# A Graphical Way to Solve the Boolean Matrix Equations $A X=B$ and $X A=B$ 

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A graphical way to find all the solutions of the Boolean matrix equations $A X=B$ and $X A=B$ is proposed and an example is given.

## 1. INTRODUCTION AND BASIC CONCEPTS

As shown by Ledley in [2, pp. 448-484] and in [3, 479-494], the determination of the solutions for the Boolean matrix equations $A X=B$ and $X A=B$ has important applications to switching theory and logical problems. A way to find all the solutions is given in the books cited above. Recently, Rudeanu [4] has derived a complete solution to the equations $A X=B$ and $X A=B$ in parametric form. In this paper we apply a well known graphtheoretic representation of a Boolean matrix to find a graphical way to determine the complete solution to the equations $A X=B$ and $X A=B$. We assume that the reader is familiar with the basic concepts in graph theory.

By a Boolean matrix $Q=\left[q_{i j}\right]$ we shall mean in this paper a $(0,1)$-matrix. The join of two Boolean $n \times m$ matrices $A$ and $B$ is the matrix $\left[a_{i j} \cup b_{i j}\right]$, and the product of the matrices $C$ and $D$ of orders $n \times p$ and $p \times m$, respectively, is an $n \times m$ matrix $C D=\left[\bigcup c_{i s} d_{s j}\right]$. Further, $A^{\mathrm{T}}$ is the transpose of $A$ and $A^{\prime}$ the complement of $A$, i.e. $A^{\mathrm{T}}=\left[a_{j i}\right]$ and $A^{\prime}=\left[a_{i j}^{\prime}\right] . A \geqq B$ if and only if $a_{i j} \geqq b_{i j}$ for any index pair $i j$.

It is well known that with every $m \times n$ Boolean matrix $Q$ one can naturally associate a bipartite graph $G_{b}(Q)$ as follows (see e.g. Hedetniemi [1]): The set of vertices $V\left(G_{b}(Q)\right)$ of $G_{b}(Q)$ consists of two disjoint subsets $\left\{u_{i} \mid i=1, \ldots, m\right\}$ and $\left\{v_{j} \mid j=1, \ldots, n\right\}$ which correspond to the rows and columns of $Q$, respectively. An edge $\left(u_{i}, v_{j}\right)$ joining $u_{i}$ and $v_{j}$, belongs to the edge set $E\left(G_{b}(Q)\right)$ only if $q_{i j}=1$
in $Q$. Conversely, every bipartite graph $G_{b}$ can be translated into a Boolean matrix according to the rules above.

In the following we shall concentrate on the equation $A X=B$. As known, the solution of $X A=B$ is analogous to that of $A X=B$.

## 2. THE BOOLEAN MATRIX EQUATION $A X=B$

Consider the product of two Boolean matrices $A$ and $B$, and let the vertex sets of the bipartite graphs $G_{b}(A)$ and $G_{b}(B)$ be $V\left(G_{b}(A)\right)=\left\{u_{A i} \mid i=1, \ldots, \ldots, m\right\} \cup$ $\cup\left\{v_{A s} \mid s=1, \ldots, k\right\}$ and $V\left(G_{b}(B)\right)=\left\{u_{B s} \mid s=1, \ldots, k\right\} \cup\left\{v_{B j} \mid j=1, \ldots, n\right\}$. Let us draw the bipartite graphs $G_{b}(A)$ and $G_{b}(B)$ such that the vertices in the sets $\left\{v_{A \varepsilon}\right\}$ and $\left\{u_{B s}\right\}$ are common, and denote the graph thus obtained by $G_{b}(A) G_{b}(B)$. Then, according to the formula $A B=\left[\bigcup_{s} a_{i s} b_{s j}\right]$, in the bipartite graph $G_{b}(A B)$ a vertex $u_{A B i}$ is connected by an edge to a vertex $v_{A B j}$ if and only if there is a path of length two from $u_{A i}$ to $v_{B j}$ in the graph $G_{b}(A) B_{b}(B)$. As an illustration, see the graphs of Fig. 1. This graphical form of the product of two Boolean matrices can be applied to the determination of a complete solution to $A X=B$.

$$
A=\left[\begin{array}{ll}
1 & 0 \\
1 & 0
\end{array}\right], \quad B=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right], \quad A B=\left[\begin{array}{ll}
0 & 1 \\
0 & 1
\end{array}\right]
$$


$G_{b}(A)$

$G_{b}(B)$

$G_{b}(A) G_{b}(B)$

$G_{b}(A B)$

Fig. 1.

