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On some Properties of Estimators of a Probability Density

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Let $X_1, X_2, \ldots, X_n$ be mutually independent and identically distributed s-dimensional random vectors with a probability density $p(x) \in L_2$.
Consider the class of estimators $\hat{p}_n(x)$ of the form

$$\hat{p}_n(x) = \frac{1}{n} \sum_{k=1}^{n} p_n(x - X_k)$$

for estimating of the density $p(x)$, where $p_n(x) \in L_2$. In this paper some consistency properties of estimators in the sense of mean integral square error are studied.

1. INTRODUCTION

Let

$$X_k = (X_{k1}, \ldots, X_{ks}), \quad k = 1, 2, \ldots, n,$$

be a random sample with a sample size n from a distribution $F(x)$, $x \in R^s$, with the probability density $p(x) \in L_2$, i.e. $X_1, \ldots, X_n$ are mutually independent and identically distributed s-dimensional random vectors with the density $p(x) \in L_2(R^s)$.

Consider the class of estimators $p_n(x)$ of the form

$$\hat{p}_n(x) = \frac{1}{n} \sum_{k=1}^{n} p_n(x - X_k)$$

for estimating of the density $p(x)$, where $p_n(x) \in L_2$. The quality of the estimators is defined by the mean integral square error (MISE)

$$I_n = E \int_{R^s} [\hat{p}_n(x) - p(x)]^2 \, dx .$$

Let

$$f_1(t) = \int_{R^s} e^{i(t,x)}\, p(x)\, dx\,, \quad f_2(t) = \int_{R^s} e^{i(t,x)}\, \hat{p}_n(x)\, dx\,,$$

$$f_3(t) = \int_{R^s} e^{i(t,x)}\, p_n(x)\, dx\,,$$

where $(t, x) = \sum_{k=1}^{s} t_k \,.\, x_k$ is the inner product of two elements t and x of R^s.

It is proved in [1] and [2] that

(1) $$I_n = \frac{1}{(2\pi)^s} \int_{R^s} \left\{\frac{1}{n} |f_3(t)|^2 \,.\, [1 - |f_1(t)|^2] + |f_1(t)|^2 \,.\, |1 - f_3(t)|^2\right\} dt$$

and if

$$f_3(t) = f_3^*(t) = \frac{n|f_1(t)|^2}{1 + (n-1)\,|f_1(t)|^2}\,,$$

$$f_3^*(t) = \int_{R^s} e^{i(t,x)}\, p_n^*(x)\, dx\,,$$

where

$$p_n^*(x) = \frac{1}{(2\pi)^s} \int_{R^s} e^{-i(t,x)} f_3^*(t)\, dt\,,$$

then the minimum of I_n is reached and

(2) $$I_n^* = \min I_n = \frac{p_n^*(0, 0, \ldots, 0)}{n} - o\left(\frac{1}{n}\right).$$

2. ASYMPTOTIC BEHAVIOUR OF I_n^*

For the sake of simplicity only the bivariate case is considered and asymptotic behaviour (if $n \to \infty$) of I_n^* is studied.

We say (see [3]), that the characteristic function $f_1(t)$ decreases exponentially with respect to coefficients $p_i > 0$ and with respect to t_i ($i = 1, 2$) if the following conditions are satisfied:

(a) $|f_1(t)| \leqq A\, e^{-(p,|t|)}$, for some constant $A > 0$ and for all t,

(b) $$\lim_{\substack{v_1 \to \infty \\ v_2 \to \infty}} \int_0^1 \int_0^1 [1 + e^{(p,v)} \,.\, |f_1(v_1 t_1, v_2 t_2)|^2]^{-1}\, dt = 0\,,$$

where

$$(p, |t|) = p_1 \,.\, |t_1| + p_2 \,.\, |t_2|\,, \quad (p, v) = p_1 v_1 + p_2 v_2\,.$$

The case when the characteristic function $f_1(t)$ decreases algebraically of degree $p_i > 0$ with respect to arguments t_i $(i = 1, 2)$ was considered in [1], [2].

