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Completing Linear Differential Games by State Dependent Strategies

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This paper deals with sufficient conditions for the completion of a linear differential game at a given time. A known condition is shown to be equivalent to a simpler one which is easier to apply.

1. PRELIMINARIES

In this paper, we shall deal with differential games, given by a differential equation

$$\dot{x} = A(t)\,x + u - v\,, \tag{1}$$

(where x, u, $v \in R^n$, $A(t)$ is locally integrable $n \times n$), two convex compact sets P, $E \subset R^n$ (the *control domains*) and a convex closed set $G \subset R^n$ (the *target set*).* We shall denote the two players by the same letters as their control domains, namely $\mathscr{P}$ (*pursuer*) and $\mathscr{E}$ (*evader*). Given any point (t_0, x_0), the aim of the pursuer (evader) is to choose at any instant its control parameter $u \in P$ ($v \in E$) so that the solution of (1) starting at x_0 for $t = t_0$, reaches G as soon (as late) as possible.

By a *$\mathscr{P}$-control* (*$\mathscr{E}$-control*) we shall understand any measurable function with values in $P(E)$. As in [1] by a *strategy* of $\mathscr{P}(\mathscr{E})$ we shall understand any upper semi-continuous set valued function $U(t, x)$ $(V(t, x))$ defined on R^{n+1} with values being closed convex subsets of $P(E)$.

Given any initial point (t_0, x_0), we consider as an *outcome* of the game under the strategies U, V of $\mathscr{P}$, $\mathscr{E}$ respectively any trajectory of the multivalued differential equation

$$\dot{x} \in A(t)\,x + U(t, x) - V(t, x) \tag{2}$$

* Let us note that the results of this paper remain true if we allow P, E to vary upper semicontinuously with t.

starting at x_0 for $t = t_0$. Note that our assumption guarantees that such a trajectory exists and also that every trajectory can be extended for all $t \geqq t_0$ (cf. [2]).

We shall say that $\mathscr{P}$ can *complete* the game (starting from (t_0, x_0)) at T by the strategy U, if every solution of (2) under $V(t, x) \equiv E$ (i.e. any solution of the equation $\dot{x} \in A(t)\,x + U(t, x) - v(t)$ for any $\mathscr{E}$-control $v(t)$), starting at x_0 for $t = t_0$, satisfies $x(T) \in G$. We shall be interested in sufficient conditions for completing a game.

Let us make some comments concerning our strategy concept. As it is known from simple examples (cf. e.g. [3], [4]), one cannot get through with considering continuous strategies only. However, if discontinuous strategies are used (which corresponds to substituting discontinuous functions of u and v into (1)), difficulties arise in the definition and existence of solution.

A general way to deal with these difficulties is to define the strategy and outcome (trajectory) of the game by some approximation scheme (cf. [4] or [5, 6]), which is a rather complicated apparatus. Sometimes however (cf. [1]) one can work directly with "exact" discontinuous strategies by using Fillipov's concept of solution [7]. As it is shown in [8, 9], this concept leads to an equivalent multivalued differential equation. In order to avoid this construction (for the details of which the reader is referred to [8, 9]) we have defined (following [1]) the strategy directly as a set-valued function.

Also let us note that unlike in [4—6, 10—15] our strategies are independent from all information of the present and past of the game except of the present state (i.e. the variable x) of the game. There are certain reasons which we do not want to go into in this paper to consider (for the game studied in this paper) all the other information about the past and present of the game as unnecessary.

Finally, let us remark that the definition of completion can be equivalently replaced by any of the following two ones:

(i) for any $\mathscr{E}$-control $v(t)$, any solution $x(t)$ of the equation $\dot{x} \in A(t)\,x + U(t, x) - v(t)$ starting at x_0 for $t = t_0$, satisfies $x(T) \in G$,

(ii) for any strategy $V(t, x)$, any solution $x(t)$ of (2) starting at x_0 for $t = t_0$, satisfies $x(T) \in G$.

2. KRASOVSKI'S CONDITION FOR THE EXISTENCE OF A COMPLETING STRATEGY

For a convex set $C \subset R^n$ denote by $h_C : D(C) \to R$ the support function of C, i.e. $h_C(\psi) = \sup\limits_{x \in C} \langle \psi, x \rangle$ and $D(C) = \{\psi \mid \sup\limits_{x \in C} \langle \psi, x \rangle < \infty\}$. Here $\langle x, y \rangle$ means the scalar product.

