KYBERNETIKA — VOLUME 9 (1973), NUMBER 5

Determining the Maximal Degree of a Tree Given by Its Distance Matrix

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In the paper a method of determining the maximal degree of a vertex of a tree given by the matrix of distances of its terminal vertices is described. The problem to give such a method was proposed by A. A. Zykov.

In [2] V. G. Vizing quotes a problem given by A. A. Zykov and concerning the trees which are given be their distance matrix.

Let T be a finite tree, let $v_1, \ldots, v_k$ be its terminal vertices. For $i = 1, \ldots, k$ and $j = 1, \ldots, k$ let d_{ij} be the distance of the vertices v_i, v_j in the tree T. The matrix $\mathbf{D} = \|d_{ij}\|$ will be called the distance matrix of the tree T. A. A. Zykov has proved that each finite tree is uniquely (up to isomorphism) determined by its distance matrix.

He asked a question to find a method of determining the maximal degree of a vertex of a tree, if the distance matrix of this tree is given. It is possible to reconstruct this tree and then to find this degree, but this is very complicated. Thus we shall give a simpler method. This problem (similarly as all problems concerning the degrees of vertices of trees) may be of importance in the applications of graph theory in the theory of communication networks.

We shall use the ternary operation $P(x, y, z)$ on the vertex set of a tree defined by L. Nebeský [1]. If x, y, z are three vertices of a tree T, then $P(x, y, z)$ is the vertex of T which is contained in the arc joining arbitrary two of the vertices x, y, z.

L. Nebeský has proved that this vertex is determined uniquely, therefore $P(x, y, z)$ is a ternary operation defined for all triples of vertices of T. Further this operation is symmetric, i.e. for any permutation π of the set $\{x, y, z\}$ we have

$$P(\pi(x), \pi(y), \pi(z)) = P(x, y, z) .$$

We shall prove a theorem.

Theorem 1. *Let x, y, z, t be four vertices of a finite tree T. Then the following two assertions are equivalent*:

$$P(x, y, z) = P(x, y, t) = P(x, z, t) = P(y, z, t) . \tag{1}$$

$$d(x, y) + d(z, t) = d(x, z) + d(y, t) . \tag{2}$$

Here $d(u, v)$ denotes the distance between u and v in T.

Proof. (1) $\Rightarrow$ (2). Let $u = P(x, y, z)$. As u lies on the arc joining x and y, we have

$$d(x, y) = d(u, x) + d(u, y) .$$

Analogously

$$d(x, z) = d(u, x) + d(u, z) ,$$

$$d(z, t) = d(u, z) + d(u, t) ,$$

$$d(y, t) = d(u, y) + d(u, t) .$$

Then

$$d(x, y) + d(z, t) = d(u, x) + d(u, y) + d(u, z) + d(u, t) = d(x, z) + d(y, t) .$$

(2) $\Rightarrow$ (1). Assume $P(x, y, z) = u$, $P(x, y, t) = v$, $u \neq v$. The vertices u, v lie both on the arc joining x and y; without the loss of generality, suppose that $d(u, x) < d(v, x)$. Then

$$d(x, y) = d(u, x) + d(u, v) + d(v, y) .$$

As $P(x, y, z) = u$, we have

$$d(x, z) = d(u, x) + d(u, z) .$$

As $P(x, y, t) = v$, we have

$$d(y, t) = d(v, y) + d(v, t) .$$

As u lies between y and z and v lies between y and u, u lies between v and z. Analogously, v lies between u and t. Therefore

$$d(z, t) = d(u, z) + d(u, v) + d(v, t) .$$

Substituting into (2) we obtain

$$d(u, x) + d(u, v) + d(v, y) + d(u, z) + d(u, v) + d(v, t) = \\ = d(u, x) + d(u, z) + d(v, y) + d(v, t) ,$$

which gives

$$d(u, v) = 0 ,$$

which is a contradiction with the assumption $u \neq v$.

We shall define a quaternary relation ϱ on the vertex set of T so that $(x, y, z, t) \in \varrho$, if and only if (1) (and therefore also (2)) holds.

We shall prove a further theorem.