Definition. Let $\varphi(n)$ be some function of n and let $\lim_{n\to\infty} \varphi(n) = \infty$. The estimator $\hat{p}_n(x)$ is said to be *integrally consistent of order* $\varphi(n)$ if

$$\lim_{n\to\infty} \varphi(n) \,.\, I_n = a \quad \text{where} \quad 0 < a < \infty$$

(see [4]).

Theorem 1. *Let the characteristic function $f_1(t)$ decrease exponentially with respect to coefficients $p_i > 0$ and arguments t_i $(i = 1, 2)$. Then the estimator $p_n^*(x)$ is integrally consistent of order $n/\ln^2 n$, namely:*

$$\lim_{n\to\infty} \frac{n}{\ln^2 n} \,.\, I_n^* = \frac{1}{8\pi^2 p_1 p_2}\,.$$

Proof.

$$\int_{R^2} \frac{|f_1(t)|^2}{1 + (n-1)\,|f_1(t)|^2}\,\mathrm{d}t - \int_{R^2} \frac{\exp\left[-2(p, |t|)\right]}{1 + (n-1)\exp\left[-2(p, |t|)\right]}\,\mathrm{d}t =$$

$$= \int_{R^2} \frac{|f_1(t)|^2 - \exp\left[-2(p, |t|)\right]}{\left[1 + (n-1)\,|f_1(t)|^2\right]\left[1 + (n-1)\exp\left[-2(p, |t|)\right]\right]}\,\mathrm{d}t = L.$$

Denote

$$v_1 = \frac{\ln(n-1)}{2p_1}, \quad v_2 = \frac{\ln(n-1)}{2p_2}.$$

Using the condition (a), we have

$$|L| \leqq 4(A^2 + 1)\int_0^\infty \int_0^\infty \frac{\exp\left[-2(p, t)\right]\mathrm{d}t_1\,\mathrm{d}t_2}{\left[1 + (n-1)\,|f_1(t)|^2\right]\left[1 + (n-1)\exp\left[-2(p, t)\right]\right]} \leqq$$

$$\leqq 4(A^2 + 1)\left\{\int_0^{v_1}\int_0^{v_2} \frac{\mathrm{d}t_1\,\mathrm{d}t_2}{(n-1)\left[1 + (n-1)\,.\,|f_1(t)|^2\right]} + \right.$$

$$\left. + \int_{v_1}^\infty \int_{v_2}^\infty \mathrm{e}^{-2(p,t)}\,\mathrm{d}t_1\,\mathrm{d}t_2 + \int_0^{v_1}\int_{v_2}^\infty \mathrm{e}^{-2(p,t)}\,\mathrm{d}t_1\,\mathrm{d}t_2 + \int_{v_1}^\infty \int_0^{v_2} \mathrm{e}^{-2(p,t)}\,\mathrm{d}t_1\,\mathrm{d}t_2\right\} = L_1\,.$$

Then, denoting

$$t_1 = v_1 s_1\,, \quad t_2 = v_2 s_2\,,$$

$$L_1 = 4(A^2 + 1)\left\{\frac{1}{n-1}\int_0^1\int_0^1 \frac{v_1 v_2\, ds_1\, ds_2}{1 + \exp[(p, v)] \,.\, |f_1(v_1 s_1, v_2, s_2)|^2} + \right.$$

$$+ \int_1^\infty \int_1^\infty v_1 v_2\, e^{-2(p,vs)}\, ds_1\, ds_2 + \int_0^1 \int_1^\infty v_1 v_2\, e^{-2(p,vs)}\, ds_1\, ds_2 +$$

$$\left. + \int_1^\infty \int_0^1 v_1 v_2\, e^{-2(p,vs)}\, ds_1\, ds_2\right\} =$$

$$= 4(A^2 + 1)\left\{\frac{\ln^2(n-1)}{4p_1 p_2(n-1)}\int_0^1\int_0^1 \frac{ds_1\, ds_2}{1 + \exp[(p, v)] \,.\, |f_1(v_1 s_1, v_2 s_2)|^2} + \right.$$

$$\left. + \frac{1}{2p_1 p_2(n-1)} - \frac{1}{4p_1 p_2(n-1)^2}\right\}.$$

Therefore, $L \to 0$ if $n \to \infty$.