Denote by $W(\tau, T)$ ($\tau \leqq T$) the set of all points $x \in R^n$ with the following property: For any $\mathscr{E}$-control $v(t)$ there is a $\mathscr{P}$-control $u(t)$ such that every solution $x(t)$ of (1) with $u = u(t)$, $v = v(t)$ starting at x for $t = \tau$, satisfies $x(T) \in G$. This set is used in [1, 10] and it is shown there that $x \in W(\tau, T)$ if and only of for all $\psi \in D(G)$

$$(3) \qquad \langle \psi, X(T, \tau)\,x \rangle \leqq h_G(\psi) + \int_\tau^T [h_P(-X'(T, t)\,\psi) - h_E(-X'(T, t)\,\psi)]\,\mathrm{d}t$$

where prime denotes transpose, $X(t, \tau)$ is the fundamental matrix of the equation $\dot{x} = A(t)\, x$ such that $X(\tau, \tau)$ is the unity matrix.

In [1] it is proven that if $x_0 \in W(t_0, T)$, the game can be completed at T provided the sets $W(\tau, T)$ are *stable* for $\tau \in [t_0, T]$, i.e. for every $x \in W(\tau, T)$, δ sufficiently small and any $\mathscr{E}$-control $v(t)$ there is a $\mathscr{P}$-control $u(t)$ such that the solution of (1) with $u = u(t)$, $v = v(t)$ starting at x for $t = \tau$, satisfies $x(t) \in W(t, T)$ for $t \in [\tau, \tau + \delta]$. The completion of the game can be accomplished by the *extremal* strategy

$$U(t, x) = \begin{cases} P & \text{if} \quad x \in W(\tau, T)\,, \\ U_e(t, x) & \text{if} \quad x \notin W(\tau, T)\,, \end{cases} \tag{4}$$

where

$$U_e(t, x) = \{u \in P \mid \langle q(t, x) - x, u \rangle = h_P(q(t, x) - x)\}$$

and $q(t, x)$ is the projection of x onto $W(t, T)$ i.e. the unique point $q \in W(t, T)$ satisfying $|x - q| = \min\limits_{y \in W(t,T)} |x - y|$, $|.|$ being the Euclidean norm.

The stability condition is not a very practical one. In [1] a following sufficient condition for the stability of the sets $W(t, T)$ is given:

For every (τ, x), $t_0 \leqq \tau \leqq T$, for which

$$\Phi(\tau, x) = \max_{\psi \in D(G) \cap S} [\langle \psi, X(T, \tau)\, x \rangle - h_G(\psi) + \\ + \int_\tau^T [h_E(-X'(T, t)\, \psi - h_P(-X'(T, t)\, \psi)]\, \mathrm{d}t > 0 \tag{5}$$

(S being the unit sphere) the maximum is achieved for a unique $\psi = \psi^*(\tau, x) \in R^n$.

We shall refer to this condition as *Condition C*.

In section 3, we shall prove that this condition is equivalent to another one, which is simpler and easier to apply. In section 4, we apply it to a well known example. The problem of optimality of the completing strategy is discussed in section 5.

We conclude this section by giving a (different to [1]) proof of the completion of the game for the case of condition C being satisfied. The strategy used here is somewhat different to the strategy (4). We first prove two lemmas, the second of which is a modification of [10, Theorem 1.3].

Lemma 1. *Let $X \subset R^p$, $Y \subset R^q$, $f : X \times Y \to R^1$ be continuous and let x_0 be an interior point of X. Let there be a neighbourhood $U \subset X$ of x_0 such that for every $x \subset U$ the set $Z(x)$ of points $y \in Y$ such that $f(x, y) = \varphi(x) = \max\limits_{y \in Y} f(x, y)$ is non-empty. Assume that $Z(x_0) = \{y_0\}$ and that for any sequence $\{x_k, y_k\}$ of points from $X \times Y$ such that $y_k \to y \in R^q \setminus Y$* or *$|y_k| \to \infty$, $f(x_k, y_k) \to -\infty$. Then,*

$$\lim_{x \to x_0} \big(\sup_{y \in Z(x)} |y - y_0| \big) = 0\,.$$

Proof. Assume the contrary. Then, there is a sequence $x_k \to x_0$, $y_k \to y^* \neq y_0$ such that $y_k \in Z(x_k)$. From the last assumption of the lemma and the continuity of f it follows that $y^* \in Y$ and $f(x_0, y^*) = \varphi(x_0)$. Thus, $y^* = y_0$ contrary to our assumption.

