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An Axiomatic Characterization of Generalized Directed-divergence

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A characterization theorem for the generalized directed-divergence defined in (1.1) is proved by assuming a set of five postulates (2.1)—(2.5).

1. INTRODUCTION

Let $P = (p_1, \ldots, p_n)$, $Q = (q_1, \ldots, q_n)$, $R = (r_1, \ldots, r_n)$, p_i, q_i, $r_i \geqq 0$, $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = \sum_{i=1}^{n} r_i = 1$ be three finite discrete probability distributions. Then we define the *generalized directed-divergence* by the expression, (refer [1]),

$$I_n(p_1, \ldots, p_n; q_1, \ldots, q_n; r_1, \ldots, r_n) = \sum_{i=1}^{n} p_i \log (q_i \mid r_i) . \tag{1.1}$$

Here the convention $0 \log 0 = 0$ is followed and logarithms will be to the base 2. Also whenever q_i or r_i is zero then the corresponding p_i is also zero and $\log (q_i \mid r_i)$ is to be taken as $(\log q_i - \log r_i)$.

For $n = 2$, (1.1) takes the following form:

$$I_2(p, 1 - p; q, 1 - q; r, 1 - r) = = p \log (q/r) + (1 - p) \log \{(1 - q)/(1 - r)\} , \tag{1.2}$$

for $p, q, r \in K$, where $K =]0, 1[\times]0, 1[\times]0, 1[\cup \{(0, y, z)\} \cup \{(1, y', z')\}$, with $y, z \in [0, 1)$ and $y', z' \in (0, 1]$.

For $P \equiv Q$, (1.1) reduces to the ordinary measure of directed-divergence ([5], [7]) as given below:

$$I_n(p_1, \ldots, p_n; r_1, \ldots, r_n) = \sum_{i=1}^{n} p_i \log (p_i/r_i) . \tag{1.3}$$

An axiomatic characterization of (1.3) was given earlier in [2] and that its theorem lacks mathematical rigour was pointed out by us in [6].

In this paper, we will prove a characterization theorem for the generalized directed-divergence defined in (1.1) by assuming a set of five postulates.

A more general measure, called the generalized directed-divergence of type β, was discussed and characterized through axioms by us in [4]. The characterization theorem in [4] was proved entirely on different lines than those of the present paper.

2. POSTULATES

In this section we give a set of five postulates which will be used in the next section to establish a characterization theorem for the generalized directed-divergence.

Postulate 1 (Recursivity).

$$I_n(p_1, \ldots, p_n; q_1, \ldots, q_n; r_1, \ldots, r_n) = \tag{2.1}$$
$$= I_{n-1}(p_1 + p_2, p_3, \ldots, p_n; q_1 + q_2, q_3, \ldots, q_n; r_1 + r_2, r_3, \ldots, r_n) +$$
$$+ (p_1 + p_2) I_2[p_1/(p_1 + p_2), p_2/(p_1 + p_2); q_1/(q_1 + q_2), q_2/(q_1 + q_2);$$
$$r_1/(r_1 + r_2), r_2/(r_1 + r_2)],$$

for $p_1 + p_2, q_1 + q_2, r_1 + r_2 > 0$.

Postulate 2 (Symmetry).

$$I_3(p_1, p_2, p_3; q_1, q_2, q_3; r_1, r_2, r_3) = I_3(p_a, p_b, p_c; q_a, q_b, q_c; r_a, r_b, r_c) \tag{2.2}$$

where $\{a, b, c\}$ is an arbitrary permutation of $\{1, 2, 3\}$.

Postulate 3 (Derivative). Let

$$f(p, q, r) = I_2(p, 1 - p; q, 1 - q; r, 1 - r), \tag{2.3}$$

for all $(p, q, r) \in K$ where K is as given in (1.2). Also let f have continuous first partial derivatives with respect to all the three variables $p, q, r \in (0, 1)$.

Postulate 4 (Nullity).

$$f(p, p, p) = 0 \quad \text{for} \quad p \in (0, 1). \tag{2.4}$$

Postulate 5 (Normalization).

$$f(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}) = \tfrac{1}{3} \quad \text{and} \quad f(\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}) = 0. \tag{2.5}$$

3. CHARACTERIZATION THEOREM

In this section we will prove the following theorem:

Theorem. *The only function I_n satisfying the postulates* 1 *to* 5 *is the generalized directed-divergence given by* (1.1).

Proof. The proof of the theorem depends on the following lemmas.