Let us consider the following integral

$$K_n = \int_{R^2} \frac{\exp[-2(p, |t|)]}{1 + (n-1)\exp[-2(p, |t|)]}\, dt =$$

$$= -\frac{2}{p_2(n-1)}\int_0^\infty dt_1 \int_0^\infty d \ln[1 + (n-1)\, e^{-2p_1 t_1}\, e^{-2p_2 t_2}] =$$

$$= \frac{2}{p_2(n-1)}\int_0^\infty \ln[1 + (n-1)\, e^{-2p_1 t_1}]\, dt_1 = \frac{2}{p_2(n-1)}\, Q_n\,.$$

Let $x = (n-1)\, e^{-2p_1 t_1}$, then

$$Q_n = \frac{1}{2p_1}\int_0^{n-1} \frac{\ln(1+x)}{x}\, dx = \frac{1}{2p_1}\int_0^1 \frac{\ln(1+x)}{x}\, dx + \frac{1}{2p_1}\int_1^{n-1} \frac{\ln(1+x)}{x}\, dx\,.$$

Calculations of the two last integral give (see [5])

$$Q_n = \frac{\pi^2}{24p_1} + \frac{\ln^2(n-1)}{4p_1} + \frac{1}{2p_1} \,.\, \psi(n)\,,$$

where

$$\psi(n) = \sum_{k=1}^{\infty} \frac{1}{(2k)^2 \,.\, (n-1)^{2k}} - \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 \,.\, (n-1)^{2k-1}}\,.$$

It is easy to verify that

$$|\psi(n)| < \frac{1}{n-2}\,.$$

 Hence

$$K_n = \frac{\ln^2 (n-1)}{2p_1p_2(n-1)} + \frac{\pi^2}{12p_1p_2(n-1)} + o\left(\frac{1}{n^2}\right).$$

Therefore if $n \to \infty$, then

$$\frac{n}{\ln^2 n}\int_{R^2} \frac{|f_1(t)|^2}{1+(n-1)\,.\,|f_1(t)|^2}\,\mathrm{d}t \to \frac{1}{2p_1p_2}.$$

Thus, from expression (2) with $s = 2$ the statement of the Theorem 1 follows.

Hence, Theorem 1 is proved.

3. ASYMPTOTIC BEHAVIOUR OF I_n

The estimator $\hat{p}_n(x)$ is said to be of an *exponential type* if

(3) $$f_3(t) = h(A_n\,.\,\mathrm{e}^{\alpha|t_1|}, A_n\,.\,\mathrm{e}^{\alpha|t_2|}),$$

where

$$\lim_{n\to\infty} A_n = 0\,, \quad |h(t)| \leqq B\,, \quad h(t) \in L_2\,, \quad t \in R^2\,.$$

Let us consider the following condition:

(4) $$|1 - h(t)| \leqq B_1|t_1t_2| \quad \text{for all} \quad |t_i| \leqq 1\,, \quad i = 1, 2\,.$$

Theorem 2. *Let $f_1(t)$ satisfy the condition (a), let the estimator $\hat{p}_n(x)$ be of an exponential type, let the function $h(t)$ satisfy (4) and let*

$$A_n = D\,.\,n^{-b}\,, \quad b > \tfrac{1}{2}\,, \quad \alpha \leqq p_i \quad (i = 1, 2)\,.$$

Then the estimator $\hat{p}_n(x)$ is integrally consistent of order $n/\ln^2 n$, namely:

$$\lim_{n\to\infty} \frac{n}{\ln^2 n}\,.\,I_n = \left(\frac{b}{\pi\alpha}\right)^2.$$

First we formulate two lemmas.

Lemma 1. *Under the conditions of Theorem 2 we have*

$$\lim_{n\to\infty} \frac{1}{\ln^2 n}\int_{Dn^{-b}}^{\infty}\int_{Dn^{-b}}^{\infty} \frac{h^2(t_1, t_2)}{t_1\,.\,t_2}\,\mathrm{d}t_1\,\mathrm{d}t_2 = b^2\,.$$

Proof.