As shown in the literature, the equation $A X=B$ has a solution if and only if the matrix $\left(A^{\mathrm{T}} B^{\prime}\right)^{\prime}$ is a solution to $A X=B$, i.e. $A\left(A^{\mathrm{T}} B^{\prime}\right)^{\prime}=B$. Moreover, the solutions of $A X=B$ form a join semilattice, denoted by $L_{\cup}(X)$, where $\left(A^{\mathrm{T}} B^{\prime}\right)^{\prime}$ is the greatest element. Hence, if $A Q=B, Q \cup\left(A^{\mathrm{T}} B^{\prime}\right)^{\prime}=\left(A^{\mathrm{T}} B^{\prime}\right)^{\prime}$. Thus, in order to obtain the complete set of solutions, one needs to determine the greatest element and the minimum elements of the semilattice $L_{\cup}(X)$, if such exist. First we consider a direct way to determine the graph $G_{b}\left(\left(A^{\mathrm{T}} B^{\prime}\right)^{\prime}\right)$, and the matrix $\left(A^{\mathrm{T}} B^{\prime}\right)^{\prime}$ as well, and then we show an obvious way to find all the solutions of $A X=B$.
Assume that the equation $A X=B$ has a solution. Now clearly a bipartite graph $G_{b}\left(X_{0}\right)$ corresponds to the greatest solution of $A X=B$, if in the graph $G_{b}(A) G_{b}\left(X_{0}^{\prime}\right)$ every vertex $u_{A i}$, corresponding to $u_{B i}$ in $G_{b}(B)$, is connected by a path of length
two to every vertex $v_{X_{0}{ }^{\prime} j}$, corresponding to $v_{B j}$ in $G_{b}(B)$, for which $\left(u_{B i}, v_{B j}\right) \notin E\left(G_{b}(B)\right)$, i.e. $\left(u_{B i}, v_{B j}\right) \in E\left(G_{b}\left(B^{\prime}\right)\right)$. Thus the following simple rule can be obtained to find the graph $G_{b}\left(X_{0}^{\prime}\right)$ :

Rule 1. Connect in $G_{b}\left(X_{0}^{\prime}\right)$ the vertices $\Gamma u_{A i}=\left\{v_{A i_{1}}, \ldots, v_{A i_{r}}\right\}=\left\{u_{X_{0} i_{1}}, \ldots, u_{X_{0} i_{r}}\right\}$, $u_{A i} \in V\left(G_{b}(A)\right)$, to all the vertices $v_{X_{0^{\prime} j}}$ for which $\left(u_{B i}, v_{B j}\right) \in E\left(G_{b}\left(B^{\prime}\right)\right)$.

It should be noted that the matrix $X_{0}$ determined by the rule above does not give any indication of the non-consistency of the equation $A X=B$.

As an illuminating example, consider the following consistent Boolean matrix equation

$$
\left[\begin{array}{lll}
1 & 0 & 1  \tag{1}\\
0 & 1 & 1
\end{array}\right] X=\left[\begin{array}{ll}
1 & 0 \\
1 & 1
\end{array}\right]
$$

The graphs $G_{b}(A)$ and $G_{b}(B)$ are given in Fig. 2, and the graph $G_{b}\left(X_{0}^{\prime}\right)$ can be seen in the graph $G_{b}(A) G_{b}\left(X_{0}^{\prime}\right)$ determined by Rule 1. Hence,

