the degree of probabilistic connective is meaningless.

2) If we compute p_Φ recursively for a given LP-expression, we must use Theorem 1, paying attention to the joint probabilities. In a given inductive step (at the beginning) we have the following situation:

We have subforms $F_1, \ldots, F_{n_l}$ as in Theorem 1 and moreover, subforms $G_1, \ldots, G_k$.

$$F' \simeq (F_1 \& \ldots \& F_{n_1}) \vee \ldots \vee (F_{n_{l-1}}, \ldots, F_{n_l})$$

is then the interior of a probabilistic connective φ_t. We must consider the joint probabilities

$$P(\boldsymbol{\sigma}, \gamma'; \omega_1^*, \ldots, \omega_{n_l}^*, \omega_1^G, \ldots, \omega_k^G, \boldsymbol{\omega}),$$

where

$$\gamma' = (\gamma_t, \ldots, \gamma_n), \quad \boldsymbol{\omega}' = (\omega_t, \ldots, \omega_n).$$

Now we have to proceed computation from point 1) of Theorem 1 for $K_1 \simeq \simeq F_1 \,\&\, \ldots \,\&\, F_{n_1}$ (variables $\omega_1^*, \ldots, \omega_{n_1}^*$) for any given value of other variables. We use matrices

$$\boldsymbol{P}_{F_1}(\gamma'; \omega_{n_1+1}^*, \ldots, \omega_k^G, \boldsymbol{\omega}')$$

and we obtain the joint probabilities

$$P(\boldsymbol{\sigma}, \gamma'; \omega_{K_1}^{**}, \omega_{n_1+1}^*, \ldots, \omega_k^G, \boldsymbol{\omega}')$$

and now we repeat this computation for K_2 etc. In the same way we apply point 2) to the joint probabilities $P(\boldsymbol{\sigma}, \gamma'; \omega_{K_1}^{**}, \ldots, \omega_{K_l}^{**}, \omega_1^G, \ldots, \omega_k^G, \boldsymbol{\omega}')$ for any given value of $\gamma', \omega_1^G, \ldots, \omega_k^G, \boldsymbol{\omega}'$.

3) The computation is simpler for an LP-expression in PDNF, obtained from a stochastically independent LP-expression. Functionally equivalent subexpressions can occur only in $K \simeq K_1 \vee \ldots \vee K_l$.

We can see that

$$P(\boldsymbol{\sigma}, \gamma'; \omega_1^*, \ldots, \omega_k^G, \omega') = P_1(\boldsymbol{\sigma}; \omega_1^*, \ldots, \omega_{n_l}^*)\, P_2(\boldsymbol{\sigma}; \omega_1^G, \ldots, \omega_k^G) \prod_{i=t}^{t^n} P(\gamma_i; \omega_i)$$

and for every $F_i \equiv_f F_j$ and for $\omega_i^* \neq \omega_s^*$

$$P_1(\boldsymbol{\sigma}; \omega_1^*, \ldots, \omega_{n_l}^*) = 0.$$

We will now turn our attention to the vectors of LP-expressions. First, we define a characteristic matrix of a probabilistic operator (for the concept of probabilistic operator see [2]) as a matrix $\boldsymbol{P} = (p_{a,b})$, where $p_{a,b} = P_a(b)$ (a is a symbol from input alphabet, b a symbol from the output alphabet). Analogously we define a characteristic matrix of a vector of LP-expressions. Let us consider a vector $\boldsymbol{\Phi} = [\boldsymbol{F}, \Omega_{\boldsymbol{F}}, \mathscr{P}_{\boldsymbol{F}}]$, where $\boldsymbol{F} = (F_1, \ldots, F_k)$ and $\boldsymbol{F}$ contains variables $x_1, \ldots, x_m$. Then we call the matrix

(4) $$\boldsymbol{P} = (p_{ij})_{j=1,\ldots,2^k}^{i=1,\ldots,2^m}, \quad \text{where} \quad p_{ij} = P(\boldsymbol{\sigma}; \Omega_{\zeta_1}, \ldots, \Omega_{\zeta_k})$$

($\boldsymbol{\sigma}$, ζ are binary forms of the numbers $i - 1$, $j - 1$), the characteristic matrix of $\boldsymbol{\Phi}$ (see Theorem 5 of [2]).

We now formulate a theorem about the computation of this matrix for a particular case.