$$(5)\qquad \frac{1}{\ln^2 n}\int_{Dn^{-b}}^{\infty}\int_{Dn^{-b}}^{\infty}\frac{h^2(t_1,t_2)}{t_1\,.\,t_2}\,dt_1\,dt_2 =$$

$$= \frac{1}{\ln^2 n}\int_{Dn^{-b}}^{1}\int_{Dn^{-b}}^{1}\frac{dt_1\,dt_2}{t_1\,.\,t_2} - \frac{1}{\ln^2 n}\int_{Dn^{-b}}^{1}\int_{Dn^{-b}}^{1}\frac{1-h^2(t_1,t_2)}{t_1\,.\,t_2}\,dt_1\,dt_2 +$$

$$+ \frac{1}{\ln^2 n}\int_{1}^{\infty}\int_{1}^{\infty}\frac{h^2(t_1,t_2)}{t_1\,.\,t_2}\,dt_1\,dt_2 + \frac{1}{\ln^2 n}\int_{Dn^{-b}}^{1}\int_{1}^{\infty}\frac{h^2(t_1,t_2)}{t_1\,.\,t_2}\,dt_1\,dt_2 +$$

$$+ \frac{1}{\ln^2 n}\int_{1}^{\infty}\int_{Dn^{-b}}^{1}\frac{h^2(t_1,t_2)}{t_1\,.\,t_2}\,dt_1\,dt_2\,.$$

The first term of (5) tends to b^2 if $n \to \infty$. It follows from conditions (4) and $h(t_1, t_2) \in L_2$, that the other terms of (5) tend to zero if $n \to \infty$.

Lemma 1 is proved.

Lemma 2. *Under the conditions of Theorem 2 we have*

$$\int_{R^2} |f_1(t)|^2\,.\,|1 - f_3(t)|^2\,dt = o\left(\frac{\ln^2 n}{n}\right).$$

Proof.

$$\frac{n}{\ln^2 n}\int_{R^2} |f_1(t)|^2\,.\,|1 - f_3(t)|^2\,dt =$$

$$= \frac{4n}{\ln^2 n}\int_0^{\infty}\int_0^{\infty}|f_1(t_1,t_2)|^2\,.\,|1 - f_3(t_1,t_2)|^2\,dt_1\,dt_2 = Q\,.$$

Denote

$$x_1 = Dn^{-b}e^{\alpha t_1}\,, \quad x_2 = Dn^{-b}e^{\alpha t_2}\,.$$

Then we have

$$Q = n\left(\frac{2}{\ln n}\right)^2\int_{Dn^{-b}}^{\infty}\int_{Dn^{-b}}^{\infty}\left|f_1\left[\frac{1}{\alpha}\ln\left(\frac{n^b}{D}x_1\right),\ \frac{1}{\alpha}\ln\left(\frac{n^b}{D}x_2\right)\right]\right|^2.$$

$$.\,\frac{|1 - h(x_1,x_2)|^2}{\alpha^2 x_1 x_2}\,dx_1\,dx_2 \leqq n\left(\frac{2}{\ln n}\right)^2\int_{Dn^{-b}}^{1}\int_{Dn^{-b}}^{1}A^2\,.$$

$$.\,\exp\left\{-2\left[\frac{p_1}{\alpha}\ln\left(\frac{x_1}{Dn^{-b}}\right) + \frac{p_2}{\alpha}\ln\left(\frac{x_2}{Dn^{-b}}\right)\right]\right\}B_1^2\,\frac{x_1x_2}{\alpha^2}\,dx_1\,dx_2 +$$

$$+ n\left(\frac{2}{\ln n}\right)^2\int_1^{\infty}\int_1^{\infty}(1 + B)^2\,A^2\exp\left\{-2\left[\frac{p_1}{\alpha}\ln\left(\frac{x_1}{Dn^{-b}}\right) + \frac{p_2}{\alpha}\ln\left(\frac{x_2}{Dn^{-b}}\right)\right]\right\}.$$

$$\cdot \frac{dx_1\,dx_2}{\alpha^2 x_1 x_2} + n\left(\frac{2}{\ln n}\right)^2 \int_1^\infty \int_{Dn^{-b}}^1 (1 + B)^2 A^2 \exp\left\{-2\left[\frac{p_1}{\alpha}\ln\left(\frac{x_1}{Dn^{-b}}\right) + \right.\right.$$