Lemma 2. *Let the assumptions of Lemma 1 be satisfied and let $\partial f/\partial x$ exist and be continuous over $X \times Y$. Then $(\partial\varphi/\partial x)(x_0)$ exists and is equal to $(\partial f/\partial x)(x_0, y_0)$.*

Proof. We have for sufficiently small $|h|$:

$$\begin{aligned}\varphi(x_0 + h) - \varphi(x_0) &= \max_{y\in Y} f(x_0 + h, y) - \max_{y\in Y} f(x_0, y) = f(x_0 + h, y_h) - f(x_0, y_0) = \\ &= f(x_0 + h, y_h) - f(x_0, y_h) + f(x_0, y_h) - f(x_0, y_0) \leqq \\ &\leqq \frac{\partial f}{\partial x}(x_0 + \vartheta h, y_h)\, h\end{aligned}$$

where $y_h \in Z(x_0 + h)$, $\vartheta \in (0, 1)$. From Lemma 1 we obtain

$$\limsup_{h\to 0} |h|^{-1} \left(\varphi(x_0 + h) - \varphi(x_0) - \frac{\partial f}{\partial x}(x_0, y_0)\, h\right) \leqq 0\,. \tag{6}$$

On the other hand we have

$$\begin{aligned}\varphi(x_0 + h) - \varphi(x_0) &= f(x_0 + h, y_h) - f(x_0, y_0) = f(x_0 + h, y_h) - f(x_0 + h, y_0) + \\ &+ f(x_0 + h, y_0) - f(x_0, y_0) \geqq \frac{\partial f}{\partial x}(x_0, y_0)\, h + o(h)\end{aligned}$$

which gives the opposite inequality to (6). This proves the lemma.

Denote

$$\omega(t, \psi, x) = \langle \psi, X(T, \tau)\, x\rangle - h_G(\psi) + \int_\tau^T [h_E(-X'(T, t)\, \psi - h_P(-X'(T, t)\, \psi)]\, \mathrm{d}t$$

for $\psi \in D(G)$. Then, $\Phi(t, x) = \max\limits_{\psi\in D(G)\cap S} \omega(t, \psi, x)$, where S is is the unit sphere in R^n.

Theorem 1. *Assume that Condition C holds. Then, for every (t_0, x_0) such that $x_0 \in W(t_0, T)$, the game can be completed at T by the strategy*

$$U(t, x) = \begin{cases} P & \text{if } x \in W(t_0, T)\,, \\ U^*(t, x) & \text{if } x \in W(t_0, T) \end{cases} \tag{7}$$

where

$$U^*(t, x) = \{u \in P \mid \langle \psi^*(t, x), u\rangle = -h_P(-X'(T, t)\, \psi^*(t, x))\,.$$

Proof. First of all we have to prove that $U(t, x)$ is actually a strategy, i.e. that $U(t, x)$ is upper semicontinuous. To do this, assume $\{(t_k, x_k)\} \to (t_0, x_0)$. Without loss of generality we may assume that the sequence $\{\psi^*(t_k, x_k)\}$ is convergent. We prove $\psi^*(t_0, x_0) = \lim_{k\to\infty} \psi^*(t_k, x_k)$.

Assume the contrary. Then, from the unicity of $\psi^*(t_0, x_0)$ and the continuity of ω in ψ it follows that for sufficiently large k, $\omega(t_k, \psi^*(t_k, x_k), x_k) < \omega(t_k, \psi^*(t_0, x_0), x_k)$ contrary to our assumption. Taking now a convergent sequence of points $\{w_k\}$ from $U(t_k, x_k)$ one sees easily that its limit has to lie in $U(t_0, x_0)$, which proves the upper semicontinuity of U.

Now, take any $\mathscr{E}$-control $v(t)$ and any point (τ, ξ) such that $\xi \notin W(\tau, T)$, i.e. $\Phi(\tau, \xi) > 0$. Denote by $x(t)$ the solution of $\dot{x} \in A(t)\,x + U(t, x) - v(t)$ satisfying $x(\tau) = \xi$. From the continuity of Φ it follows that $\Phi(t, x(t)) > 0$ for $t \in [\tau, \tau + \eta]$, $\eta > 0$. Therefore, $\psi^*(t, x(t))$ is unique for $t \in [\tau, \tau + \eta]$ and, consequently, Φ is differentiable at the points $(t, x(t))$, $t \in [\tau, \tau + \eta]$. Hence, by Lemma 2,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\Phi(t, x(t)) &= \frac{\partial}{\partial x}\,\Phi(t, x(t))\,\dot{x}(t) + \frac{\partial}{\partial t}\,\Phi(t, x(t)) = \\ &= \langle X'(T, t)\,\psi^*(t, x(t)), A(t)\,x(t) - v(t)\rangle - h_P(-X'(T, t)\,\psi^*(t, x(t)) - \\ &\quad - \langle X'(T, t)\,\psi^*(t, x(t)), A(t)\,x(t)\rangle + h_P(-X'(T, t)\,\psi^*(t, x(t)) - \\ &\quad - h_E(-X'(T, t)\,\psi^*(t, x(t)) \leqq 0\end{aligned}$$

