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Automatic Listing of Important Observational Statements II

PETR HÁJEK

This is a direct continuation of the first part — Problems and Solutions — of the present paper. (See the previous number of this journal.)

Part II — Functor calculi

7. BASIC NOTIONS

We shall now consider the structure of sentences and the ways in which sentences take values. The notions we are going to introduce are generalizations of notions studied in the classical predicate calculus and are all essentially described in [2] – Introduction. We shall keep Church's terminology as much as possible; our deviation consists in working systematically with abstract values and in different (more detailed) notion of operators (generalized quantifiers).

Recall that describing the predicate calculus one defines formulas of some language, in particular closed formulas; the meaning of closed formulas is given by giving a relational structure of an appropriate type. The meaning of a predicate is the corresponding relation on the field of the structure or – equivalently – the characteristic function of that relation, hence a two-valued function on the field of the structure. Let now V be a set of abstract values; for every set M, each mapping of M^n into V (n natural) will be called an n-ary *V-valued function on M*. A V-valued function is understood as a generalized relation; instead of asking whether an n-tuple is in the relation (yes – no), we ask how it is in the relation. (Compare e.g. the question "are x and y related?" with the question "what is the relationship of x and y?".)

7.1. Definition. (i) Let a fixed non-empty set V of abstract values be given. A *type* is a finite non-empty sequence of natural numbers. A *V-structure* of the type $\langle n_1, \ldots, \ldots, n_k \rangle$ is a $(k + 1)$-tuple $\boldsymbol{M} = \langle M, f_1, \ldots, f_n \rangle$ where $M \neq \emptyset$ (the *field* of $\boldsymbol{M}$) and each f_i is an n_i-ary V – valued function on M. (If $n_i = 0$ then $f_i \in V$.)

(ii) Let $\boldsymbol{M}^{(i)} = \langle M^{(i)}, f_1^{(i)}, \ldots, f_k^{(i)} \rangle$ $(i = 1, 2)$ be V-structures of the same type. An *isomorphism* of $\boldsymbol{M}^{(1)}$, $\boldsymbol{M}^{(2)}$ is a one-one mapping ι of $M^{(1)}$ onto $M^{(2)}$ such that $\iota(f_j^{(1)}(m_1, \ldots, m_{n_j})) = f_j^{(2)}(\iota(m_1), \ldots, \iota(m_{n_j}))$ for each $j = 1, \ldots, k$ and each $m_1, \ldots \ldots, m_{n_j} \in M^{(1)}$.

We shall now describe languages used to speak about V-structures, formulas of these languages and the meaning of a formula in a V-structure. *Functors* will be names of the V-valued functions. The *meaning* (value) of a formula containing free variables will depend on a given V-structure and on that which values are assigned to the free variables. In particular, the value of a closed formula (i.e. a formula without free variables) will depend only on a given V-structure. A *junctor* (connective) *joins* a certain number of formulas (components) into a new formula whose free (bound) variables are the union of free (bound) variables of the components. The value of the compound formula with the junctor in question results from values of the components by applying to them a function called the *associated function* of the junctor. An *operator joins* some formulas (components) and in each component *binds* some variables. Let φ be one of the components and suppose the meaning of the variables not bound by the operator to be fixed. Associate with each possible meaning of the variables not bound by the operator the corresponding value of φ. In this way we obtain a V-valued function and doing this for all the components we obtain a V-structure. The *associated function* of the operator in question is the function which associated with the V-structure just described the value of the compound formula with the operator. Exact definitions read as follows:

7.2. Definition. A *language* is a quadruple

$$L = \langle Var, (Ar(F))_{F \in |Ft|}, (Ar(\iota))_{\iota \in Jct}, (Tp(q))_{q \in Opt} \rangle ,$$

where the sets Var, $|Ft|$, Jct, Opt are non-empty and pairwise disjoint, Ft is a non-empty finite one-one sequence ($|Ft|$ denotes the set of its members), Ar is a function mapping $|Ft| \cup Jct$ into natural numbers and Tp is a function mapping Opt into the set of all types. If $Ft = \langle F_1, \ldots, F_k \rangle$ then the *type od* L is $\langle Ar(F_1), \ldots, Ar(F_k) \rangle$.

(Var is the set of *variables*, Ft is the sequence of *functors*, Jct is the set of *junctors*, Opt is the set of *operators*, Ar is the function assigning *arities* and Tp is the function assigning *types*.)

7.3. Definition. (a) If F is an n-ary functor and if $x_1, \ldots, x_n$ are variables then $F(x_1, \ldots, x_n)$ is an (atomic) *formula*, $FV\ (F_1(x_1, \ldots, x_n)) = \{x_1, \ldots, x_n\}$, $BV(F(x_1, \ldots \ldots, x_n)) = \emptyset$. ($FV(\varphi)$ will denote the set of *free variables of* φ, $BV(\varphi)$ the set of *bound variables of* φ.)

(b) If $\varphi_1, \ldots, \varphi_n$ are formulas and if ι is an n-ary junctor then $\iota(\varphi_1, \ldots, \varphi_n)$ is a *formula*,

$$FV(\iota(\varphi_1, \ldots, \varphi_n)) = \bigcup_{i=1}^{n} FV(\varphi_i)\,, \quad BV(\iota(\varphi_1, \ldots, \varphi_n)) = \bigcup_{i=1}^{n} BV(\varphi_i)\,.$$

(c) If $\boldsymbol{q}$ is an operator of the type $\langle n_1, \ldots, n_k \rangle$, if $\varphi_1, \ldots, \varphi_k$ are formulas and if $\{x_{ij}\}_{j=1,\ldots,n_i}^{i=1,\ldots,k}$ is a system of variables such that, for each i, the variables $x_{i1}, \ldots, x_{in_i}$ are pairwise distinct then

$$\boldsymbol{q}((x_{11}, \ldots, x_{1n_1})\ \varphi_1, \ldots, (x_{k1}, \ldots, x_{kn_k})\ \varphi_k)$$

is a formula; denoting it by Ψ we have

$$FV(\Psi) = \bigcup_{i=1}^{k} (FV(\varphi_i) - \{x_{i1}, \ldots, x_{in_i}\}),$$

$$BV(\Psi) = \bigcup_{i=1}^{k} (BV(\varphi_i) \cup \{x_{i1}, \ldots, x_{in_i}\}).$$

(d) The set Fm of all formulas is the least set containing all atomic formulas and closed w.r.t. (b), (c). A formula φ is *closed* if $FV(\varphi) = \emptyset$; $ClFm$ is the set of all closed formulas. Formulas not containing operators (i.e. obtained from atomic ones by applying only (b)) are called open.

7.4. Definition. A *functor calculus* is a quintuple

$$\mathfrak{F} = \langle V, L, \mathfrak{M}, (Asf_\iota)_{\iota \in Jct}, (Asf_q)_{q \in Opt} \rangle,$$

where L is a language; V is a set of abstract values; $\mathfrak{M}$ is a non-empty set of V-structures whose type equals to the type of L; for each n-ary junctor ι of L, Asf_ι is a mapping of V^n into V; and, for each operator q of L of the type $t = \langle n_1, \ldots, n_k \rangle$, Asf_q is a mapping of $\boldsymbol{Mod}(\mathfrak{M}, t)$ into V. $\boldsymbol{Mod}(\mathfrak{M}, t)$ is the set of all V-structures $\boldsymbol{M}$ of the type t such that there is a $\boldsymbol{N} \in \mathfrak{M}$ with the same field as $\boldsymbol{M}$.

7.5. Definition. Let $\mathfrak{F}$ be a functor calculus and let $\boldsymbol{M} \in \mathfrak{M}$.

(1) Let $\varphi \in Fm$. An $\boldsymbol{M}$-sequence for φ is a mapping of $FV(\varphi)$ into M (the field of $\boldsymbol{M}$). If the domain of e is $x_1, \ldots, x_n$ and if $e(x_i) = m_i$ $(i = 1, \ldots, n)$ we write $e = (x_1, \ldots, x_n)/(m_1, \ldots, m_n)$.

(2) (Auxiliary). Let e_i be a mapping of v_i into M $(i = 1, 2)$. We put $e_1 \oplus e_2 = e_1 \cup (e_2 \upharpoonright (v_2 - v_1))$, i.e. $(e_1 \oplus e_2)(x) = e_1(x)$ for $x \in v_1$ and $(e_1 \oplus e_2)(x) = e_2(x)$ for $x \in v_2 - v_1$.

(3) We define $\|\varphi\|_{\boldsymbol{M}}[e]$ (the *meaning of* φ *in* $\boldsymbol{M}$ for e) inductively as follows ($\varphi \in Fm$, $\boldsymbol{M} \in \mathfrak{M}$, $\boldsymbol{M} = \langle M, f_1, \ldots, f_k \rangle$, e is an $\boldsymbol{M}$-sequence for φ):

(i) $\|F_i(x_1, \ldots, x_n)\|_{\boldsymbol{M}}[(x_1, \ldots, x_n)/(m_1, \ldots, m_n)] = f_i(m_1, \ldots, m_n)$.