$$\left.\left. + \frac{p_2}{\alpha}\ln\left(\frac{x_2}{Dn^{-b}}\right)\right]\right\} \frac{dx_1\,dx_2}{\alpha^2 x_1 x_2} + n\left(\frac{2}{\ln n}\right)^2 \int_{Dn^{-b}}^1 \int_1^\infty (1 + B)^2 A^2 \,.$$

$$.\exp\left\{-2\left[\frac{p_1}{\alpha}\ln\left(\frac{x_1}{Dn^{-b}}\right) + \frac{p_2}{\alpha}\ln\left(\frac{x_2}{Dn^{-b}}\right)\right]\right\} \frac{dx_1\,dx_2}{\alpha^2 x_1 x_2} = Q_1 + Q_2 + Q_3 + Q_4\,.$$

Since $p_i \geqq \alpha$, $2b > 1$ we have

$$Q_1 \leqq n\left(\frac{2}{\ln n}\right)^2 \frac{A^2 B_1^2}{\alpha^2} \int_{Dn^{-b}}^1 \exp\left[-2\ln\left(\frac{x_1}{Dn^{-b}}\right)\right] x_1\,dx_1\,.$$

$$.\int_{Dn^{-b}}^1 \exp\left[-2\ln\left(\frac{x_2}{Dn^{-b}}\right) x_2\,dx_2 = n\left(\frac{2}{\ln n}\right)^2 \frac{A^2 B_1^2}{\alpha^2} \frac{D^4}{n^{4b}} (b\ln n - \ln D)^2 \xrightarrow[n\to\infty]{} 0\,,\right.$$

$$Q_2 \leqq n\left(\frac{2}{\ln n}\right)^2 \frac{(1 + B)^2 A^2}{\alpha^2} \int_1^\infty \exp\left[-2\ln\left(\frac{x_1}{Dn^{-b}}\right)\right] dx_1\,.$$

$$.\int_1^\infty \exp\left[-2\ln\left(\frac{x_2}{Dn^{-b}}\right)\right] dx_2 = n\left(\frac{2}{\ln n}\right)^2 \frac{(1 + B)^2 A^2}{\alpha^2} \frac{D^4}{n^{4b}} \xrightarrow[n\to\infty]{} 0\,,$$

$$Q_3 \leqq n\left(\frac{2}{\ln n}\right)^2 \frac{(1 + B)^2 A^2}{\alpha^2} \int_{Dn^{-b}}^1 \exp\left[-2\ln\left(\frac{x_1}{Dn^{-b}}\right)\right] \frac{dx_1}{x_1}\,.$$

$$.\int_1^\infty \exp\left[-2\ln\left(\frac{x_2}{Dn^{-b}}\right)\right] dx_2 = n\left(\frac{2}{\ln n}\right)^2 \frac{(1 + B)^2 A^2}{2\alpha^2} \frac{D^4}{n^{4b}} \left(\frac{n^{2b}}{D^2} - 1\right) \xrightarrow[n\to\infty]{} 0\,.$$

Also

$$Q_4 \xrightarrow[n\to\infty]{} 0\,.$$

Lemma 2 is proved.

Proof of Theorem 2. It follows from (1) that

$$n\left(\frac{2\pi}{\ln n}\right)^2 I_n = \frac{1}{\ln^2 n}\int_{R^2} |f_3(t)|^2 \left[1 - |f_1(t)|^2\right] dt + \frac{n}{\ln^2 n}\int_{R^2} |f_1(t)|^2\, |1 - f_3(t)|^2\, dt =$$

$$= \frac{1}{\ln^2 n}\int_{R^2} h^2(Dn^{-b} e^{\alpha|t_1|}, Dn^{-b} e^{\alpha|t_2|})\, dt - \frac{1}{\ln^2 n}\int_{R^2} h^2(Dn^{-b} e^{\alpha|t_1|}, Dn^{-b} e^{\alpha|t_2|})\,.$$

$$.\, |f_1(t)|^2\, dt + \frac{n}{\ln^2 n}\int_{R^2} |f_1(t)|^2\, |1 - f_3(t)|^2\, dt\,.$$

The first term of the last expression tends to $(2b/\alpha)^2$ if $n \to \infty$ according to Lemma 1.