for allmost all $t \in [\tau, \tau + \eta]$. Consequently, $\Phi(t, x(t))$ is nonincreasing whenever $\Phi(t, x(t)) > 0$. But from this it follows obviously that $\Phi(t_0, x(t_0)) \leqq 0$ implies $\Phi(t, x(t)) \leqq 0$ for $t \geqq t_0$. This proves the theorem, since $\Phi(T, x(T)) = 0$ if and only if $x \in G$.

3. AN EQUIVALENT CONDITION

Let $U, V \subset R^n$ be two non-empty convex closed sets and let $K \subset R^n$ be a convex cone with vertex 0. We say that *V is convexly contained in U with respect to K*, if $K \cap D(U) \subset K \cap D(V)$ and the function $k(\psi) = h_U(\psi) - h_V(\psi)$ is convex on $Q = K \cap D(U)$. If $K = R^n$ we say that V is *convexly contained* in U.

For $K = R^n$ and given U, V denote

$$\begin{aligned}(8)\qquad Z_c(\psi) &= \{x \mid \langle\psi, x\rangle \leqq k(\psi) + c|\psi|\} \quad \text{for} \quad \psi \in D(U)\,, \\ Z_c &= \bigcap_{\psi\in D(U)} Z_c(\psi) = \bigcap_{\psi\in D(U)\cap S} Z_c(\psi)\,.\end{aligned}$$

Note that Z_c is a convex set.

Proposition 1. *V is convexly contained in U if and only if for every $c > 0$, all boundary points of the sets Z_c are regular.*

 For the proof we need two lemmas:

Lemma 3. *Let M, $N \subset R^n$ be two convex closed sets such that $N \neq \emptyset$ and all boundary points of N are regular. Then, all boundary points of the set $M + N$ are regular.*

Proof. Every boundary point z of $M + N$ satisfies $\langle \psi_0, z \rangle = h_{M+N}(\psi_0)$ for some $\psi_0 \in S$ and $z = x + y$, where x, y are boundary points of M, N respectively and $\langle \psi_0, x \rangle = h_M(\psi_0)$, $\langle \psi_0, y \rangle = h_N(\psi_0)$. Since y is a regular boundary point of N, we have for every $\psi \in S \setminus \{\psi_0\}$:

$$\langle \psi, z \rangle = \langle \psi, x \rangle + \langle \psi, y \rangle < h_M(\psi) + h_N(\psi) = h_{M+N}(\psi),$$

q.e.d.

For $C \subset R^n$ convex, denote $\Delta(C)$ the set of such $\psi \in R^n$ for which $h_C(\psi) = \langle \psi, x \rangle$ for some $x \in C$.

Lemma 4. *Let $C \subset R^n$, $C \neq R^n$ be convex closed. Then, $\Delta(C)$ contains the relative interior of $D(C)$.*

Proof follows from [18, Theorem 2.3.4 and Corollary 2.3.5.3].

Lemma 5. *Let f be convex and lower semicontinuous on a convex set D. Then, for every $y \in D$, $\lim_{\lambda \to 0} f((1 - \lambda) x + \lambda y) = f(x)$.*

Proof. From $f((1 - \lambda) x + \lambda y) \leqq (1 - \lambda) f(x) + \lambda f(y)$ we have

$$\lim_{\lambda \to 0} f((1 - \lambda) x + \lambda y) \leqq \lim_{\lambda \to 0} [(1 - \lambda) f(x) + \lambda f(y)] = f(x).$$

The opposite inequality follows from the lower semicontinuity of f.

Proof of proposition 1. Assume that V is convexly contained in U. Then, for every $c \geqq 0$, $k(\psi) + c|\psi|$ is a convex and homogeneous function defined on the convex cone $D(U)$ and, therefore, its closure is the support function of some convex closed set Y (cf. [18, Theorem 13.2]).

We prove $Y = Z_c$. Since $Y \subset Z_c(\psi)$ for all $\psi \in D(U)$, also $Y \subset \bigcap_{\psi \in D(U)} Z_c(\psi) = Z_c$. The converse inclusion follows from the obvious fact, that, $h_{Z_c}(\psi) \leqq k(\psi) + c|\psi|$ for all $\psi \in D(U)$.