(ii) $\|\iota(\varphi_1, \ldots, \varphi_n)\|_{\boldsymbol{M}}[e] = Asf_\iota(\|\varphi_1\|_{\boldsymbol{M}}[e \upharpoonright FV(\varphi_2)], \ldots, \|\varphi_n\|_{\boldsymbol{M}}[e \upharpoonright FV(\varphi_n)])$.

(iii) (Auxiliary.) Let $x_1, \ldots, x_n \in Var$, let s be a mapping of $v \subseteq Var$ into M and let $(FV(\varphi) - \{x_1, \ldots, x_n\}) \subseteq v$. Then $|||(x_1, \ldots, x_n)\ \varphi|||_{\boldsymbol{M}}^{s}$ is an n-ary V-function defined as follows:

$$|||(x_1, \ldots, x_n)\ \varphi|||_{\boldsymbol{M}}^{s}(m_1, \ldots, m_n) = \|\varphi\|_{\boldsymbol{M}}[(((x_1, \ldots, x_n)/(m_1, \ldots, m_n)) \oplus s) \upharpoonright FV(\varphi)].$$

(iv) Let $x_i = \langle x_{i1}, \ldots, x_{in_i} \rangle$ $(i = 1, \ldots, k)$; then

$$\|q((x_1)\,\varphi_1, \ldots, (x_k)\,\varphi_k)\|_M [e] = Asf_q(\langle M, |||(x_1)\,\varphi_1|||^e_M, \ldots, |||(x_k)\,\varphi_k|||^e_M \rangle) .$$

(4) If φ is a closed formula then we put $\|\varphi\|_M = \|\varphi\|_M [\emptyset]$.

7.6. Remark. We shall formulate a "lemma on renaming bound variables" in the spirit of the classical predicate calculus; this lemma will not be explicitly used in the sequel (so that the reader may omit it) but could be useful for better understanding of (3) (iv) of the preceding definition (so that the reader is recommended to prove it).

(Lemma.) Let Ψ be a formula $q((x_{11}, \ldots, x_{1n_1})\,\varphi_1, \ldots, (x_{k1}, \ldots, x_{kn_k})\,\varphi_k)$ and let, for $i = 1, \ldots, k$, $y_{i1}, \ldots, y_{in_i}$ be a one-one sequence of variables not occurring in Ψ. For $i = 1, \ldots, k$, denote by Θ_i the formula resulting from φ_i by replacing each free occurrence of x_{ij} by y_{ij} $(j = 1, \ldots, n_i)$ and let Θ be the formula $q((y_{11}, \ldots, y_{1n_1})\,\Theta_1, \ldots$ $\ldots, (y_{k1}, \ldots, y_{kn_k})\,\Theta_k)$. Then we have $\|\Psi\|_M [e] = \|\Theta\|_M [e]$ for each $M \in \mathfrak{M}$ and each M-sequence e for Ψ (i.e., Ψ and Θ are strongly equivalent). In particular, we may request that the "prefixes" of φ_i, φ_j are disjoint for $i \neq j$ or, on the other hand, that all the "prefixes" are segments of a fixed sequence of variables.

7.7. Theorem. If $\mathfrak{F}$ is a functor calculus, if *Sent* is a non-empty set of closed formulas and if we put $Val(\varphi, M) = \|\varphi\|_M$ for $\varphi \in Sent$ and $M \in \mathfrak{M}$ then $\mathfrak{S}(\mathfrak{F}) =$ $= \langle Sent, \mathfrak{M}, V, Val \rangle$ is a semantical system. (Obvious.)

7.8. Convention. "*V_0-tautology in $\mathfrak{F}$*", "*V_0-equivalent in $\mathfrak{F}$*", "*strongly equivalent in $\mathfrak{F}$*" etc. means "V_0-tautology in the semantical system $\mathfrak{S}(\mathfrak{F})$", "*$V_0$-equivalent* in $\mathfrak{S}(\mathfrak{F})$", "strongly equivalent in $\mathfrak{S}(\mathfrak{F})$" etc. respectively.

7.9. Discussion. If we want to call the semantical system $\mathfrak{S}(\mathfrak{F})$ an *observational semantical system* then it is natural to request the following conditions to be satisfied: (1) The set *Sent* is calculable, (2) the field of each $M \in \mathfrak{M}$ is finite, (3) associated functions of junctors and operators are calculable. The notion of calculability is here not precise; it may mean recursivity or primitive recursivity in the case of *Sent* (a fixed Gödel numeration assumed); if V consists of natural numbers then recursivity may be used also in the case of associated functions of junctors and operators (one may suppose that fields of elements of $\mathfrak{M}$ consist of natural numbers since – evidently – isomorphisms preserve meanings). Otherwise (e.g. if V is the set of real numbers) one must look for a useful notion of calculability. Let us recall that the intuitive meaning of calculability is here the request that the computer is to generate elements of *Sent* and, given a fixed structure, find their values.

(a) The classical predicate calculus

8.1. Let Ω be a set of sets such that, for each natural n, there is a member of Ω having the cardinality n. $\mathfrak{M}$ is the set of all $\{0, 1\}$-structures of a given type t whose field is in Ω. Let $\mathfrak{F}$ be the functor calculus with the following properties:

Var is countable, the type of the language L is t,

$$Jct = \{\neg, \&, \vee\}, \quad Opt = \{\forall\}, \quad Ar(\neg) = 1, \quad Ar(\&) = Ar(\vee) = 2,$$
$$Tp(\forall) = \langle 1 \rangle, \quad Asf\neg(\varepsilon) = 1 - \varepsilon, \quad Asf_{\&}(\varepsilon, \eta) = \min(\varepsilon, \eta),$$
$$Asf_{\vee}(\varepsilon, \eta) = \max(\varepsilon, \eta) \quad (\varepsilon, \eta = 0, 1);$$

if f is a $\{0, 1\}$-valued unary function on M then

$$Asf\forall(\langle M, f \rangle) = 1 \text{ if } (\forall m \in M)(f(m) = 1),$$
$$Asf\forall(\langle M, f \rangle) = 0 \text{ otherwise}.$$

Then $\mathfrak{F}$ is the *classical predicate calculus* with the language L, logical connectives negation, conjunction and disjunction, with the universal quantifier and with the Tarski's notion of satisfaction and truth:

For each formula φ, structure $\boldsymbol{M}$ and $\boldsymbol{M}$-sequence e for φ, e satisfies φ in $\boldsymbol{M}$ iff $\|\varphi\|_{\boldsymbol{M}}[e] = 1$. We note in passing the following fundamental results of mathematical logic:

8.2. Theorem. Suppose that Ω contains at least one infinite set M_0. (1) (Gödel-Löwenheim-Skolem) A closed formula φ is a $\{1\}$-tautology iff φ is $\{1\}$-true in each $\boldsymbol{M} \in \mathfrak{M}$ with the field M_0. (2) (Gödel-Robinson-Tarski-Hanf): If there is an $F \in |Ft|$ whose arity is >1 then the set of all $\{1\}$-tautologies is a recursively enumerable non-recursive set.

(b) The classical monadic predicate calculus

8.3. In addition to the assumptions in 8.1 we assume that t (the type of L) is $\langle 1, \ldots, 1 \rangle$, i.e. all the functors are unary. Let x_0 be a fixed variable in *Var*. The following theorem is easy to prove:

8.4. Theorem. For each closed formula φ of the calculus $\mathfrak{F}$ there is a closed formula φ containing no variable except x_0 and strongly equivalent to φ in $\mathfrak{F}$; this φ can be found primitively recursively from φ.

Consequently, let L_0 be the language that differs from L only by the set of variables: L has Var and L_0 has $Var_0 = \{x_0\}$. Then for each closed formula φ of L there is a closed formula of L_0 strongly equivalent to φ. Let $\mathfrak{F}_0$ be the functor calculus that differs from $\mathfrak{F}$ only in the language: $\mathfrak{F}$ has L and $\mathfrak{F}_0$ has L_0 (just described). (We continue to assume that all predicates are unary.) $\mathfrak{F}_0$ is called the *classical monadic predicate calculus*. We have the following (well known) theorems:

8.5. Theorem. For each closed formula φ of $\mathfrak{F}$ there is a closed formula ψ of $\mathfrak{F}_0$ such that, for each $\boldsymbol{M} \in \mathfrak{M}$, $\|\varphi\|_{\boldsymbol{M}}$ (calculated in $\mathfrak{F}$) equals to $\|\psi\|_{\boldsymbol{M}}$ (calculated in $\mathfrak{F}_0$).

8.6. Theorem. Let $M_0 \in \Omega$ have cardinality $\geqq 2^n$ where $t = \langle \underbrace{1, \ldots, 1}_{n\ times} \rangle$ and let φ be a closed formula of $\mathfrak{F}_0$. φ is a $\{1\}$-tautology iff φ is $\{1\}$-true in each $\boldsymbol{M} \in \mathfrak{M}$ with the field M_0.