$$
X_{0}=\left[\begin{array}{ll}
1 & 0 \\
1 & 1 \\
1 & 0
\end{array}\right]
$$



Fig. 2.
Consider now a way to find all the solutions of $A X=B$. We construct a solution matrix base, denoted by $Z_{1}, Z_{2}, \ldots, Z_{t}$, where every $Z_{w}, w=1, \ldots, t$, is a Boolean matrix of the order of $X$ and corresponds io an edge, say $\left(u_{B i}, v_{B j}\right)$, of $G_{b}(B)$ such that $G_{b}\left(Z_{w}\right)$ contains any edge which gives in $G_{b}(A) G_{b}\left(Z_{w}\right)$ a path of length two from $u_{A i}$ to $v_{Z_{w j}}\left(=v_{B j}\right)$ and no edges such that there would be a path of length two in $G_{b}(A) G_{b}\left(Z_{w}\right)$ determining an edge of $G_{b}\left(B^{\prime}\right)$. Since the matrix product is distributive with respect to the join operation and $A Z_{w} \leqq B, A\left(Z_{1} \cup Z_{2} \cup \ldots \cup Z_{t}\right)=B$ according to the definition of the matrices $Z_{w}$, if $Z_{w}>0$ for any $w, w=1, \ldots, t$. Furthermore, as every $G_{b}\left(Z_{w}\right)$ contains all the edges giving in $G_{b}(A) G_{b}\left(Z_{w}\right)$ the edge of $G_{b}(B)$ which determines $G_{b}\left(Z_{w}\right), Z_{1} \cup \ldots \cup Z_{t}=\left(A^{\mathrm{T}} B^{\prime}\right)^{\prime}=X_{0}$, the greatest element of the solution join semilattice $L_{\cup}(X)$. According to the definition of $Z_{w}$, the matrix equation $A X=B$ is consistent if and only if $Z_{w}>0$, i.e. $E\left(G_{b}\left(Z_{w}\right)\right) \neq \emptyset$, for any $w, w=1, \ldots, t$.

64 A matrix $Q$ is a solution of $A X=B$, if $Q \cap Z_{w}>0$ for every $w$, and $Q_{0}$ is a minimum element of $L_{\cup}(X)$ if and only if the equation $Q_{00} \cap Z_{w}>0$ does not hold for any matrix $Q_{00}<Q_{0}, w=1, \ldots, t$.
For the determination of a matrix $Z_{w}$ corresponding to an edge $\left(u_{B i}, v_{B j}\right) \in E\left(G_{b}(B)\right)$ we obtain the following simple rule:

Rule 2. Connect in $G_{b}\left(Z_{w}\right)$ the vertices of $\Gamma u_{A i}=\left\{v_{A i_{1}}, \ldots, v_{A i_{r}}\right\}=\left\{u_{Z_{w i_{1}}}, \ldots\right.$ $\left.\ldots, u_{Z_{w i r}}\right\}, u_{A i} \in V\left(G_{b}(A)\right)$, to $v_{Z_{w} j}$ and remove then the edges which belong to $G_{b}\left(X_{0}^{\prime}\right)$.


$$
G_{b}(A) G_{b}\left(Z_{1}\right)
$$



$$
G_{b}(A) G_{b}\left(Z_{2}\right)
$$



$$
G_{b}(A) G_{b}\left(Z_{3}\right)
$$

Fig. 3.

Consider as an example the matrix equation in (1). Fig. 3 shows the determinations of the basis matrices $Z_{1}, Z_{2}$, and $Z_{3}$ corresponding to the edges ( $u_{B 1}, v_{B 1}$ ), ( $u_{B 2}, v_{B 1}$ ), and ( $u_{B 2}, v_{B 2}$ ), respectively. The dotted lines in Fig. 3 mean the edges of $G_{b}\left(X_{0}^{\prime}\right)$. Since $Z_{1}, Z_{2}, Z_{3}>0$, the equation in (1) is consistent.