The regularity of the boundary points of Z_c follows now from $Z_c = Z_0 + B_c$ and Lemma 3, where $B_c = \{x \mid |x| \leqq c\}$.

For the proof of the reverse implication of the proposition, we first prove $h_{Z_c(\psi)}(\psi) = h_{Z_c}(\psi)$ for $c > 0$. Obviously, $h_{Z_c}(\psi) \leqq h_{Z_c(\psi)}(\psi)$. Thus, $\{\psi \mid h_{Z_c(\psi)}(\psi) < \infty\} \subset D(Z_c)$. First, we prove the equality $h_{Z_c(\psi)}(\psi) = h_{Z_c}(\psi)$ for all $\psi \in \Delta(Z_c)$.

Let $\psi_0 \in \Delta(C) \cap S$. Then, there is a point $x_0 \in Z_c$ such that $\langle \psi, x_0 \rangle = h_{Z_c}(\psi_0)$. Since $Z_c = \bigcap_{\psi \in D(U) \cap S} Z_c(\psi)$ and x_0 is a boundary point of Z_c, there is a sequence $\{\psi_k\} \in D(U) \cap S$ such that $\varrho(\partial Z_c(\psi_k), x_0) \to 0$ where ∂Z_c stands for the boundary of Z_c. Passing to a subsequence if necessary we may assume $\psi_k \to \psi^*$, $|\psi^*| = 1$. Obviously, $x_0 \in \partial Z_c(\psi^*)$ and $\langle \psi^*, x_0 \rangle = h_{Z_c}(\psi^*)$. But from the regularity of x_0 it follows $\psi_0 = \psi^*$.

Thus, $x_0 \in \partial Z_c(\psi_0)$, and, consequently, $\langle \psi_0, x_0 \rangle = h_{Z_c(\psi_0)}(\psi_0)$.

Now, take any point $\psi_0 \in D(Z_c)$. We take another point ψ_1 from the relative interior of $D(Z_c)$. Then, $\psi_\lambda = (1 - \lambda)\,\psi_0 + \lambda\psi_1$ is also contained in the relative interior of $D(Z_c)$ (and thus, by lemma 4, in $\Delta(Z_c)$) for all $\lambda \in (0, 1]$. h_U, h_V, h_{Z_c} as support functions are convex and lower semicontinuous (cf. [18, Theorem 13.2]) which by Lemma 5 implies

$$h_{Z_c}(\psi_0) = \lim_{\lambda \to 0} h_{Z_c}(\psi_\lambda) = \lim_{\lambda \to 0} h_{Z_c(\psi_\lambda)}(\psi_\lambda) = \lim_{\lambda \to 0} h_U(\psi_\lambda) -$$
$$- \lim_{\lambda \to 0} h_V(\psi_\lambda) + \lim_{\lambda \to 0} c|\psi_\lambda| = h_U(\psi_0) - h_V(\psi_0) + c|\psi_0| = h_{Z_c(\psi_0)}(\psi_0)\,.$$

As a consequence we obtain that $h_{Z_c(\psi)}(\psi) = k(\psi) + c|\psi|$ is convex for every $c > 0$, which is possible only if k is convex q.e.d.

Proposition 2. *Let V be convexly contained in U with respect to K. Then, λV is convexly contained in U with respect to K for all $\lambda \in [0, 1]$.*

Proof. Since both $h_U(\psi) - h_V(\psi)$ and $(1 - \lambda)\,h_V(\psi)$ are convex on $D(U) \cap K$, the same is true for $h_U(\psi) - h_{\lambda V}(\psi) = h_U(\psi) - \lambda\, h_V(\psi) = h_U(\psi) - h_V(\psi) + (1 - \lambda)\,h_V(\psi)$, q.e.d.

Theorem 2. *Condition C is satisfied if and only if for every $\tau \in [t_0, T]$, the function*

$$\int_\tau^T [h_P(-X'(T, t)\,\psi) - h_E(-X'(T, t)\,\psi)]\,\mathrm{d}t + h_G(\psi) \tag{9}$$

is convex, i.e. if $F(\tau)$ is convexly contained in $Q(\tau)$, where

$$F(\tau) = \bigcap_{\psi \in R^n} \left\{x \mid \langle \psi, x \rangle \leqq \int_\tau^T h_E(-X'(T, t)\,\psi)\,\mathrm{d}t\right\},$$

$$Q(\tau) = G + \bigcap_{\psi \in R^n} \left\{x \mid \langle \psi, x \rangle \leqq \int_\tau^T h_P(-X'(T, t)\,\psi)\,\mathrm{d}t\right\}.$$

Proof. We note first that the sets $\{x \mid \Phi(\tau, x) \leqq c\}$ are convex, the sets $\{x \mid \Phi(\tau, x) = c\}$ being their boundaries. Condition C says the same as that all boundary points of the set $\{x \mid \Phi(\tau, x) \leqq c\}$ are regular.