8.7. Corollary. The set of $\{1\}$-tautologies of the classical monadic predicate calculus is primitive recursive.

(c) Monadic functor calculi

8.8. Let $\mathfrak{F}$ be a functor calculus. If Var is a one-element set and all functors are unary (of arity 1) we shall call F a *monadic functor calculus*; if, moreover, $V = \{0, 1\}$ then we shall call $\mathfrak{F}$ a *monadic predicate calculus*. In the rest of this section we assume that $\mathfrak{M}$ contains only structures with finite fields.

8.9. *Examples of operators in monadic predicate calculi.*

(1) The operator of *equivalence* (*type* $\langle 1, 1 \rangle$):

$$Asf_{\equiv}(\langle M, f, g \rangle) = 1 \Leftrightarrow f = g .$$

(2) The operator of *implication* (type $\langle 1, 1 \rangle$):

$$Asf_{\rightarrow}(\langle M, f, g \rangle) = 1 \Leftrightarrow (\forall m \in M)\,(f(m) \leqq g(m)) .$$

(3) The operator "*almost all*" (type $\langle 1 \rangle$): let $0 < p \leqq 1$, let

$$fr_{\boldsymbol{M}}(f) = \frac{\text{card}\,\{m \in M; f(m) = 1\}}{card\ M}$$

(frequency; *card* denotes cardinality). Then

$$Asf\forall^{\bullet}_{p}(\langle M, f \rangle) = 1 \Leftrightarrow fr_{\boldsymbol{M}}(f) \geqq p .$$

(4) Analogously for the operator "*almost no*".

(5) The operator "*middle*": let $0 < p \leqq \frac{1}{2}$.

$$Asf_{\otimes_p}(\langle M, f \rangle) = 1 \Leftrightarrow p \leqq fr_M(f) \leqq 1 - p\,.$$

(6) The operator "*relatively more*" (type $\langle 1, 1 \rangle$). Put

$$fr_M(f) = r, fr_M(g) = k, fr_M(\min(f, g)) = a\,.$$

Then

$$Asf_{\sim}(\langle M, f, g \rangle) = 1 \Leftrightarrow a > r\,.\,k\,.$$

(Cf. [8] where an operator "*relatively significantly more*" is defined by means of the so-called exact Fisher's test.)

8.10. *Three-valued monadic functor calculi.* In the paper [9] one describes – *mutatis mutandis* – a monadic functor calculus with $V = \{0, 1, \times\}$, $Jct = \{\neg, \&, \vee\}$. Abstract values are considered to the truth values, $\times$ meaning "unknown" (absence of information). The system of associated functions of connectives is due by Kleene. For example, $Asf_{\&}(p, q) = 1 \Leftrightarrow p, q = 1$, $Asf_{\&}(p, q) = 0 \Leftrightarrow p = 0 \vee q = 0\,(p, q \in \{0, 1, \times\})$. One defines $Asf\forall(\langle M, f \rangle) = 1 \Leftrightarrow (\forall m \in M)(f(m) = 1)$, $Asf\forall(\langle M, f \rangle) = 0 \Leftrightarrow (\exists m \in M)(f(m) = 0)$, the operator "*relatively significantly more*" is defined in some more complicated way.

8.11. *Monadic functor calculi with nominal quantities.* A quantity is nominal if its value on an object indicates that the object belongs to some class of a partition but it is meaningless to compare different values (no ordering is given). Suppose that the quantity can take only finitely many values and let the values be $0, 1, \ldots, k$ ($k \geqq 1$). The numbers 0, 1 are both possible values of the nominal quantity and also truth values. For each $X \subseteq V$ we have a unary junctor (X) whose associated function is the characteristic function of X. Monadic functor calculi with nominal quantities will be considered in a separate paper [12].

8.12. *Monadic functor calculi with real values.* We have e.g. the operator "*correlation coefficient*" (type $\langle 1, 1 \rangle$) with the associated function defined as follows:

$$Asf_{\varrho}(\langle M, f, g \rangle) = \frac{\sum_{m \in M}(f(m) - \bar{f})(g(m) - \bar{g})}{\sqrt{(\sum_{m \in M}(f(m) - \bar{f})^2 \sum_{m \in M}(g(m) - \bar{g})^2)}}\,,$$

where $\bar{f} = \sum_{m \in M} f(m)/card(M)$ and analogously for $\bar{g}$.

(d) Pocket monadic functor calculi

Limited possibilities of computers seem to force one to restrict himself to monadic functor calculi. We saw in previous examples that abstract values are often numbers (not only coded by numbers, but dealt with as with numbers!). In the present example we describe some very simple and limited monadic functor calculi (therefore "pocket" calculi) generalizing the classic monadic predicate calculus in a natural way. It seems that almost each reasonable calculus with number values will contain one of pocket calculi (in some sense). On the other hand, we show in § 10 that the set of V_0-tautologies of a pocket calculus is not recursive (even not recursively enumerable) for each nontrivial V_0. (We shall also state the meaning of thet result.)

8.13. Definition. I is the set of integers (positive, negative and zero) and Re is the set of reals. For $V = I$ or $V = Re$ we denote by $\mathfrak{F}_V^n$ the monadic functor calculus defined as follows: The set of abstract values if V. $\mathfrak{M}$ contains precisely all the V-structures with hereditarily finite fields. The language consists of n unary functors, one variable, junctors $+$, $\cdot$, $\leqq$ (binary) and Z (unary), operators Σ and Π (of type $\langle 1 \rangle$). The associated function of $+$ and $\cdot$ is the addition and the multiplication respectively; $Asf_{\leqq}(p, q) = 1$ if p is less than or equal to q, otherwise $= 0$, $Asf_Z(0) = 1$, otherwise $= 0$. $Asf_\Sigma(\langle M, f \rangle) = \sum_{m \in M} f(m)$, $Asf_\Pi(\langle M, f \rangle) = \prod_{m \in M} f(m)$ (sum and product over the model respectively).

8.14. *Examples of formulas* (the unique variable is omitted in all occurrences). Let F be a fixed functor (e.g. F_1). We have e.g. the following formulas: ΣF (sum over the model), $ZF . ZZF$ (an open formula with the value 0 for each object), $ZF + ZZF$ (an open formula with the value 1 for each object), $\Pi(ZF . ZZF)$ (a closed formula with the value 0 for each model), $\Pi(ZF + ZZF)$ (analogously, value 1). $\varphi = \psi$ is the abbreviation for $(\varphi \leqq \psi) . (\psi \leqq \varphi)$ (caution: both $\leqq$ and . are junctors!). Let Op_0 be the formula $ZF . ZZF$ and let Op_{k+1} be $Op_k + (ZF + ZZF)$. Then Op_k is an open formula with the value k for each object. One defines in a similar way a sequence of closed formulas Cl_k such that the value of Cl_k is k for each model. Fr_k is the (closed) formula $\Sigma(F = Op_k)$; its value for a model is the number of objects for which the value of F is k. (k is an arbitrary natural number).

With each formula φ of the classical monadic predicate calculus $\mathfrak{P}^n$ with n predicates we associate a formula φ^* of $\mathfrak{F}_V^n$ by the following induction: $(F_i)^*$ is F_i, $(\neg \varphi)^*$ is $Z(\varphi^*)$, $(\varphi \,\&\, \psi)^*$ is $\varphi^* . \psi^*$, $(\varphi \vee \psi)^*$ is $(\neg(\neg \varphi \,\&\, \neg \psi))^*$, $(\forall \varphi)^*$ is $\Pi(\varphi^*)$. Then the following theorem holds:

8.15. Theorem (on embedding). If $M \in \mathfrak{M}$ is a $\{0, 1\}$ – structure and if φ is a formula of $\mathfrak{P}^n$ then, for each M-sequence e for φ, $\|\varphi\|_M [e]$ in the sense of $\mathfrak{P}^n$ equals to $\|\varphi^*\|_M [e]$ in the sense of $\mathfrak{F}_V^n$ ($V = I$ or $V = Re$). (Obvious)

9. EXAMPLES OF PROBLEMS AND SOLUTIONS

(a) Some problems with a simple relation of i.c.