Lemma 1. *I_2 is symmetric.*

Proof. The postulate 1 for $n = 3$, $p_1 + p_2, q_1 + q_2, r_1 + r_2 > 0$, give

$$I_3(p_1, p_2, p_3; q_1, q_2, q_3; r_1, r_2, r_3) = \tag{3.1}$$
$$= I_2(p_1 + p_2, p_3; q_1 + q_2, q_3; r_1 + r_2, r_3) +$$
$$+ (p_1 + p_2) I_2 \left[p_1/(p_1 + p_2), p_2/(p_1 + p_2);\right.$$
$$\left.\frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2}; \frac{r_1}{r_1 + r_2}, \frac{r_2}{r_1 + r_2}\right],$$

and

$$I_3(p_2, p_1, p_3; q_2, q_1, q_3; r_2, r_1, r_3) = \tag{3.2}$$
$$= I_2(p_2 + p_1, p_3; q_2 + q_1, q_3; r_2 + r_1, r_3) +$$
$$+ (p_2 + p_1) I_2 \left[p_2/(p_2 + p_1), p_1/(p_2 + p_1); \frac{q_2}{q_1 + q_2}, \frac{q_1}{q_1 + q_2}; \frac{r_2}{r_1 + r_2}, \frac{r_1}{r_1 + r_2}\right]$$

Thus postulate 2, (3.1) and (3.2) prove lemma 1, which is equivalent to

$$f(p, q, r) = f(1 - p, 1 - q, 1 - r), \tag{3.3}$$

for $(p, q, r) \in K$.

In particular, (3.3) gives

$$f(0, 0, 0) = f(1, 1, 1). \tag{3.4}$$

Lemma 2. *f defined by* (2.3) *satisfies the functional equation*

$$f(x, y, z) + (1 - x) f\left(\frac{x}{1 - u}, \frac{y}{1 - v}, \frac{z}{1 - w}\right) = \tag{3.5}$$
$$= f(u, v, w) + (1 - u) f\left(\frac{u}{1 - x}, \frac{v}{1 - y}, \frac{w}{1 - z}\right)$$

for $x, y, z, u, v, w \in [0, 1[$ *with* $x + u, y + v, z + w \in]0, 1]$ *and that*

$$(3.6) \qquad f(x, y, z) = x \log \frac{y}{z} + (1 - x) \log \frac{1 - y}{1 - z},$$

for $(x, y, z) \in K$.

Proof. The postulate 2 gives

$$(3.7) \qquad I_3(x_1, x_2, x_3; y_1, y_2, y_3; z_1, z_2, z_3) =$$
$$= I_3(x_2, x_3, x_1; y_2, y_3, y_1; z_2, z_3, z_1) = I_3(x_3, x_1, x_2; y_3, y_1, y_2; z_3, z_2, z_1).$$

The equations (3.7), (2.3) (3.3) and the postulate 1 yield,

(3.8)

$$f(x_1 + x_2, y_1 + y_2, z_1 + z_2) + (x_1 + x_2) f\left(x_1/(x_1 + x_2), \frac{y_1}{y_1 + y_2}, \frac{z_1}{z_1 + z_2}\right) =$$
$$= f(x_1, y_1, z_1) + (1 - x_1) f\left(x_2/(1 - x_1), \frac{y_2}{1 - x_2}, \frac{z_2}{1 - z_1}\right) =$$
$$= f(x_2, y_2, z_2) + (1 - x_2) f\left(x_1/(1 - x_2), \frac{y_1}{1 - y_2}, \frac{z_1}{1 - z_2}\right),$$

for $x_1, x_2, y_1, y_2, z_1, z_2 \in [0, 1)$, $x_1 + x_2$, $y_1 + y_2$, $z_1 + z_2 \in (0, 1]$ and with the convention of section 1.

From the second and third equation pairs in (3.8), we see that f satisfies the functional equation (3.5).

Let f_1 denote the partial derivative of f with respect to the first variable. Then differentiating partially the first and third equation pairs in (3.8) with respect to x_1, we get

$$(3.9) \qquad f_1(x_1 + x_2, y_1 + y_2, z_1 + z_2) +$$
$$+ f\left[x_1/(x_1 + x_2), \frac{y_1}{y_1 + y_2}, \frac{z_1}{z_1 + z_2}\right] + \{x_2/(x_1 + x_2)\} =$$
$$= f_1\left[x_1/(x_1 + x_2), \frac{y_1}{y_1 + y_2}, \frac{z_1}{z_1 + z_2}\right] = f_1\left[x_1/(1 - x_2), \frac{y_1}{1 - y_2}, \frac{z_1}{1 - z_2}\right],$$

for $x_1, y_1, z_1 \in (0, 1)$, $x_2, y_2, z_2 \in [0, 1)$ and $x_1 + x_2, y_1 + y_2, z_1 + z_2 \in (0, 1]$.