Denote $Y_c(\tau) = X(\tau, T)\,\{x \mid \Phi(\tau, x) \leqq c\}$. Obviously, all boundary points of the sets $Y_c(\tau)$ are also regular. Further, we have

$$Y_c(\tau) = \bigcap_{\psi \in D(G) \cap S} \Big\{x \mid \langle \psi, x\rangle \leqq h_G(\psi) +$$

$$+ \int_\tau^T [h_P(-X'(T, t)\,\psi) - h_E(-X'(T, t)\,\psi)\,\mathrm{d}t] + c\Big\}$$

for $c \geqq 0$. Hence, the theorem follows from Proposition 1 if we replace U, V, Z_c by $Q(\tau)$, $F(\tau)$, $Y_c(\tau)$ respectively.

Corollary. *If $F(\tau)$ is convexly contained in $Q(\tau)$ for all $\tau \in [t_0, T]$ then for every $x_0 \in W(t_0, T)$, the game can be completed at T.*

Remarks. 1. If for all $t \in [\tau, T]$, $-X(T, t)\,E$ is convexly contained in $-X(T, t)\,P$ with respect to $D(G)$ (or, equivalently, E is convexly contained in P with respect to $-X(T, t)\,D(G)$) then $F(\tau)$ is convexly contained in $Q(\tau)$.

This follows from the fact that $D(F(\tau)) = R^n$, $D(Q(\tau)) = D(G)$ and

$$h_{Q(\tau)}(\psi) - h_{F(\tau)}(\psi) = \int_\tau^T [h_P(-X'(T, t)\,\psi) - h_E(-X'(T, t)\,\psi)]\,\mathrm{d}t + h_G(\psi) =$$

$$= \int_\tau^T [h_{-X(T,t)P}(\psi) - h_{-X(T,t)E}(\psi)]\,\mathrm{d}t + h_G(\psi)$$

for all $\psi \in D(G)$.

2. The condition (9) of Theorem 2 appeared recently in [15] where it is also used as a sufficient condition for completing the game. However, the strategy concept of [15] is different in that the $\mathscr{P}$-strategy can depend not only on the state, but also on a small piece of the $\mathscr{E}$-control in the future.

4. AN EXAMPLE

In this section, we apply the results of section 2 to the example of [11] (cf. also [10]) the pursuit-evasion game of isotropic rockets, with a more general type of constraints.

Consider a pursuit-evasion problem, defined by two differential equations

$$(10) \qquad \ddot{x} = -\alpha\dot{x} + u\,,$$
$$\ddot{y} = -\beta\dot{y} - v$$

where $x,\ y \in R^n$, $u \in P_0 \subset R^n$, $v \in E_0 \subset R^n$ (P_0, E_0 convex compact) and α, β are positive numbers. The target set is the plane $x = y$. By substituting $x_1 = x$, $x_2 = \dot{x}$,

$x_3 = y$, $x_4 = \dot{y}$ we turn (10) into the standard form

$$\dot{x}_1 = x_2 ,$$
$$\dot{x}_2 = -\alpha x_2 + u ,$$
$$\dot{x}_3 = x_4 ,$$
$$\dot{x}_4 = -\beta x_4 - v ,$$
$$P = \{(0, u, 0, 0) \mid u \in P_0\} , \quad E = \{(0, 0, 0, v) \mid v \in E_0\} .$$

The system being autonomous, we can without loss of generality assume $T = 0$, and write $X(t) = X(0, t) = X(-t, 0)$. Then we have for $t \geqq 0$

$$X(t) = \begin{pmatrix} 1 & e_1(t) & 0 & 0 \\ 0 & e^{-\alpha t} & 0 & 0 \\ 0 & 0 & 1 & f_1(t) \\ 0 & 0 & 0 & e^{-\beta t} \end{pmatrix}$$

where

$$e_1(t) = \int_0^t e^{-\alpha\xi} \, d\xi , \quad f_1(t) = \int_0^t e^{-\beta\xi} \, d\xi$$

and

$$D(G) = \{(\chi, 0, -\chi, 0) \mid \chi \in R^n\} .$$

For $t \geqq 0$ and $\psi \in D(G)$, $\psi = (\chi, 0, -\chi, 0)$, we have

$$h_P(-X'(t)\,\psi) - h_E(-X'(t)\,\psi) = e_1(t)\, h_{P_0}(-\chi) - f_1(t)\, h_{E_0}(\chi) .$$