Theorem 2. *Let T be a tree, let $x_1, \ldots, x_k, y$ be some its $k + 1$ pairwise different vertices. Let $P(a, b, c) = u$ (where u is some vertex of T) for arbitrary three pairwise different elements a, b, c of the set $\{x_1, \ldots, x_k\}$. Let $(x_1, x_2, x_i, y) \in \varrho$ for $i = 3, \ldots, k$. Then $P(a, b, y) = u$ for an arbitrary pair of different elements a, b of $\{x_1, \ldots, x_k\}$.*

Proof. As $(x_1, x_2, x_i, y) \in \varrho$ for $i = 3, \ldots, k$ and $P(x_1, x_2, x_i) = u$, we have also $P(x_1, x_2, y) = P(x, x_i, y) = P(x_2, x_i, y) = u$. This means that u lies on the arc joining x_i and y for $i = 1, \ldots, k$. If a, b are arbitrary two different elements of the set $\{x_1, \ldots, x_k\}$, the vertex u lies between a and y and between b and y. It lies also between a and b according to the assumption and therefore $P(a, b, y) = u$.

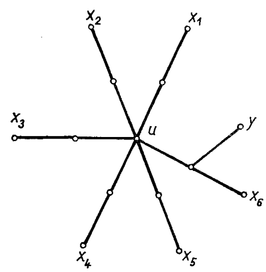

Fig. 1.

Fig. 1 shows the case when $k = 6$, $P(a, b, c) = u$ for arbitrary three pairwise different elements of $\{x_1, \ldots, x_6\}$, $(x_1, x_2, x_i, y) \in \varrho$ for $i = 3, 4, 5$, but not for $i = 6$. We see that then $P(x_1, x_6, y) \neq u$.

Theorem 3. *Let T be a finite tree, let u be its vertex with the degree $k \geqq 3$. Then there exist k terminal vertices $x_1, \ldots, x_k$ of T such that for any three pairwise different elements a, b, c of the set $\{x_1, \ldots, x_k\}$ we have $P(a, b, c) = u$.*

Proof. Let $e_1, \ldots, e_k$ be the edges incident with u. For each $i = 1, \ldots, k$ consider an arc A_i of the maximal length such that one of its terminal vertices is u and its terminal edge incident with u is e_i. Let x_i be the other terminal vertex of A_i; the maximality of the length of A_i implies that x_i is a terminal vertex of T. For $i \neq j$ arcs A_i, A_j have only one common vertex u, because otherwise their union would form a circuit, which is impossible in a tree. Therefore x_i for $i = 1, \ldots, k$ are pairwise different and the arc joining x_i and x_j for $i \neq j$ is the union of the arcs A_i, A_j and therefore it contains u. This implies that for any three pairwise different elements a, b, c of the set $\{x_1, \ldots, x_k\}$ we have $P(a, b, c) = u$.

For finding the maximal degree of a vertex of T we need to find the set $\{x_1, \ldots, x_k\}$ described in Theorem 3 with the maximal possible cardinality. This is the basis of the following algorithm.

Algorithm. Let the distance matrix $\mathbf{D} = \|d_{ij}\|$ of a finite tree T be given and let it have n rows. Let $N = \{1, 2, \ldots, n\}$. We shall use an integral variable k.

(A) If $n = 2$, then the maximal degree of a vertex of T is 2, if $d_{12} > 1$, or 1 otherwise. If not go to (B).

(B) If $n = 3$, then the maximal degree of a vertex of T is 3. If not, go to (C).

(C) Put $k := 4$. Go to (D).

(D) For any four distinct elements p, q, r, s of N we find $f(p, q, r, s) = d_{pq} + d_{rs} - d_{pr} - d_{qs}$. Let $\mathfrak{M}_4 = \{\{p, q, r, s\} \mid f(p, q, r, s) = 0\}$. Go to (E).

(E) If $\mathfrak{M}_k = \emptyset$, go to (G), else put $k := k + 1$ and go to (F).

(F) Put $\mathfrak{M}_k = \{\{p_1, \ldots, p_{k-1}, q\} \mid \{p_1, \ldots, p_{k-1}\} \in \mathfrak{M}_{k-1}, \{p_1, p_2, p_i, q\} \in \mathfrak{M}_4$ for $i = 3, \ldots, k - 1\}$. Go to (E).

(G) The maximal degree of a vertex of T is $k - 1$.

(Received November 10, 1972.)

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