Theorem 4. *Let us take a vector* $\Phi = [F, \Omega_F, \mathscr{P}_F]$, *where* $F = (F_1, \ldots, F_k)$ *and for every* γ

$$P(\gamma; \omega) = \prod_{l=1}^{k} P_l(\gamma_l; \omega_l),$$

where

$$\gamma_l = (\gamma_{i_1}, \ldots, \gamma_{is}) \quad and \quad \omega_l = (\omega_{i_1}, \ldots, \omega_{is})$$

for F_l *containing* $\varphi_{i_1}, \ldots, \varphi_{is}$. *(We call such a vector weakly stochastically independent). We put* $\sigma_l = \sigma_{i_1}, \ldots, \sigma_{i_s})$ *for* F_l *containing* $x_{i_1}, \ldots, x_{i_r}$. *Let* $p_l = (p_{l;\sigma_l})$ *be the characteristic vector of* $\Phi_l = [F_l, \Omega_{F_l}, P_{F_l}]$. *Then*

$$p_{ij} = \prod_{l=1}^{k} (p_{l;\sigma_l})^{\zeta_l} (1 - p_{l;\sigma_l})^{1-\zeta_l}.$$

Proof. We know that $p_{ij} = P(\sigma; \Omega_{\zeta_1}, \ldots, \Omega_{\zeta_l})$; stochastic independence implies

$$p_{ij} = P_1(\sigma_1; \Omega_{\zeta_1}), \ldots, P_k(\sigma_k; \Omega_{\zeta_k}).$$

We have

$$P_l(\sigma_l; \Omega_1) = p_{l;\sigma_l}, P_l(\sigma_l; \Omega_0) = 1 - p_{l;\sigma_l}$$

and thus

$$p_{ij} = \prod_{l=1,\zeta_l=1}^{k} p_{l;\sigma_l} \prod_{l=1,\zeta_l=0}^{k} (1 - p_{l;\sigma_l}). \qquad \square$$

In other cases we have to compute these probabilities inductively and simultaneously for all LP-expressions in the vector analogously as for a single LP-expression.

A stochastically independent LP-net with more then one output can be transformed into a vector of LP-nets. The method is analogous to constructing the canonical LP-expression in [2]. We transform our net into a logical net (remembering the position of probabilistic elements). And now we transform. this net into a vector of logical nets in the way described, e.g., in [3]. Then we can put probabilistic elements in these nets. Since we needed new subnets structurally equivalent to the original ones in the preceeding step, we have to use some new functionally equivalent elements.

Then we have a vector of LP-nets, where two different nets N_i, N_j can contain functionally equivalent subnets, i.e. $F(N_i)$, $F(N_j)$ obtain subexpressions $F_1 \equiv_f F_2$, $F_1 \prec F(N_i)$, $F_2 \prec F(N_j)$. We know how to compute the characteristic vectors of LP-expressions which can contain subexpressions $F_1 \equiv_f F_2$. The characteristic matrix of the vector of LP-expressions is then computed similarly to the subsequent example.

Example. Consider a vector $\Phi = [F, \Omega_F, \mathscr{P}_F]$, where

$$F \simeq \begin{pmatrix} F_1 \ (x) \\ F_2 \ (x) \end{pmatrix}$$

and

$$F_1 \simeq G_1(F_1'(x), x), \quad F_2 \simeq G_2(F_2'(x), x).$$

Let

$$F_1' \equiv_f F_2'$$

and let

$$P(\gamma; \omega) = P_{F'_1}(\gamma_1; \omega_1) P_{G_1}(\gamma_2; \omega_2) P_{G_2}(\gamma_3; \omega_3).$$

If we denote $\zeta = (\zeta_1, \zeta_2)$, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m)$ the binary forms of the numbers $j - 1$, $i - 1$, then

$$p_{ij} = (p_{G_1;0,\sigma})^{\zeta_1} (1 - p_{G_1;0,\sigma})^{1-\zeta_1} (p_{G_2;0,\sigma})^{\zeta_2} (1 - p_{G_2;0,\sigma})^{1-\zeta_2} .$$
$$. (1 - p_{F;\sigma}) + (p_{G_1;1,\sigma})^{\zeta_1} (1 - p_{G_1;1,\sigma})^{1-\zeta_1} (p_{G_2;1,\sigma})^{\zeta_2} (1 - p_{G_2;1,\sigma})^{1-\zeta_2} p_{F;\sigma},$$

where $\boldsymbol{p}_{G_1}, \boldsymbol{p}_{G_2}, \boldsymbol{p}_F$ are the characteristic vectors of $G_1(y, \boldsymbol{x})$, $G_2(y, \boldsymbol{x})$, and $F_1'(\boldsymbol{x})$.