9.1. Let $\mathfrak{F}$ be a monadic predicate calculus with n predicates; suppose that $\mathfrak{M}$ contains precisely all structures of the corresponding type whose field is herediatrily finite. There are junctors of disjunction and negation with usual associated functions and an operator H of the type $\langle 1 \rangle$ with properties formulated below. An *elementary disjunction* is an open formula $\varepsilon_1 F_{i_1} \vee \ldots, \vee \varepsilon_k F_{i_k}$ where $1 \leqq i_1 < \ldots < i_k \leqq n$ and each ε_i is either the negation sign or the empty symbol. ED is the set of all the elementary disjunctions. (Note that $card(ED) = 3^n - 1$.) For $\varphi, \psi \in ED$ write $\varphi \subseteq \psi$ if φ is a subdisjunction of ψ; write $\varphi \subset \psi$ if $\varphi \subseteq \psi$ but not $\psi \subseteq \varphi$ (i.e. φ is a proper subdisjunction of ψ). Let $\leqq$ be a linear ordering of ED extending $\subseteq$. Put $Sent = \{H(\varphi); \varphi \in ED\}$ and let $\mathfrak{S} = \langle Sent, \mathfrak{M}, \{0, 1\}, Val \rangle$. Put $IC = \{H(\varphi)/H(\psi); \varphi \subset \psi\}$ and suppose that IC is $\{1\}$ – sound for $\mathfrak{S}$. (This is the case e.g. for H being $\forall$ or $\forall^{\bullet}_{p}$, cf. [6], [7].) Put $\boldsymbol{P} = \langle Sent, \{1\}, IC, S \rangle$ where $H(\varphi) \leqq_S H(\psi) \Leftrightarrow \varphi \leqq \psi$. $\boldsymbol{P}$ is a l.o. $\mathfrak{S}$-problems; one looks for its (direct) solution.

It follows directly from the definition of IC that $\langle Sent, IC \rangle$ is simple; $H(\varphi) \leqq_{IC} H(\psi) \Leftrightarrow \varphi \subseteq \psi$ holds for each $\varphi, \psi \in ED$ and we see that $\leqq_{IC}$ is an ordering. By Corollary 6.6, having found a (direct) solution increasingly independent w.r.t. S one has the least (both $\subseteq$-least and card-least) solution. This solution is strongly independent and determines a solution of each hierarchical problem that results from $\boldsymbol{P}$ by adding a hiararchy $\boldsymbol{H}$ satisfying $R_{IC} \subseteq R_H$.

(b) Problems with the operator of equivalence

9.2. Let $\mathfrak{F}$ be a monadic predicate calculus with n predicates, let $\mathfrak{M}$ be as above; suppose that in $\mathfrak{F}$ there are junctors of conjunction and negation with usual associated function and some operators, besides others the operator of implication and equivalence (cf. 8.9). Define *elementary conjunctions* in the obvious way. Let $P \subseteq |Ft|$ be a non-empty set of predicates and let EK_P be the set of all the elementary conjunctions built up only from predicates on P. Let K be a one-one mapping associating with each $\varphi \in EK_P$ a closed formula K_φ such that, for each $M \in \mathfrak{M}$, (1) truthfulness of K_φ implies satisfiability of φ, i.e. $\|K_\varphi\|_M = 1$ implies $(\exists a \in M)(\|\varphi\|_M [x/a] = 1)$, (2) $\|K_\varphi\|_M = 1$ and $\|\varphi\|_M = 1$ implies $\|K_\psi\|_M = 1$. We further assume that no K_φ has the form of equivalence.

For example, $P = |Ft|$ and K_φ is $\oplus_p \varphi$ (see 8.9); or $P = \{F_1, \ldots, F_{n-1}\}$ and $K\varphi$ is $\varphi \sim F_n$ (φ is relatively more frequented in F_n than in $\neg F_n$; cf. [8] where one uses the operator "relatively significantly more").

9.3. Put

$$Sent = \{K_\varphi; \varphi \in EK_\varphi\} \cup \{\varphi \equiv \psi; \varphi, \psi \in EK_P \,\&\, \varphi \subset \psi\},$$

$$IC_1 = \left\{\frac{K_\varphi, \varphi \equiv \psi}{K_\chi}; \varphi \subset \chi \subseteq \psi \,\&\, \varphi, \chi, \psi \in EK_P\right\}.$$

Then IC_1 is $\{1\}$-sound for $\mathfrak{S} = \langle Sent, \mathfrak{M}, \{0, 1\}, Val \rangle$. Let the *characteristic formula* of K_φ and of $\varphi \equiv \psi$ be φ. Finally, put $\boldsymbol{P}_1 = \langle \{K_\varphi; \varphi \in EK_P\}, \{1\}, IC_1 \rangle$.

9.4. Lemma. The quasiordering R_{IC_1} induced by IC_1 has the following properties: If $\varphi \subset \psi$ and $\varphi, \psi \in EK_P$ then each sentence whose characteristic formula is φ precedes each sentence whose characteristic formula is ψ. Any two distinct sentences with the same characteristic formula are incomparable.

Hence R_{IC_1} is an ordering and therefore each solution of $\boldsymbol{P}_1$ in a $\boldsymbol{M} \in \mathfrak{M}$ increasingly independent w.r.t. some linearization S of R_{IC_1} is strongly independent (see 5,1 and 4.9) and determines a solution of each reasonable hierarchical problem $\langle \boldsymbol{P}_1, \boldsymbol{H} \rangle$ by 5.9.

9.5. Definition. (1) $\varphi \in EK_P$ is *prime in* $\boldsymbol{M}$ if $\|\varphi \equiv \psi\|_M = 0$ for each $\psi \in EK_P$ such that $\psi \subset \varphi$. (2) $\varphi \in EK_P$ is *regular* in $\boldsymbol{M}$ if $\|\varphi \equiv \psi\|_M = 0$ for each $\psi \in EK$ such that $\varphi \subset \psi$.

9.6. Lemma. For each $M \in \mathfrak{M}$ and each $\varphi \in EK_P$ satisfiable in M there is a uniquely determined regular $\bar{\varphi} \in EK_P$ such that $\varphi \subseteq \bar{\varphi}$. ($\bar{\varphi}$ is called the *regulator* of φ.)

Proof. If φ is regular then $\bar{\varphi}$ is φ. Suppose φ is not regular and let $F_{i1}, \ldots, F_{ik}$ be all predicates in P not occurring in φ and such that either $\|\varphi \rightarrow F_{ij}\|_M = 1$ or $\|\varphi \rightarrow \neg F_{ij}\|_M = 1$. Since φ is satisfiable in M the last two equalities cannot hold

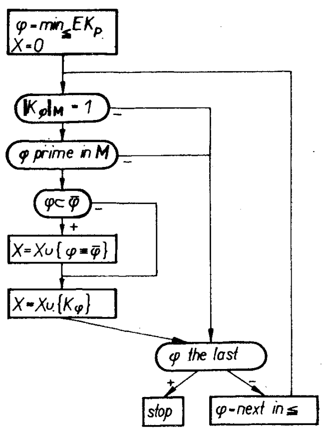

Fig. 1.

simultaneously; hence let ε_j be such that $\|\varphi \rightarrow \varepsilon_j F_{ij}\|_M = 1$ and let $\bar{\varphi}$ be $\varphi \,\&\, \varepsilon_1 F_{i1} \,\&$ $\& \ldots \& \,\varepsilon_k F_{i_k}$ (rearranged in order to be an elementary conjunction). Then $\bar{\varphi} \in EK_P$, $\varphi \subset \bar{\varphi}$, $\|\varphi \equiv \bar{\varphi}\|_M = 1$ and, for each ψ, $\varphi \subseteq \psi$ and $\|\varphi \equiv \psi\|_M = 1$ implies $\psi \subseteq \bar{\varphi}$.

9.7. Note that the above proof yields a method for obtaining $\bar{\varphi}$ from φ in an easy way. Let $\leqq$ be a linearization of $\subseteq$ (on EK_P) and let S be a linear ordering of *Sent* with the following properties: If $\varphi < \psi$ then each sentence whose characteristic formula is φ prededes each sentence whose characteristic formula is ψ. Each equivalence with the characteristic formula φ precedes K_φ.

9.8. Consider the following flow-diagram (Fig. 1.) defining a set X (a $\boldsymbol{M} \in \mathfrak{M}$ being given).

9.9. Lemma. For each $\boldsymbol{M} \in \mathfrak{M}$, the set X defined by the diagram 9.7 is a strongly independent solution of $\boldsymbol{P}_1$.

Proof. We show that X is a solution increasingly independent w.r.t. S (cf. 9.4) Let $K_\psi \in F \cap Tr_{\{1\}}(\boldsymbol{M})$. If ψ is prime then $K_\psi \in X$ and K_ψ cannot follow immediately from some preceding elements of S: if we had $K_\varphi <_S K_\psi$, $(\varphi \equiv \bar{\varphi}) <_S K_\psi$, $K_\varphi \in X$, $(\varphi \equiv \bar{\varphi}) \in X$, $(K_\varphi, \varphi = \bar{\varphi})/K_\psi \in IC_1$ then we would have $\varphi \subset \psi \subseteq \bar{\varphi}$, $\|\varphi \equiv \psi\|_{\boldsymbol{M}} = 1$, which would contradict the assumption that ψ is prime. If ψ is not prime then there is a $\varphi \subset \psi$, φ prime and such that $\|\varphi \equiv \psi\|_{\boldsymbol{M}} = 1$. Hence we have $\varphi \subset \psi \subseteq \bar{\varphi}$, $K_\varphi \in X$, $(\varphi \equiv \bar{\varphi}) \in X$, i.e. $K_\psi \in IC_1(X)$.