Now differentiating partially with respect to x_2 the first and second equation pairs in (3.8), we have

$$(3.10) \qquad f_1(x_1 + x_2, y_1 + y_2, z_1 + z_2) +$$
$$+ f\left[x_1/(x_1 + x_2), \frac{y_1}{y_1 + y_2}, \frac{z_1}{z_1 + z_2}\right] - \{x_1/(x_1 + x_2)\} =$$
$$= f_1\left[x_1/(x_1 + x_2), \frac{y_1}{y_1 + y_2}, \frac{z_1}{z_1 + z_2}\right] = f_1\left[x_2/(1 - x_1), \frac{y_2}{1 - y_1}, \frac{z_2}{1 - z_1}\right],$$

for $x_2, y_2, z_2 \in (0, 1)$, $x_1, y_1, z_1 \in [0, 1)$, $x_1 + x_2, y_1 + y_2, z_1 + z_2 \in (0, 1]$.

Thus subtracting (3.10) from (3.9), we have

(3.11)

$$f_1\left[x_1/(x_1 + x_2), \frac{y_1}{y_1 + y_2}, \frac{z_1}{z_1 + z_2}\right] + f_1\left[x_2/(1 - x_1), \frac{y_2}{1 - y_1}, \frac{z_2}{1 - z_1}\right] =$$
$$= f_1\left[x_1/(1 - x_2), \frac{y_1}{1 - y_2}, \frac{z_1}{1 - z_2}\right],$$

for $x_1, x_2, y_1, y_2, z_1, z_2 \in (0, 1)$ wtih $x_1 + x_2, y_1 + y_2, z_1 + z_2 \in (0, 1]$.

Substituting $x_1 = xu/(1 + x + xu)$, $x_2 = x/(1 + x + xu)$, $y_1 = yv/(1 + y + yv)$, $x_2 = y/(1 + y + yv)$, $z_1 = zw/(1 + z + zw)$, and $z_2 = z/(1 + z + zw)$ in (3.11), the equation (3.11) takes the following form:

$$(3.12) \quad f_1\left[u/(1 + u), \frac{v}{1 + v}, \frac{w}{1 + w}\right] + f_1\left[x/(1 + x), \frac{y}{1 + y}, \frac{z}{1 + z}\right] =$$
$$= f_1\left[ux/(1 + ux), \frac{vy}{1 + vy}, \frac{wz}{1 + wz}\right],$$

for $x, y, z, u, v, w \in (0, \infty)$.

Define

$$(3.13) \quad F(x, y, z) = f_1\left[x/(1 + x), \frac{y}{1 + y}, \frac{z}{1 + z}\right], \quad \text{for} \quad x, y, z \in (0, \infty),$$

so that (3.12) reduces to

$$(3.14) \quad F(u, v, w) + F(x, y, z) = F(xu, yv, zw), \quad \text{for} \quad x, y, z, u, v, w \in (0, \infty).$$

Since f_1 is continuous due to postulate 3, F is also continuous. By letting $y = z = v = w = 1$, we get from (3.14), that

$$F(u, 1, 1) = a \log u,$$

so that, this in (3.14) for $y = z = 1$ gives

$$F(u, v, w) + a \log x = F(xu, v, w) = F(x, v, w) + a \log u ,$$

and hence

$$F(u, v, w) - a \log u = F(x, v, w) - a \log x =$$
$$= \text{a function of } v \text{ and } w \text{ alone} = A(v, w) \text{ (say)} .$$

This in (3.14) gives

$$A(v, w) + A(y, z) = A(yv; zw) .$$

Repeating the above argument, it is easy to see that $A(v, w) = b \log v + c \log w$, so that the continuous solution of (3.14) is given by

$$F(x, y, z) = a \log x + b \log y + c \log z , \tag{3.15}$$

for $x, y, z \in (0, \infty)$, where a, b, c are arbitrary constants.

Hence (3.15) with the help of (3.13) gives

$$f_1(x, y, z) = a \log \{x/(1 - x)\} + b \log \{y/(1 - y)\} + c \log \{z/(1 - z)\} \tag{3.17}$$

for $x, y, z \in (0, 1)$.