Thus, according to Remark 1 of section 2, the game can be completed at 0 from every point of $W(-t, 0)$, provided the set $-f_1(t)\, E_0$ is convexly contained in $e_1(t)\, P_0$ for every $t \geqq 0$. Since $f_1 e_1$ allways lies between 1 and $\alpha\beta^{-1}$, in virtue of proposition 2 we can replace this condition by: $-\max\{1, \alpha\beta^{-1}\}\, E_0$ is convexly contained in P_0.

As a consequence, we obtain the following result:

If $-\max\{1, \alpha\beta^{-1}\}\, E_0 \subset \operatorname{int} P_0$, *then the game can be completed from every point of* R^n.

To prove this, we note that there is an $\eta > 0$ such that

$$P_1 = -(\max\{1, \alpha\beta^{-1}\} + \eta)\, E_0 \subset P_0$$

and that it suffices to prove that the game can be completed from every point with $\mathscr{P}$-strategies with values from the smaller set P_1. For this auxiliary game, obviously $-f_1 E_0$ is convexly contained is $e_1 P_1$ for every t so that the game can be completed from every point of $\bigcup_{t \geqq 0} W(-t, 0)$. We have for every $\psi = (\chi, 0, -\chi, 0)$:

$$\lim_{\tau \to \infty} \gamma(\tau, \psi) = \infty ,$$

where

$$\gamma(\tau, \psi) = \int_0^\tau [h_P(-X'(t)\,\psi) - h_E(-X'(t)\,\psi)]\,\mathrm{d}t =$$

$$= \int_0^\tau [e_1(t)\, h_{P_1}(-\chi) - f_1(t)\, h_{E_0}(\chi)]\,\mathrm{d}t \geqq \tau\eta\,.$$

Since (cf. (3))

$$X(\tau)\, W(-\tau, 0) = \bigcap_{\psi\in S} \{x \mid \langle\psi, x\rangle \leqq h_G(\psi) + \gamma(\tau, \psi)\}$$

and $X(\tau)$ is bounded, we have

$$\bigcup_{\tau\leqq 0} W(\tau, 0) = R^n\,,$$

q.e.d.

5. OPTIMALITY OF THE COMPLETING STRATEGY

So far we have not been concerned with the problem of optimality of the pursuit strategy, in the sense of minimizing the time at which the outcome trajectories enter G.

To define the optimality concept precisely we need the notion of the ***value*** of a game. Since the definition of it is rather lengthy, we refer the reader to the papers [4–6].

Given an initial point (t_0, x_0), we shall call a strategy $U(t, x)$ of $\mathscr{P}$ optimal, if for every control $v(t)$ every trajectory of the system $\dot{x} \in A(t)\, x + U(t, x) - v(t)$ starting at (t_0, x_0) satisfies $x(\tau) \in G$ for some $\tau \in [t_0, T]$, where $T - t_0$ is the value of the game in the sense of [4–6].

In general, the strategies, (4), (7) have no relation to the optimal ones except in a special case, described in the following

Theorem 3. *Let for all* $t, t' \in [t_0, T]$, $t \leqq t'$, $W(t_0, t) \subset W(t_0, t')$ *is valid and let* $x_0 \in W(t_0, T) \setminus \bigcup_{t\in[t_0, T)} W(t_0, t)$. *Assume that Condition C is satisfied. Then, the strategy* $U(t, x)$ *given by* (7) *is* ***optimal*** *and* $T - t_0$ *is the value of the game.*

Proof. It suffices to prove that for every $\tau \in [t_0, T)$ there is an $\mathscr{E}$-control $v_\tau(t)$ such that under any $\mathscr{P}$-control $u(t)$ the solution of (1) with $u = u(t)$, $v = v(t)$ satisfies $x(t) \notin G$ for $t \in [t_0, \tau]$.

Since for $\tau \in [0, T)$, $x_0 \notin W(t_0, \tau)$, there is a $\psi_\tau \in S$ such that

$$\langle\psi_\tau, x_0\rangle > h_{W(t_0,\tau)}(\psi_\tau) \geqq h_{W(t_0,t)}(\psi_t) \tag{11}$$

for all $t \in [t_0, \tau]$.