9.10. Remark. (1) This solution is in fact described in [8].

(2) The solution X need not be card-minimal (the following counter-example is due to Dr. K. Bendová): Let $Ft = \langle F_1, F_2 \rangle$, $\|F_1 \subseteq F_2\|_{\boldsymbol{M}} = 1$, $\|K_{F_1}\|_{\boldsymbol{M}} = 1$, $\|K_{\neg F_1}\|_{\boldsymbol{M}} = 0$. Then we have the direct solution K_{F_1}, K_{F_2}, $K_{F_1 \& F_2}$; the (indirect) solution X consists of $F_1 \equiv F_1 \,\&\, F_2$, K_{F_1}, $F_2 \equiv F_1 \,\&\, F_2$, K_{F_2}.

We see that it is desirable to have a stronger relation of i.e. that would enable us to derive the last sentence in X from the three sentences preceding it. So let us make the following definition:

9.11. Definition.

$$IC_2 = \left\{ \frac{K_{\chi_1}, \varphi_1 \equiv \psi, \varphi_2 \equiv \psi}{K_{\chi_2}} ; \varphi_1 \subseteq \chi_1 \subseteq \psi, \varphi_2 \subseteq \chi_2 \subseteq \psi \right\}.$$

Note that $K_\varphi IC_1 e \Rightarrow K_\varphi IC_2 e$ (put $\varphi_1 = \varphi_2 = \chi_1$ in IC_2!) and that IC_2 is $\{1\}$-sound for S by the assumption (2) on K_φ. Put $\boldsymbol{P}_2 = \langle \{K_\varphi; \varphi \in K_P\}, \{1\}, IC_2 \rangle$. We shall now succeed to describe a card-minimal solution.

9.12. Theorem. Let $\boldsymbol{M} \in \mathfrak{M}$. Let X be a set of sentences with the following elements: for each $\psi \in EK_P$ regular in $\boldsymbol{M}$ and such that $\|K_\psi\|_{\boldsymbol{M}} = 1$, (1) all equivalences $\varphi \equiv \psi$ true in $\boldsymbol{M}$ and such that φ is prime in $\boldsymbol{M}$ and $\varphi \subset \psi$, (2) exactly one formula K_φ such that φ is prime in $\boldsymbol{M}$, $\varphi \subseteq \psi$ and $\varphi \equiv \psi$ is true in $\boldsymbol{M}$. Then X is a card-minimal solution of $\boldsymbol{P}_2$ in $\boldsymbol{M}$.

Proof. X is a solution: If $\|K_\varphi\|_M = 1$ and $\psi = \bar{\varphi}$ then either ψ is prime and φ is ψ, i.e. $K_\varphi \in X$; or ψ is not prime and then there are φ_0, φ_1 prime and such that $\|\varphi_i \equiv \psi\|_M = 1$ $(i = 0, 1)$, $K_{\varphi_0} \in X$, $\varphi_1 \subseteq \varphi$. We have $(\varphi_i \equiv \psi) \in X$ $(i = 0, 1)$ and $K_\varphi IC_2\{K_{\varphi_0}, \varphi_0 \equiv \psi, \varphi_1 \equiv \psi\}$.

X is a card-minimal solution: Let Y be a solution of $\boldsymbol{P}_2$ in $\boldsymbol{M}$, let ψ be regular and such that $\|K_\psi\|_M = 1$. Denote by X_ψ and Y_ψ the set of all sentences Φ in X (in Y) such that $\overline{cf(\Phi)}$ is ψ. If ψ is prime then $X_\psi = Y_\psi = \{K_\psi\}$. Let ψ be not prime and let $\varphi_0, \ldots, \varphi_n$ be prime subformulas of φ equivalent to ψ (everything in $\boldsymbol{M}$!). Then X_ψ has $(n + 1)$ elements. For each φ_i, Y_ψ must contain K_{φ_i} or some $\varphi_i = \varphi_i'$ (or both). Hence if Y_ψ contains at least one sentence whose characteristic formula is distinct from all the φ_i's then Y_ψ has at least $(n + 1)$ elements. If Y_ψ contains only sentences with characteristic formulas $\varphi_0, \ldots, \varphi_n$ then for some i Y_ψ must contain both K_{φ_i} and $\varphi_i \equiv \psi$ (since $K_\psi \in IC_2(Y)$!), Consequently, Y_ψ has also in this case $(n + 1)$ elements. Since X is a disjoint union of all the X_ψ's and similarly for Y, we see that *card* $(X) \leqq$ *card* (Y) and hence X is a card-minimal solution.

9.13. Let us consider the flow-diagram (S is as in 9.7) shown in Fig. 2.

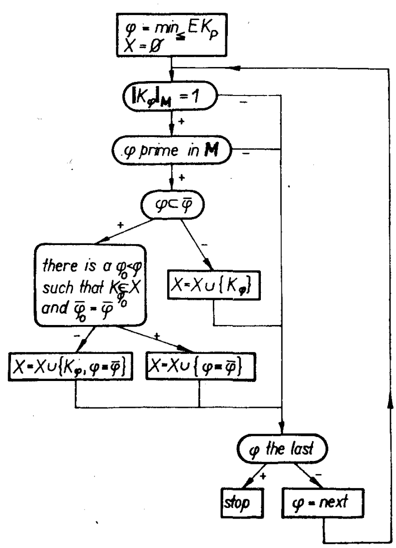

Fig. 2.

9.14. Lemma. Let $\boldsymbol{M} \in \mathfrak{M}$. The set X defined by 9.12 is a card-minimal strongly independent solution of $\boldsymbol{P}_2$ in $\boldsymbol{M}$. The system $\{X_k, h \in \boldsymbol{H}\}$ is a solution of $\langle \boldsymbol{P}_2, \boldsymbol{H} \rangle$

for each hierarchy $\boldsymbol{H}$ such that $R_{cf} \subseteq R_{\boldsymbol{H}}$ where $\langle \Phi, \Psi \rangle \in R_{cf} \Leftrightarrow cf(\Phi) \leqq cf(\Psi)$ ($cf(\Phi)$ denotes the characteristic formula of Φ).

Proof. X differs from the set described in 9.7 only in the following: One omits in the output of 9.7 each K_φ such that (1) $\|K_\varphi\|_M = 1$, (2) φ is prime but (3) $\bar{\varphi}$ has already been reached, i.e. there is a $\varphi_0 < \varphi$ such that $\bar{\varphi}_0 = \bar{\varphi}$, $K_{\varphi_0} \in X$ and hence $(\varphi_0 = \bar{\varphi}) \in X$, i.e. $K_\varphi \in IC_2(X)$. Suppose now that $\|K_\varphi\|_M = 1$, φ is not prime and $\bar{\varphi}$ is ψ. Then there are prime subconjunctions φ_0, φ_1 of ψ equivalent to φ and such that $\varphi_1 \subseteq \varphi \subset \psi$ and φ_0 is the $\leqq$-least prime subconjunction of ψ equivalent to ψ. Then K_{φ_0}, $(\varphi_0 \equiv \psi)$, $(\varphi_1 \equiv \psi) \in X$ and consequently $K_\varphi \in IC_2(X)$. Hence X is a solution. The reader easily verifies that $R^w_{IC_2}$ is an ordering and that the ordering S (described in 9·7) extends $R^w_{IC_2}$; he further verifies that X is increasingly independent w.r.t. S. Hence X is strongly independent by 5.13 and 4.9. For each ψ regular and such that $\|K_\varphi\| = 1$, X contains all equivalences $\varphi \equiv \psi$ where φ is a prime proper subconjunction of ψ equivalent to ψ; in addition, X contains K_φ where φ is the least prime subconjunction of ψ equivalent to ψ. By 9·11, X is a card-minimal solution. The proof of the statement concerning hierarchicity is left to the reader.

9.15. Remark. Let us note that in [9] (and in [12]) one can find examples of problems for semantical systems given by functor calculi that are not predicate calculi (are not two-valued).

(c) Some other problems with a simple relation of i.c.

The aim of this subsection is to criticize some aspects of [6]. Part II and of the corresponding parts of [7] (as well as of the "statistical interpretation" in [10]). We have no comments to the listing of "(almost) prime disjunctions" of a given model; this corresponds to Subsection (a) above. But by [6], the computer should convert obtained disjunctions into certain implications (in terminology of [7], find all the maximal good antecedents of each (almost) prime disjunction); we show in 9.25 below that from the point of view of the ALIOS theory this is not well theoretically founded. We describe a modified method in terms of solutions of S-problems; the knowledge of [6] is not necessary for understanding the following text.

