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## A Graphical Way to Solve the Boolean Matrix Equations AX=B and XA=B

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A graphical way to find all the solutions of the Boolean matrix equations AX = B and XA = B is proposed and an example is given.

## 1. INTRODUCTION AND BASIC CONCEPTS

As shown by Ledley in [2, pp. 448-484] and in [3, 479-494], the determination of the solutions for the Boolean matrix equations AX = B and XA = B has important applications to switching theory and logical problems. A way to find all the solutions is given in the books cited above. Recently, Rudeanu [4] has derived a complete solution to the equations AX = B and XA = B in parametric form. In this paper we apply a well known graphtheoretic representation of a Boolean matrix to find a graphical way to determine the complete solution to the equations AX = B and XA = B. We assume that the reader is familiar with the basic concepts in graph theory.

By a Boolean matrix  $Q = [q_{ij}]$  we shall mean in this paper a (0, 1)-matrix. The join of two Boolean  $n \times m$  matrices A and B is the matrix  $[a_{ij} \cup b_{ij}]$ , and the product of the matrices C and D of orders  $n \times p$  and  $p \times m$ , respectively, is an  $n \times m$  matrix  $CD = [\bigcup c_{ij}d_{sj}]$ . Further,  $A^{T}$  is the transpose of A and A' the complement of A, i.e.  $A^{T} = [a_{ji}]$  and  $A' = [a'_{ij}]$ .  $A \ge B$  if and only if  $a_{ij} \ge b_{ij}$  for any index pair ij.

It is well known that with every  $m \times n$  Boolean matrix Q one can naturally associate a bipartite graph  $G_b(Q)$  as follows (see e.g. Hedetniemi [1]): The set of vertices  $V(G_b(Q))$  of  $G_b(Q)$  consists of two disjoint subsets  $\{u_i \mid i = 1, ..., m\}$  and  $\{v_j \mid j = 1, ..., n\}$  which correspond to the rows and columns of Q, respectively. An edge  $(u_i, v_j)$  joining  $u_i$  and  $v_j$ , belongs to the edge set  $E(G_b(Q))$  only if  $q_{ij} = 1$ 

in Q. Conversely, every bipartite graph  $G_b$  can be translated into a Boolean matrix according to the rules above.

In the following we shall concentrate on the equation AX = B. As known, the solution of XA = B is analogous to that of AX = B.

## 2. THE BOOLEAN MATRIX EQUATION AX = B

Consider the product of two Boolean matrices A and B, and let the vertex sets of the bipartite graphs  $G_b(A)$  and  $G_b(B)$  be  $V(G_b(A)) = \{u_{Ai} \mid i = 1, ..., m\} \cup$  $\cup \{v_{As} \mid s = 1, ..., k\}$  and  $V(G_b(B)) = \{u_{Bs} \mid s = 1, ..., k\} \cup \{v_{Bj} \mid j = 1, ..., n\}$ . Let us draw the bipartite graphs  $G_b(A)$  and  $G_b(B)$  such that the vertices in the sets  $\{v_{As}\}$ and  $\{u_{Bs}\}$  are common, and denote the graph thus obtained by  $G_b(A) G_b(B)$ . Then, according to the formula  $AB = [\bigcup a_{is}b_{sj}]$ , in the bipartite graph  $G_b(AB)$  a vertex  $u_{ABi}$  is connected by an edge to a vertex  $v_{ABj}$  if and only if there is a path of length two from  $u_{Ai}$  to  $v_{Bi}$  in the graph  $G_b(A) B_b(B)$ . As an illustration, see the graphs of Fig. 1. This graphical form of the product of two Boolean matrices can be applied to the determination of a complete solution to AX = B.



As shown in the literature, the equation AX = B has a solution if and only if the matrix  $(A^{T}B')'$  is a solution to AX = B, i.e.  $A(A^{T}B')' = B$ . Moreover, the solutions of AX = B form a join semilattice, denoted by  $L_0(X)$ , where  $(A^TB')'$  is the greatest element. Hence, if AQ = B,  $Q \cup (A^TB')' = (A^TB')'$ . Thus, in order to obtain the complete set of solutions, one needs to determine the greatest element and the minimum elements of the semilattice  $L_{o}(X)$ , if such exist. First we consider a direct way to determine the graph  $G_b((A^TB'))$ , and the matrix  $(A^TB')$  as well, and then we show an obvious way to find all the solutions of AX = B.

Assume that the equation AX = B has a solution. Now clearly a bipartite graph  $G_b(X_0)$  corresponds to the greatest solution of AX = B, if in the graph  $G_b(A) G_b(X'_0)$ every vertex  $u_{Ai}$ , corresponding to  $u_{Bi}$  in  $G_b(B)$ , is connected by a path of length

two to every vertex  $v_{X_0'j}$ , corresponding to  $v_{Bj}$  in  $G_b(B)$ , for which  $(u_{Bi}, v_{Bj}) \notin E(G_b(B))$ , i.e.  $(u_{Bi}, v_{Bj}) \in E(G_b(B'))$ . Thus the following simple rule can be obtained to find the graph  $G_b(X'_0)$ :

**Rule 1.** Connect in  $G_b(X'_0)$  the vertices  $\Gamma u_{Ai} = \{v_{Ai_1}, \dots, v_{Ai_r}\} = \{u_{X_0'i_1}, \dots, u_{X_0'i_r}\}, u_{Ai} \in V(G_b(A))$ , to all the vertices  $v_{X_0'j}$  for which  $(u_{Bi}, v_{Bj}) \in E(G_b(B'))$ .