As one can readily check, the minimum elements of $L_{\mathrm{V}}(X)$ are $X_{1}=\left[\begin{array}{ll}0 & 0 \\ 0 & 1 \\ 1 & 0\end{array}\right]$ and
$X_{2}=\left[\begin{array}{ll}1 & 0 \\ 1 & 1 \\ 0 & 0\end{array}\right] . Z_{1} \cup Z_{2} \cup Z_{3}=\left[\begin{array}{ll}1 & 0 \\ 1 & 1 \\ 1 & 0\end{array}\right]=X_{0}=\left(A^{\mathrm{T}} B^{\prime}\right)^{\prime}$. The other solutions to $A X=$
$=B$, which are between $X_{1}$ and $X_{0}$ in $L_{\cup}(X)$, are $X_{3}=\left[\begin{array}{ll}1 & 0 \\ 0 & 1 \\ 1 & 0\end{array}\right]$ and $X_{4}=\left[\begin{array}{ll}0 & 0 \\ 1 & 1 \\ 1 & 0\end{array}\right]$.
There is an other way to construct a solution matrix base. After determining the matrices $Z_{1}, \ldots, Z_{t}$ defined above, we substitute the matrix $Z_{w}, w=1, \ldots, t$, by a set $\left\{Y_{1 w}, Y_{2 w}, \ldots, Y_{s_{w} w}\right\}$ of matrices, where $Y_{1 w} \cup \ldots \cup Y_{s_{w} w}=Z_{w}, Y_{k w}>0$ and $Y_{k w}$ contains a single one for any $k, k=1, \ldots, s_{w}$. Every solution to $A X=B$ is obtained by forming all possible joins ( $U Y$ ) of the matrices in the sets $\left\{Y_{1 w}, \ldots, Y_{s_{w} w}\right\}$ such that $(U Y) \cap Z_{w}>0$ for any value of $w$.
In the example considered before,
$Y_{11}=\left[\begin{array}{ll}1 & 0 \\ 0 & 0 \\ 0 & 0\end{array}\right], \quad Y_{12}=\left[\begin{array}{ll}0 & 0 \\ 0 & 0 \\ 1 & 0\end{array}\right], \quad Y_{21}=\left[\begin{array}{ll}0 & 0 \\ 1 & 0 \\ 0 & 0\end{array}\right], \quad Y_{22}=\left[\begin{array}{ll}0 & 0 \\ 0 & 0 \\ 1 & 0\end{array}\right], \quad$ and $\quad Y_{31}=\left[\begin{array}{ll}0 & 0 \\ 0 & 1 \\ 0 & 0\end{array}\right]$.
Thus $\quad X_{1}=Y_{12} \cup Y_{22} \cup Y_{31}=Y_{12} \cup Y_{31}=Y_{22} \cup Y_{31}, \quad X_{2}=Y_{11} \cup Y_{21} \cup Y_{31}$, $X_{3}=Y_{11} \cup Y_{22} \cup Y_{31}=Y_{11} \cup Y_{12} \cup Y_{22} \cup Y_{31}, \quad X_{4}=Y_{12} \cup Y_{21} \cup Y_{31}=Y_{12} \cup$ $\cup Y_{21} \cup Y_{22} \cup Y_{31}$, and $X_{0}=Y_{11} \cup Y_{12} \cup Y_{21} \cup Y_{31}=Y_{11} \cup Y_{21} \cup Y_{22} \cup Y_{31}=$ $=Y_{11} \cup Y_{12} \cup Y_{21} \cup Y_{22} \cup Y_{31}$.

In the case of the equation $X A=B$, Rule 1 and Rule 2 can be expressed as follows:
Rule $\mathbf{1}^{\prime}$. Connect in $G_{b}\left(X_{0}^{\prime}\right)$ the vertices $\Gamma v_{A i}=\left\{u_{A i_{1}}, \ldots, u_{A i_{r}}\right\}=\left\{v_{X_{0} i_{1}}, \ldots, v_{X_{0} \prime_{r} i_{r}}\right\}$, $v_{A i} \in V\left(G_{b}(A)\right)$, to all the vertices $u_{X_{0}{ }^{\prime} j}$ for which $\left(u_{B j}, u_{B i}\right) \in E\left(G_{b}\left(B^{\prime}\right)\right)$.

Rule $2^{\prime}$. Connect in $G_{b}\left(Z_{w}\right)$ the vertices of $\Gamma v_{A j}=\left\{u_{A j_{1}}, \ldots, u_{A j_{r}}\right\}=\left\{v_{Z_{w} j_{1}}, \ldots\right.$ $\left.\ldots, v_{Z_{w} j_{r}}\right\}, v_{A j} \in V\left(G_{b}(A)\right)$, to $u_{Z_{w} i}$ and remove then the edges which beiong to $G_{b}\left(X_{0}^{\prime}\right)$.
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