9.16. Let $\mathfrak{F}$ be a monadic predicate calculus with n predicates, with usual junctors &, $\vee$, $\neg$ and with an operator $\rightarrow^*$ (quasi-implication) of the type $\langle 1, 1 \rangle$. Let $\mathfrak{M}$ be as above. The set GEK of *generalized elementary conjunctions* consists of all elementary conjunctions and of the empty conjunction $\bigwedge\emptyset$ whose value for each object is 1. Similarly, GED consists of all elementary disjunctions and of $\bigvee\emptyset$ whose value for each object is 0. If $\varkappa \in GEK$ then $neg(\varkappa)$ denotes the GED logically equivalent to $\neg\varkappa$; similarly $neg(\delta)$ for $\delta \in GED$. *Sent* is the set of all (closed) formulas of the form $\varkappa \rightarrow^* \delta$ where $\varkappa \in GEK$, $\delta \in GED$, $\varkappa$ and δ have no predicates in common and at least one of $\varkappa$, δ is non-empty. The characteristic formula $cf(\varkappa \to^* \delta)$ of $\varkappa \to^* \delta$ is the elementary disjunction $neg(\varkappa) \vee \delta$. Let IC be the following relation of i.c. on $Sent$:

$$IC = \left\{ \frac{\varkappa \,\&\, neg(\delta') \to^* \delta}{\varkappa \to^* \delta \vee \delta''} \,;\, \delta' \subseteq \delta'' \,\&\, (\delta' \neq \mathsf{V}\emptyset \vee \delta' \neq \delta'') \right\}.$$

We supposed that IC is $\{1\}$-sound for $\mathfrak{S} = \langle Sent, \mathfrak{M}, \{0, 1\}, Val \rangle$. We consider the problem $\boldsymbol{P} = \langle Sent, \{1\}, IC \rangle$.

9.17. Let $\Phi = \varkappa_1 \to^* \delta_1$ and $\Psi = \varkappa_2 \to^* \delta_2$. We say that Ψ is *reducible to* Φ if $\varkappa_1 = \varkappa_2$ and $\delta_1 \subseteqq \delta_2$; and that Ψ is *specifiable to* Φ if $\varkappa_1 \subseteq \varkappa_2$ and $cf(\Phi) = cf(\Psi)$. Hence $\Psi \in IC(\{\Phi\})$ iff Φ results from Ψ by reduction and specification.

9.18. *Examples of operators for which IC is sound.*

(1) Implication $\to$, see 8.9.

(2) Good almost-implication $\to^{\circ}_{p,s}$ of [6]: $\|\varkappa \to^{\circ}_{p,s} \delta\|_M = 1$ if there are at least s objects in $\boldsymbol{M}$ having $\varkappa$ and if at least $100p$ % of them have also δ (s natural, $0 < p \leqq 1$).

(3) Probable almost-implication $\to^{!}_{p,\alpha}$ of [10]: Let $m_\varkappa$ be the number of objects in $\boldsymbol{M}$ having $\varkappa$ and let $m_{\varkappa\&\delta}$ be the number of objects in $\boldsymbol{M}$ having $\varkappa \,\&\, \delta$. Then $\|K \to^{!}_{p,s} \delta\|_M = 1$ if $p \leqq p_*(m_\varkappa, m_{\varkappa\&\delta})$ where $p_*(x, y)$ is the lower limit of the one-sided upper confidence interval with confidence coefficient $1 - \alpha$ for the number of investigations equal to x and the number of investigations with positive result equal to y.

(4) Suscipiousness of almost-implication $\to^{?}_{p,\alpha}$ of [10]: Here one requires $p \geqq p^*(m_\varkappa, m_{\varkappa\&\delta})$ where $p^*(x, y)$ is the upper limit of the one-sided lower confidence interval.

(IC is sound in the case (2) by [6] Theorem 3 or by [7] Theorem 8 and in cases (3), (4) by [10] Theorem 5 and Note 2.)

9.19. One sees immediately that IC is simple and that $\leqq_{IC}$ is an ordering. Consequently, by 6.6, for each model $\boldsymbol{M}$ the problem $\boldsymbol{P}$ described in 9.16 has a uniquely determined both $\subseteq$-minimal and card-minimal solution consisting of all the $\leqq$ minimal elements of $Tr(\boldsymbol{M})$.

9.20. Let $\boldsymbol{M}$ be a model and let $\Psi \in Tr(\boldsymbol{M})$. Ψ is *reducible in* $\boldsymbol{M}$ if there is a $\Phi \in Tr(\boldsymbol{M})$, $\Phi \neq \Psi$, such that Ψ is reducible to Φ. Similarly for "*specifiable in* $\boldsymbol{M}$". Ψ is *prime in* $\boldsymbol{M}$ if Ψ is true but neither reducible nor specifiable in $\boldsymbol{M}$. Evidently, the least solution described in 9.19 consists precisely of all sentences prime in $\boldsymbol{M}$.

9.21. Lemma. Let Φ, $\Psi \in Tr(\boldsymbol{M})$ and let Ψ be specifiable to Φ. If Φ is reducible in $\boldsymbol{M}$ then Ψ is also reducible in $\boldsymbol{M}$. (Let $\Phi = \varkappa_1 \,\&\, \varkappa_2 \to^* \delta_1 \vee \delta_2$, $\Psi = \varkappa_1 \to^* \delta_1 \vee$

$\vee\ \delta_2 \cup neg(\varkappa_2)$ and let $(\varkappa_1 \,\&\, \varkappa_2 \to^* \delta_1) \in Tr(M)$. Then also $(\varkappa_1 \to^* \delta_1 \vee neg(\varkappa_2)) \in$
$\in Tr(M)$ and Ψ is reducible in M.)

9.22. Corollary. If Ψ is not reducible in M and if Ψ is specifiable to Φ then Φ is prime iff Φ is true and not specifiable in M.

9.23. Remark. We show that the implication of 9.21 cannot be converted (for $\to^*$ being $\to^{o}_{p,s}$; similar examples could be constructed for $\to^{!}_{p,\alpha}$ and for $\to^{?}_{p,\alpha}$). Let $p = 0{,}9$ and $s = 10$. Put $\Phi = \neg P_1 \,\&\, \neg P_2 \to^* P_3 \vee P_4$, $\Psi = \neg P_1 \to^* P_2 \vee P_3 \vee P_4$ $\Psi_0 = \neg P_1 \to^* P_2 \vee P_3$. Let frequencies in M be given by the following table:

P_1	P_2	P_3	P_4	
0	0	1	1	5
0	0	1	0	2
0	0	0	1	2
0	0	0	0	1
0	1	anything		20
...	...	...	...	..

Then Φ, $\Psi \in Tr(M)$, Ψ is specifiable to Φ, Ψ is reducible in M (since $\Psi_0 \in Tr(M)$) but Φ is not reducible in M.

The reader shows easily that the implication of 9.21 is convertible for $\to^{o}_{p,s}$ if $p = 1$ (and consequently is convertible for $\to$).

9.25. Discussion. The algorithm described in [6] and the corresponding algorithm of [10] can be understood as an algorithm constructing the least solution of a problem P_0 described in Subsection (a) of the present section and hence working with an operator H. In addition, one considers there an operator $\to^*$ for which our present IC is sound and such that $\{(\varkappa \to^* \delta)/H(\varphi);\ \varphi = cf(\varkappa \to^* \delta)\}$ is a sound relation of i.c. In fact, if $\varphi = cf(\varkappa \to^* \delta)$ then $H(\varphi)$ is semantically equivalent to $\bigwedge\emptyset \to^* \varphi$. For each member $H(\varphi)$ of the solution of P_0 the algorithm finds all formulas $\varkappa \to^* \delta$ with $cf(\varkappa \to^* \delta) = \varphi$ that are not specifiable in M. The corresponding $\varkappa$ is called a good antecedent of φ. From the point of view of P_0 this is some additional information; but 9.24 shows that we do not obtain the solution of the problem P (of 9.16) since it is possible that the quasiimplication $\varkappa \to^* \delta$ is prime in M (i.e. neither reducible nor specifiable, take $\neg P_1 \,\&\, \neg P_2 \to^* P_3 \vee P_4$), but $\bigwedge\emptyset \to^* \delta \vee neg(\varkappa)$ is reducible in M (i.e. $\forall^{\bullet}_{p}(P_1 \vee P_2 \vee P_3) \in Tr(M)$) and hence $H(\delta \vee neg(\varkappa))$ is not a member of the solution of P_0. This leads us to following conclusion:

Whenever $p < 1$ then P is to be considered independently from P_0; good antecedents of members of the least solution of P_0 are of little interest. In the rest of this subsection we make some remarks concerning algorithms constructing the solution of P.