Proof. We have

$$p_{ij} = P(\boldsymbol{\sigma}; \Omega_{\zeta_1}, \Omega_{\zeta_2}) =$$

(5)
$$= \sum_{l,k \in \{0,1\}^2} P'(\boldsymbol{\sigma}, l, \boldsymbol{\sigma}, k, \boldsymbol{\sigma}; \Omega_l^{F'_1}, \Omega_k^{F'_2}, \Omega_{\zeta_1}^{G_1}, \Omega_{\zeta_2}^{G_2}).$$

If we consider that $P'(.; \Omega_l^{F'_1}, \Omega_k^{F'_2}, .) = 0$ for $l \neq k$, then the right-hand side of (5) equals

$$\sum_{l \in \{0,1\}} P''(\boldsymbol{\sigma}, l, \boldsymbol{\sigma}; \Omega_l^{F'_1}, \Omega_{\zeta_1}^{G_1}, \Omega_{\zeta_2}^{G_2}),$$

and if we apply the independence condition we complete the proof. □

Lastly, we will pay attention to probabilistic automata. Let us have a vector of LP-expressions

$$\boldsymbol{\Phi} = [\boldsymbol{P}, \Omega_{\boldsymbol{F}}, \mathscr{P}_{\boldsymbol{F}}],$$

where

$$\boldsymbol{F} \simeq (\boldsymbol{G}_1, \boldsymbol{G}_2), \quad \boldsymbol{G}_1 \simeq (F_1(\boldsymbol{x}, {}_z), \ldots, F_k(\boldsymbol{x}, {}_z))$$

and

(6)
$$\boldsymbol{G}_2 \simeq (G_1(\boldsymbol{x}, {}_z), \ldots, G_s(\boldsymbol{x}, {}_z)).$$

We can compute

$$P(func_{\boldsymbol{F}}(\boldsymbol{x}, {}_z) = \boldsymbol{\delta}, \zeta / \boldsymbol{x} = \boldsymbol{\sigma}, z = \xi).$$

If we have an output alphabet $\{|\Omega_{\delta,\xi}|_1\}$ ($|\Omega_{\delta,\xi}|_1 = \boldsymbol{\delta}$) and a state alphabet $\{|\Omega_{\delta,\xi}|_2\}$ ($|\Omega_{\delta,\xi}|_2 = \zeta$), we have a probabilistic automaton. If $P(\gamma; \omega) = P_{G_1}(\gamma_1; \omega_1) P_{G_2}(\gamma_2; \omega_2)$ we have a probabilistic Mealy automaton. We have these automata with the output alphabet $\{\zeta\}$, the input alphabet $\{\boldsymbol{\sigma}\}$ and the state alphabet $\{\boldsymbol{\delta}\}$, where $\boldsymbol{\sigma}$ are values of variables $\boldsymbol{x}(t - 1)$, $\boldsymbol{\delta}$ are values of $z(t)$ and ξ are values of $z(t - 1)$. If we include a new kind of primitive elements — delay elements and the rules for their connection (as in [1] or [3]), we have a net realizing a probabilistic automaton. We can transform this net by the elimination of elements of forkjunction into a net which can be described by the vector $\boldsymbol{\Phi} = [\boldsymbol{F}, \Omega_N, \mathscr{P}_N]$, $\boldsymbol{F} = \boldsymbol{F}(N)$, where $\boldsymbol{F}$ is as in (6), and where both $func_{G_1}(\boldsymbol{x}(t), z(t-1)) = y(t)$ and $(\boldsymbol{x}(t), z(t-1)) = z(t)$ are analogous to the canonical equations of the deterministic automata (see [3]). We can calculate the characteristic matrix $\boldsymbol{P} = (p_{\sigma,\xi;\zeta,\delta})$ in the same way as in the calculation of characteristic matrix of the vector of LP-expressions.

The results which are described in this paper can be applied to the problem of realization of probabilistic automata. These problems will be considered in another paper [4].

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