This on integration with respect to x gives $f(x, y, z) = a[x \log x + (1 - x) . \log (1 - x)] + bx \log \{y/(1 - y)\} + cx \log \{z/(1 - z)\} + g(y, z)$, for $x, y, z \in \in (0, 1)$, where g is a function of y and z only, that is,

$$f(x, y, z) = a\, S(x) + bx \log \frac{y}{1 - y} + cx \log \frac{z}{1 - z} + g(y, z) , \tag{3.17}$$

for $x, y, z \in]0, 1[$, where $S(x)$ is the Shannon function,

$$S(x) = -x \log x - (1 - x) \log (1 - x) . \tag{3.18}$$

For $x = y$, the postulates 1, 2, 3, 4 and 5 give due to [3] that,

$$f(x, y, z) = x \log \frac{x}{z} + (1 - x) \log \frac{1 - x}{1 - z} ,$$

whereas (3.17) gives,

$$f(x, x, z) = -a\, S(x) + bx \log \frac{x}{1 - x} + cx \log \frac{z}{1 - z} + g(x, z) ,$$

so that, these with (3.17) yield,

$$f(x, y, z) = a[-S(x) + S(y)] + \tag{3.19}$$

$$+ b(x - y) \log \frac{y}{1-y} + c(x - y) \log \frac{z}{1-z} + y \log \frac{y}{z} + (1 - y) \log \frac{1-y}{1-z},$$

for $x, y, z \in]0, 1[$.

For $u = v = w = t$, the equation (3.5), with (3.19) becomes

$$(a + b)[t \log t + (1 - y - t) \log (1 - y - t) - (1 - y) \log (1 - y)] + \tag{3.20}$$

$$+ c[t \log t + (1 - y - t) \log (1 - z - t) - (1 - y) \log (1 - z)] +$$

$$+ (1 - y) \log \frac{1-y}{1-z} - (1 - y - t) \log \frac{1-y-t}{1-z-t} = 0,$$

provided $x - y \neq 0$, which can very well be chosen like that.

For $t = 1 - y$, (3.20) gives with the convention $0 \log 0 = 0$, that $c = -1$. For $y = 0 = z$, (3.20) gives $a + b = 0$, provided $S(t) \neq 0$, which can be had for proper t. Thus

$$f(x, y, z) = a\left[-S(x) + S(y) - (x - y) \log \frac{y}{1-y}\right] +$$

$$+ x \log \frac{y}{z} + (1 - x) \log \frac{1-y}{1-z},$$

for $x, y, z \in]0, 1[$, that is,

$$f(x, y, z) = a\left[x \log \frac{x}{y} + (1 - x) \log \frac{1-x}{1-y}\right] + \tag{3.21}$$

$$+ x \log \frac{y}{z} + (1 - x) \log \frac{1-y}{1-z}$$

for $x, y, z \in]0, 1[$.

By postulate 5, taking $x = \frac{2}{3}$, $y = \frac{1}{3}$, $z = \frac{1}{3}$ in (3.21), we get $a = 0$, so that f has the form given by (3.6) for $x, y, z \in]0, 1[$.

With little manipulation and the use of (3.5) and (3.21), it can be shown that, f indeed has the form (3.6) for $(x, y, z) \in K$.

The proof of Lemma 2 is now complete.

Proof of the Theorem. Applying successively the postulate 1, we have

$$I_n(p_1, \ldots, p_n; q_1, \ldots, q_n; r_1, \ldots, r_n) = \sum_{i=2}^{n} P_i f(p_i/P_i, q_i/Q_i, r_i/R_i), \tag{3.22}$$

where $P_i = p_1 + \ldots + p_i$, $Q_i = q_1 + \ldots + q_i$, $R_i = r_1 + \ldots + r_i$ for $i = 1, 2, \ldots, n$ with $P_n = Q_n = R_n = 1$.

Hence (3.22) and (3.6) give

(3.23)
$$I_n(p_1, \ldots, p_n; q_1, \ldots, q_n; r_1, \ldots, r_n) =$$
$$= \sum_{i=2}^{n} P_i \left[\frac{p_i}{P_i} \log\left(\frac{q_i R_i}{Q_i r_i}\right) + \left(1 - \frac{p_i}{P_i}\right) \log\left\{\left(1 - \frac{q_i}{Q_i}\right)\Big|\left(1 - \frac{r_i}{R_i}\right)\right\}\right] =$$
$$= \sum_{i=2}^{n} p_i \log\left(\frac{q_i}{r_i}\right) + \sum_{i=2}^{n} p_i \log\left(\frac{R_i}{Q_i}\right) + \sum_{i=2}^{n} P_{i-1} \log\left(\frac{Q_{i-1}}{R_{i-1}}\right) =$$
$$= \sum_{i=2}^{n} p_i \log\left(\frac{q_i}{r_i}\right) + P_1 \log\left(\frac{Q_1}{R_1}\right) = \sum_{i=2}^{n} p_i \log\left(\frac{q_i}{r_i}\right),$$

which proves the theorem.

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