9.26. First convert P into a l.o. problem by taking a linear ordering extending $\leqq_{IC}$. Since $\Phi \leqq_{IC} \Psi$ implies $cf(\Phi) \subseteq cf(\Psi)$, one could use a linear ordering S_1 extending $\leqq_{IC}$ and such that for each ED φ the set $\{\Phi; cf(\Phi) = \varphi\}$ is an interval. But it seems that other linear orderings are more appropriate. The linear orderings S_2, S_3 are defined as follows (the *length* of Φ, denoted by $lh(\Phi)$, is the number of predicates occuring in Φ): Let $\Phi = \varkappa_1 \to^* \delta_1$, $\Psi = \varkappa_2 \to^* \delta_2$. $\Phi \leqq_{S_2} \Psi$ if (1) $lh(\Phi) < lh(\Psi)$ or (2) $lh(\Phi) = lh(\Psi)$ and $lh(\varkappa_1) > lh(\varkappa_2)$ or (3) $lh(\Phi) = lh(\Psi)$, $lh(\varkappa_1) = lh(\varkappa_2)$ and $\varkappa_1$ precedes lexicographically $\varkappa_2$ or (4) $lh(\Phi) = lh(\Psi)$, $\varkappa_1 = \varkappa_2$ and δ_1 precedes lexicographically δ_2. Note that in (2) the condition $lh(\varkappa_1) > lh(\varkappa_2)$ is equivalent to $lh(\delta_1) < lh(\delta_2)$. Now, $\Phi \leqq_{S_3} \Psi$ if (1) or (2) as above or (3') $lh(\Phi) = lh(\Psi)$, $lh(\delta_1) = lh(\delta_2)$ and δ_1 precedes lexicographically δ_2 or (4') $lh(\Phi) = lh(\Psi)$, $\delta_1 = \delta_2$ and $\varkappa_1$ precedes lexicopraphically $\varkappa_2$. Evidently both S_2 and S_3 extend $\leqq_{IC}$.

9.27. For each tested sentence $\Phi = \varkappa_1 \to^* \delta$ the computer can decide whether Φ is prime in $\boldsymbol{M}$ in the following steps (in the rest, $\to^*$ is assumed to be one of the operators described in 9.18 (2)–(4); the sign (!) denotes that some "jumping", i.e. omission of an interval of sentences from the testing is possible, cf. e.g. [7] p. 305):

(1) The frequency of $\varkappa$ is so small that no formula with this antecedent can be true (!);
(2) Φ is not true in $\boldsymbol{M}$;
(3) Φ is true but reducible in $\boldsymbol{M}$ (!);
(4) Φ is true, irreducible but specifiable in $\boldsymbol{M}$;
(5) Φ is prime in $\boldsymbol{M}$.

9.28. One can consider restrictions of P to some subsets of *Sent*. Let a natural l be given. Put

$$Sent_1 = \{\varkappa \to^* \delta; lh(\delta) \leqq l\}\ ; \quad Sent_2 = \langle \varkappa \to^* \delta; lh(\varkappa) \geqq l\}\ ;$$
$$Sent_3 = \{\varkappa \to^* \delta; lh(\varkappa) \leqq l\}\ ; \quad Sent_4 = \{\varkappa \to^* \delta; lh(\delta) \geqq l\}\ .$$

By taking an appropriate l one can restrict oneself e.g. to formulas with non-empty antecedent as well as to formulas with empty antecedent; this latter case reduces – mutatis mutandis – to the listing of true formulas of the form $\forall_p^{\bullet}\delta$. Note that both $Sent_1$ and $Sent_2$ are initial segments of *Sent* w.r.t. $\leqq_{IC}$ and consequently if X is the least solution of P in $\boldsymbol{M}$ then $X \cap Sent_i$ is the least solution of $P \upharpoonright Sent_i$ ($i = 1, 2$). This is not the case for $i = 3, 4$. In particular, if $Sent_3$ consists of formulas $\Phi = \varkappa \to^* \delta$ with empty $\varkappa$ then deciding whether Φ is prime one has to make steps (2), (3) (5) of 9.27 (since (1) is never true if the cardinality of $\boldsymbol{M}$ is reasonable and since no specification in $Sent_3$ is possible).

9.29. Further restrictions on *Sent* in the style of GUHA-probes are possible besides the restrictions of 9.28. One can have some predicates with determined form (for each tested Φ if P occurs in Φ then its occurrence in $cf(\Phi)$ must have the determined form — positive or negative); one can have e.g. the set B of important predicates (each tested Φ must contain at least one predicate from B) and the set *Ant* of predicates such that for each tested Φ if a member of *Ant* occurs in Φ then it must occur in the antecedent of etc. Note that each of the restrictions described here (i.e. in 9.29) defines in each $\leqq_{IC}$-segment of *Sent* a $\leqq_{IC}$-subsegment — cf. 9.28.

9.30. The programmer has to decide if the computer will use a table of "critical values" ($\{crit(x)\}_{x=x_{\min},\ldots,card(M)}$ such that $\|\varkappa \to^* \delta\|_M = 1$ iff $m_{\varkappa\&\delta} \geqq crit(m_\varkappa)$) or not cf. [10] p. 19 and also [8] p. 513.

10. Unsolvability of the pocket calculi

The pocket monadic functor calculi $\mathfrak{F}_V^n$ were defined in 8.13 (V is either I or Re). Let $V_0 \subseteq V$ be given. If we have a semantical system $\mathfrak{S}$ defined with the help of $\mathfrak{F}_V^n$ (or some more powerful functor calculus) and an $\mathfrak{S}$-problem. $\boldsymbol{P} = \langle F, V_0, IC\rangle$ then V_0-tautologies as elements of solutions of $\boldsymbol{P}$ in a $\boldsymbol{M} \in \mathfrak{M}$ are unwanted since their V_0-truth says nothing relevant for the particular model M (since they are true in all models). Consequently, the relation IC should detect as many V_0-tautologies as possible, i.e. $\varphi IC\emptyset$ should hold for many tautologies. This suggest the question whether the set of all V_0-tautologies is decidable (recursive). Note that the answer is positive for the classical monadic perdicate calculus, see 8.7. The following theorem provides the full answer for the calculi $\mathfrak{F}_V^n$ (V is either I or Re, N is the set of natural numbers, i.e. non-negative integers.)

10.1. Theorem. (i) All closed formulas are V_0-tautologies iff $V_0 = V$.

(ii) No closed formula is a V_0-tautology iff $V_0 \cap N = \emptyset$.

(iii) If $V_0 \cap N = \emptyset$ and $V_0 \subset V$ then the set $Taut_{V_0}$ of all the V_0-tautologies is not recursively enumerable (a fortiori, not recursice, i.e. the problem of V_0-tautologicity is undecidable).

10.2. Remark. The rest of the present section is devoted to the proof of Theorem 10.1. But let us first point out the meaning of this theorem. We see that in the non trivial case (iii) we cannot have a recursive relation of i.c. which would be V_0-sound and would yield all the V_0-tautologies as immediate consequences of the empty set of premises. Generally we cannot prevent the computer from constructing solutions containing some V_0-tautologies since our relation of i.o. cannot detect all of them (since it is recursive). Consequently, our choice of *Sent* and F (subsets of the set of closed formulas) becomes very important. It can happen that choosing our $\mathfrak{S}$ and $\boldsymbol{P}$ appropriately (and, of course, adequately w.r.t. to the subject of research) the set $Taut_{V_0} \cap Sent$ (or $Taut_{V_0} \cap F$) *is* decidable (e.g. empty).

10.3. The proof of 10.1 is based on the recent and famous Matiasevič's result [4] on the unsolvability of the 10. Hilbert's problem. The reader is supposed to be familiar with the notions of recursive and recursively enumerable sets. We define necessary notions in a version useful for our purpose (cf. [3]).

(1) An n-ary *polynomial* is an arbitrary mapping $P(x_1, \ldots, x_n)$ of N^n into N having the following form:

$$\sum_{\substack{0 \leqq i_1 \leqq k_1 \\ 0 \leqq i_n \leqq k_n}} a_{i_1} \ldots a_{i_n} x_1^{i_1} \ldots x_n^{i_n} \quad (a_{i1}, \ldots, a_{in} \in N) .$$

(2) A set $A \subseteq N$ is *diophantine* if there are polynomials $P(y, x_1, \ldots, x_n)$, $Q(y, x_1, \ldots, x_n)$ such that

$$A = \{y; (\exists x_1, \ldots, x_n) (P(y, x_1, \ldots, x_n) = Q(y, x_1, \ldots, x_n))\} .$$

(A is said to be the diophantine set *corresponding to* P, Q.)

10.4. Lemma. The set Pol_n of all n-ary polynomials is the least set of mappings of N^n into N containing for each $i = 1, \ldots, n$ the function $I_n^i(x_1, \ldots, x_n) = x_i$, for each $a \in N$ the function $K_n^a(x_1, \ldots, x_n) = a$ and closed under sums and product of functions. (Obvious.)

10.5. Lemma. (Matiasevič). A set $A \subseteq N$ is recursively enumerable iff it is diophantine.

I apologize for my calling this long-expected result a lemma; but in the present context we only need the following.

10.6. Corollary. There is a diophantine non-recursive set of integers.

Let us now recall the pocket calculi: We use denotations from 8.14.