Choose $v_\tau(t)$ so that $\langle X'(t_0, t)\,\psi_\tau, v_\tau(t)\rangle = h_E(-X'(t_0, t)\,\psi_\tau)$. From the upper semicontinuity of the set $\{v \in E \mid \langle X'(\tau, t)\,\psi_\tau, v_\tau(t)\rangle = h_E(-X'(T, t)\,\psi_\tau)$ and Fillippov's implicit function lemma [16] it follows that v_τ can be choosen measurable. By multiplying both sides of (3) by $X'(\tau, T)$ we obtain

$$W(t_0, \tau) = \left\{x \mid \langle \psi, x\rangle = h_G(X'(t_0, \tau)\,\psi) + \int_{t_0}^{\tau} [-h_E(-X'(t_0, t)\,\psi) + \right.$$
$$\left. + h_P(-X'(t_0, \tau)\,\psi]\,\mathrm{d}t \text{ for all } \psi \in X'(\tau, t_0)\,D(G)\right\}.$$

In virtue of Condition C we have

$$h_{W(t_0,\tau)}(\psi) = h_G(X'(t_0, \tau)\,\psi) + \int_{t_0}^{\tau} [-h_E(-X'(t_0, t)\,\psi) +$$
$$+ h_P(-X'(t_0, t)\,\psi)]\,\mathrm{d}t\,.$$

Thus

$$\langle \psi_\tau, x_0\rangle > h_G(X'(t_0, \tau)\,\psi_\tau) + \int_{t_0}^{\tau} [-h_E(-X'(t_0, t)\,\psi_\tau) + h_P(-X'(t_0, t)\,\psi_\tau)]\,\mathrm{d}t\,.$$

But we have

$$x(\tau) = X(\tau, t_0)\,x_0 + \int_{t_0}^{\tau} X(t, t_0)\,[-v_\tau(t) + u(t)]\,\mathrm{d}t\,,$$

$$\langle X'(t_0, \tau)\,\psi_\tau, x_\tau\rangle = \langle \psi_\tau, x_0\rangle + \int_{t_0}^{\tau} [\langle -X'(t_0, t)\,\psi_\tau, v_\tau(t)\rangle +$$
$$+ \langle X'(t_0, t)\,\psi_\tau(t), u(t)\rangle]\,\mathrm{d}t = \langle \psi_\tau, x_0\rangle + \int_{t_0}^{\tau} [h_E(-X'(t_0, t)\,\psi_\tau) +$$
$$+ \langle X'(t_0, t)\,\psi_\tau, u(t)\rangle]\,\mathrm{d}t > h_G(X'(t_0, \tau)\,\psi_\tau)\,.$$

Consequently, $x(\tau) \notin G$. Further, using (11) we obtain by a similar computation $x(t) \notin G$ for $t \leqq \tau$.

The following theorem gives a sufficient condition for the inclusion assumption of Theorem 3.

Theorem 4. *Assume that for every* $t \in [t_0, T]$, $x \in G$, $\psi \in M(x)$,

$$\min_u \max_v \langle \psi, A(t)\,x + u - v\rangle \leqq 0 \tag{12}$$

(*where* $M(x)$ *is the set of all support normals to* G *at* x) *is satisfied. Then, for every* $t_1 \leqq t_2$ *from* $[t_0, T]$, $W(t_0, t_1) \subset W(t_0, t_2)$.

Proof. From [17, § 3] it follows that G is selectively invariant, i.e. there is an upper semicontinuous set-valued function $U(t, x)$ such that all solutions of the equation

$$\dot{x} \in A(t)\,x + U(t, x) - E \tag{13}$$

starting in G remain in G.

Assume now $x_0 \in W(t_0, t_1)$. Then, for any $\mathscr{E}$-control $v(t)$ on $[t_0, t_2]$, there is a $\mathscr{P}$-control $u(t)$ on $[t_0, t_1]$ such that the solution $x(t)$ of (1) with $u = u(t)$, $v = v(t)$ passing through x_0 at $t = t_0$ satisfies $x(t_1) \in G$. Extend $x(t)$ to the interval $[t_0, t_2]$ by taking any solution of the equation $\dot{x} \in A(t)\,x + U(t, x) - v(t)$ on $[t_1, t_2]$ starting at $x(t_1)$. Since this is also a solution of (13), we have $x(t_2) \in G$. Using Fillipov's implicit function lemma we find that there is a control $u(t) \in U(t, x(t)) \subset P$ such that $x(t)$ on $[t_1, t_2]$ is a solution of (1) with $u = u(t)$, $v = v(t)$. Thus, for $v(t)$ we have constructed a control $u(t)$ such that $x(t_2) \in G$, q.e.d.

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