It should be noted that the matrix  $X_0$  determined by the rule above does not give any indication of the non-consistency of the equation AX = B.

As an illuminating example, consider the following consistent Boolean matrix equation

(1) 
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The graphs  $G_b(A)$  and  $G_b(B)$  are given in Fig. 2, and the graph  $G_b(X'_0)$  can be seen in the graph  $G_b(A) G_b(X'_0)$  determined by Rule 1. Hence,



Consider now a way to find all the solutions of AX = B. We construct a solution matrix base, denoted by  $Z_1, Z_2, ..., Z_t$ , where every  $Z_w$ , w = 1, ..., t, is a Boolean matrix of the order of X and corresponds to an edge, say  $(u_{Bi}, v_{Bj})$ , of  $G_b(B)$  such that  $G_b(Z_w)$  contains any edge which gives in  $G_b(A) G_b(Z_w)$  a path of length two from  $u_{Ai}$  to  $v_{Z_wj} (=v_{Bj})$  and no edges such that there would be a path of length two in  $G_b(A) G_b(Z_w)$  determining an edge of  $G_b(B')$ . Since the matrix product is distributive with respect to the join operation and  $AZ_w \leq B$ ,  $A(Z_1 \cup Z_2 \cup ... \cup Z_t) = B$ according to the definition of the matrices  $Z_w$ , if  $Z_w > 0$  for any w, w = 1, ..., t. Furthermore, as every  $G_b(Z_w)$  contains all the edges giving in  $G_b(A) G_b(Z_w)$  the edge of  $G_b(B)$  which determines  $G_b(Z_w), Z_1 \cup ... \cup Z_t = (A^TB')' = X_0$ , the greatest element of the solution join semilattice  $L_{\cup}(X)$ . According to the definition of  $Z_w$ , the matrix equation AX = B is consistent if and only if  $Z_w > 0$ , i.e.  $E(G_b(Z_w)) \neq \emptyset$ , for any w, w = 1, ..., t.

A matrix Q is a solution of AX = B, if  $Q \cap Z_w > 0$  for every w, and  $Q_0$  is a minimum element of  $L_{\cup}(X)$  if and only if the equation  $Q_{00} \cap Z_w > 0$  does not hold for any matrix  $Q_{00} < Q_0$ , w = 1, ..., t.

For the determination of a matrix  $Z_w$  corresponding to an edge  $(u_{Bi}, v_{Bj}) \in E(G_b(B))$  we obtain the following simple rule:

**Rule 2.** Connect in  $G_b(Z_w)$  the vertices of  $\Gamma u_{Ai} = \{v_{Ai_1}, ..., v_{Ai_r}\} = \{u_{Z_w i_1}, ..., u_{Z_w i_r}\}, u_{Ai} \in V(G_b(A))$ , to  $v_{Z_w j}$  and remove then the edges which belong to  $G_b(X'_0)$ .





Consider as an example the matrix equation in (1). Fig. 3 shows the determinations of the basis matrices  $Z_1$ ,  $Z_2$ , and  $Z_3$  corresponding to the edges  $(u_{B1}, v_{B1})$ ,  $(u_{B2}, v_{B1})$ , and  $(u_{B2}, v_{B2})$ , respectively. The dotted lines in Fig. 3 mean the edges of  $G_b(X'_0)$ . Since  $Z_1, Z_2, Z_3 > 0$ , the equation in (1) is consistent.

As one can readily check, the minimum elements of  $L_{\cup}(X)$  are  $X_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$  and

$$X_{2} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad Z_{1} \cup Z_{2} \cup Z_{3} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = X_{0} = (A^{\mathsf{T}}B')'. \text{ The other solutions to } AX = B, \text{ which are between } X_{1} \text{ and } X_{0} \text{ in } L_{\cup}(X), \text{ are } X_{3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } X_{4} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

There is an other way to construct a solution matrix base. After determining the matrices  $Z_1, ..., Z_t$  defined above, we substitute the matrix  $Z_w$ , w = 1, ..., t, by a set  $\{Y_{1w}, Y_{2w}, ..., Y_{sww}\}$  of matrices, where  $Y_{1w} \cup ... \cup Y_{sww} = Z_w$ ,  $Y_{tw} > 0$  and  $Y_{tw}$  contains a single one for any  $k, k = 1, ..., s_w$ . Every solution to AX = B is obtained by forming all possible joins (UY) of the matrices in the sets  $\{Y_{1w}, ..., Y_{sww}\}$  such that  $(UY) \cap Z_w > 0$  for any value of w.

In the example considered before,

REFERENCES

$$Y_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad Y_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad Y_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Y_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad Y_{31} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

In the case of the equation XA = B, Rule 1 and Rule 2 can be expressed as follows:

**Rule 1'.** Connect in  $G_b(X'_0)$  the vertices  $\Gamma v_{Ai} = \{u_{Ai_1}, \dots, u_{Ai_r}\} = \{v_{X_0'i_1}, \dots, v_{X_0'i_r}\}, v_{Ai} \in V(G_b(A))$ , to all the vertices  $u_{X_0'j}$  for which  $(u_{Bj}, u_{Bi}) \in E(G_b(B'))$ .

**Rule 2'.** Connect in  $G_b(Z_w)$  the vertices of  $\Gamma v_{Aj} = \{u_{Aj_1}, \ldots, u_{Aj_r}\} = \{v_{Z_wj_1}, \ldots, v_{Z_wj_r}\}, v_{Aj} \in V(G_b(A))$ , to  $u_{Z_wi}$  and remove then the edges which belong to  $G_b(X'_0)$ .

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