10.7. Lemma. Put $x_i^* = Fr_i$ $(i \in N)$ note that x_i^* is a closed formula of F_V^n. For each polynomial $P(x_1, \ldots, x_n)$ there is a closed formula π of $\mathfrak{F}_V^n$ such that

$$(*) \qquad (\forall \boldsymbol{M} \in \mathfrak{M}) (\|\pi\|_{\boldsymbol{M}} = P(\|x_1^*\|_{\boldsymbol{M}}, \ldots, \|x_n^*\|_{\boldsymbol{M}}) .$$

Proof by induction using 10.4. In the case of I_n^i, take x_i^* for π; in the case of K_n^a, take Cl_a for π. If $P = P_1 + P_2$, if π_1 corresponds to P_1 and π_2 to P_2, then take $\pi_1 + \pi_2$ for π (+ in $\pi_1 + \pi_2$ is a junctor!). The validity of (*) for π follows from the validity of (*) for π_1 and π_2 by the definition of the associated function of $+$. Similarly for $P = P_1 \cdot P_2$.

10.8. Lemma. For each $m_1, \ldots, m_k \in N$ and each $\alpha \in V$, there is a $\boldsymbol{M} \in \mathfrak{M}$ such that $\|x_i^*\|_{\boldsymbol{M}} = m_i$ $(i = 1, \ldots, k)$ and $\|\Sigma F\|_{\boldsymbol{M}} = \alpha$.

Proof. Let M contain for each $i = 1, \ldots, k$ exactly m_i objects a such that $f(a) = i$ (f is the meaning of F in the model to be constructed). Put $\beta = \sum_{i=1}^{k} i \, . \, m_i$ and let γ, δ be elements of V distinct from all the numbers $1, \ldots, k$ and such that $\gamma - \delta = \alpha - \beta$. We put into M a new object c such that $f(c) = \gamma$ and another object d such that $f(d) = \delta$. This completes the description of M; evidently, M has the desired properties.

10.9. Lemma. For each diophantine set A there is a recursive sequence of closed formulas $\{\varphi_k;\ k \in N\}$ of $\mathfrak{F}_V^n$ such that the following holds for each $k \in N$:

$$k \in A \Leftrightarrow \varphi_k \text{ is not a } \{0\}\text{-tautology} .$$

Proof. Let A correspond to $P(x_0, x_1, \ldots, x_m)$, $Q(x_0, x_1, \ldots, x_m)$ and let π, ϱ be closed formulas such that 10.7. (*) holds for P, π and for Q, ϱ. Let π_k and ϱ_k be formulas resulting from π and ϱ by replacing each subformula x_0^* by the formula Cl_k respectively ($k \in N$). Then we have

$$(\forall k)\,(\forall M)\,(\|\pi_k\|_M = P(k, \|x_1^*\|_M, \ldots, \|x_m^*\|_M)$$

and similarly for Q, ϱ.

Let φ_k be the formula $\pi_k = \varrho_k$ (i.e. $(\pi_k \leqq \varrho_k) \, . \, (\varrho_k \leqq \pi_k)$, cf. 8.14). Evidently, the sequence $\{\varphi_k; k \in N\}$ is recursive. If $k \notin A$ then for each $n_1, \ldots, n_m$ we have $P(k, n_1 \ldots) \neq Q(k, n_1, \ldots)$, hence $(\forall M)\,(\|\pi_k = \varrho_k\|_M = 0)$ and φ_k is a $\{0\}$-tautology. If $n \in A$ then there are $n_1, \ldots, n_m$ such that $P(k, n_1, \ldots) = Q(k, n_1, \ldots)$; by the preceding lemma, there is an $M \in \mathfrak{M}$ such that $\|x_i^*\|_M = n_i$ $(i = 1, \ldots, m)$, i.e. $\|\pi_k = \varrho_k\|_M = 1$ and φ_k is not a $\{0\}$-tautology. If $\alpha \in V$ we could even find the last model in such a way that $\|\Sigma F\|_M = \alpha$. This will be used in the proof of the following lemma.

10.10. Lemma. Suppose $V_0 \cap N \neq \emptyset$, $V_0 \subset V$ and let A be a diophantine set. Then there is a recursive sequence $\{\psi_k, k \in N\}$ of closed formulas of $\mathfrak{F}_V^n$ such that the following holds for each $k \in N$:

$$k \in A \Leftrightarrow \psi_k \text{ is not a } V_0\text{-tautology} .$$

Proof. Let φ_n be as in the proof of the preceding lemma, let $q \in V_0 \cap N$ and let ψ_k be $(Cl_q \, . \, Z\varphi_k) + ((\Sigma F) \, . \, \varphi_k)$. The sequence $\{\psi_k; k \in N\}$ is recursive and we have

$$\|\varphi_k\|_M = 0 \Rightarrow \|\psi_k\|_M = q \,, \quad \|\varphi_k\|_M = 1 \Rightarrow \|\psi_k\|_M = \|\Sigma F\|_M \,.$$

Hence if $k \notin A$ then ψ_k is a V_0-tautology (since $q \in V_0$). If $k \in A$ then there is a $M \in \mathfrak{M}$ such that $\|\varphi_k\|_M = 1$ and $\|\Sigma F\|_M = \alpha$ for some $\alpha \notin V_0$. Hence ψ_k is not a V_0-tautology.

10.11. Corollary. If $V_0 \cap N \neq \emptyset$ and $V_0 \subset V$ then $Taut_{V_0}$ is not recursively enumerable.

Proof. Let A be a diophantine non-recursive set; then $N - A$ is not recursively enumerable. If $\{\psi_k; k \in N\}$ is the sequence corresponding to A by the preceding lemma then $k \in N - A \Leftrightarrow \psi_k$ is a V_0-tautology. Hence $Taut_{v_0}$ is not recursively enumerable; if it were r.e., then so would be $N - A$ as a preimage of a r.e. set by means of a recursive function.

The part (iii) of 10.1 has been proved. It remains to prove (i) and (ii), but this is easy.

10.12. Lemma. If $V_0 \subset V$ then there is a closed formula which is not a V_0-tautology. (For example, ΣF.)

10.13. Lemma. If $V_0 \cap N \neq \emptyset$ then there is a closed formula which is a V_0-tautology. (For example, Cl_k for some $k \in V_0 \cap N$.)

10.14. Lemma. If $V_0 \cap N = 0$ then no closed formula is a V_0-tautology.

Proof. For every model, if ranges of all functions are included in N then values of all closed formulas are in N.

This completes the proof of 10.1.

11. CONCLUDING REMARKS

(i) The aim of the present paper was to analyze some general notions relevant to the idea of ALIOS but not to provide direct instructions for construction of programmes seeking good solutions. But *I* sincerely hope that new programmes seeking good solutions will be realized. The efficiency of such programmes will depend on the particular definitions of a semantical system and a system of problems (cf. 1.6 (4)) and also on the programmer's skill. One can find some examples of economizing the computing time in the GUHA papers. The matter can be described in present terminology roughly as follows: The programme is a cycle with parameters $h \in H$ (H being a hierarchy). After the computer has processed a h one does not take mechanically the next element of the hierarchy but, in dependence on the results X_h and the processed model $\boldsymbol{M}$, it finds a longer or shorter (possibly empty) interval in H that can be skipped. The programmer must decide whether the results (elements of the solution, maybe together with their values) will be kept in the computer's memory during the computation or not (they were not kept in the existing GUHA programmes).

For the time being one cannot decide whether one will succeed in building a modular programme that could be arranged into many programmes finding solutions of a broad variety of problems. If so then the analysis of notions such as those considered in the present paper will be one of the necessary assumptions.

(ii) Throughout the paper we did not mention *statistics.* At the same time many research workers seem not to be able to imagine any point of view other than statistical, i.e. observational statement are statistics used for testing of statistical hypotheses (= theoretical statements). I do not want to argue against the statistical point of view; it is tenable and perhaps the most widespread. But we simply did not need to speak about statistics. Methods of testing statistical hypotheses yield important particular cases of semantical systems and problems (cf. [8], [10] and below). Personally I believe that the statistical point of view is not the only possible one and that notions of the ALIOS theory are idependent of statistical notions.

(iii) The next step for the development of the ALIOS theory should consists in the study of particular classes of semantical systems and problems satisfying some reasonable restricting conditions and admitting a deeper particular theory. (An attempt in this direction has been accomplished by my student P. Horák). In particular, methods of testing statistical hypotheses should be examined from this point of view. One can also try to formulate general properties of correspondence rules between observation languages and theoretical languages using the apparatus of the ALIOS theory. (An attempt of a statistical nature is being made by T. Havránek.) More practical experience with particular computer programme is indispensable.

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Dr. Petr Hájek, CSc.: *Matematický ústav ČSAV (Mathematical Institute — Czechoslovak Academy of Sciences), Žitná 25, 115 67 Praha 1. Czechoslovakia.*