The second term tends to zero since $|h(t)| \leqq B$ for all (t_1, t_2).
The third term tends also to zero according to Lemma 2.
Thus

$$\lim_{n\to\infty} \frac{n}{\ln^2 n} I_n = \left(\frac{b}{\pi\alpha}\right)^2 .$$

Hence Theorem 2 is proved.

4. EXAMPLES

The following examples show that the order of consistency of an estimator can be reached.

Example 1. Let us consider the probability density of the form

$$p(x, y) = \frac{1}{\pi^2(1 + x^2)(1 + y^2)} .$$

Obviously,

$$f(t_1, t_2) = e^{-|t_1| - |t_2|} .$$

Then

$$p_n^*(0, 0) = \frac{4n}{(2\pi)^2} \int_0^\infty \int_0^\infty \frac{\exp[-2t_1 - 2t_2]}{1 + (n - 1)\exp[-2t_1 - 2t_2]} \mathrm{d}t_1 \, \mathrm{d}t_2 =$$

$$= \frac{n \ln^2 (n - 1)}{8\pi^2(n - 1)} + \frac{n}{48(n - 1)} + O\left(\frac{1}{n}\right).$$

Thus if $n \to \infty$

$$I_n^* = \frac{1}{8\pi^2} \frac{\ln^2 n}{n} + \frac{1}{48} \frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

Example 2. Now, we also consider the density of Cauchy distribution

$$p(x, y) = \frac{1}{(2\pi)^2} \frac{1}{(1 + x^2 + y^2)^{3/2}} .$$

In this case

$$f(t_1, t_2) = \frac{1}{(2\pi)^2} \int_{R^2} \exp[i(t_1 x + t_2 y)] \, p(x, y) \, \mathrm{d}x \, \mathrm{d}y = \exp[-\sqrt{(t_1^2 + t_2^2)}] ,$$

and

$$p_n^*(0, 0) = \frac{4n}{(2\pi)^2} \int_0^\infty \int_0^\infty \frac{\exp[-2\sqrt{(t_1^2 + t_2^2)}]}{1 + (n - 1)\exp[-2\sqrt{(t_1^2 + t_2^2)}]} \mathrm{d}t_1 \, \mathrm{d}t_2 .$$

Using the polar coordinates $t_1 = \varrho \cos \theta$, $t_2 = \varrho \sin \theta$ we have

$$p_n^*(0, 0) = \frac{n}{4\pi(n-1)}\left[\frac{\pi^2}{24} + \frac{\ln^2(n-1)}{4} + O\left(\frac{1}{n}\right)\right].$$

Hence

$$I_n^* = \frac{1}{16\pi}\frac{\ln^2 n}{n} + \frac{\pi}{96}\frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

Example 3. At last we consider an example where the characteristic function decreases more rapidly then a characteristic function with properties (a), (b). Denote the density of the bivariate normal distribution by $p(x, y)$, i.e.

$$p(x, y) = \frac{1}{2\pi\sqrt{(1-r^2)}} \exp\left\{-\frac{1}{2(1-r^2)}\left[x^2 - 2rxy + y^2\right]\right\}, \quad r^2 \neq 1.$$

Obviously

$$f(t_1, t_2) = \exp\left\{-\tfrac{1}{2}\left[t_1^2 + 2rt_1t_2 + t_2^2\right]\right\}.$$

Then

$$p_n^*(0, 0) = \frac{4n}{(2\pi)^2}\int_0^\infty\int_0^\infty \frac{|f(t_1, t_2)|^2}{1 + (n-1)|f(t_1, t_2)|^2}\, dt_1\, dt_2 =$$

$$= \frac{n}{\pi^2}\int_0^{\pi/2} d\theta \int_0^\infty \frac{\varrho \exp\left[-(\varrho^2 + r\varrho^2 \sin 2\theta)\right]}{1 + (n-1)\exp\left[-(\varrho^2 + r\varrho^2\sin 2\theta)\right]}\, d\varrho.$$

Thus (see [5])

$$I_n^* = \frac{1}{2\pi^2\sqrt{(1-r^2)}}\left[\frac{\pi}{2} - \operatorname{arctg}\frac{r}{\sqrt{(1-r^2)}}\right]\frac{\ln n}{n} + O\left(\frac{1}{n}\right).$$

The example 3 shows that the order of consistency of the estimator is the best possible in this case.

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