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# Automatic Listing of Important Observational Statements I

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The theory of automatic listing of important observational statements (the ALIOS theory) is a part of applied mathematical logic which introduces and studies general notions of abstract semantics and of objective epistemology from the point of view of applications to computer programmes processing observational data and formulating observational statements.

## 0. INTRODUCTION

I present here a theory which can be considered as a part of applied mathematical logic. To specify the subject of this theory, I shall try to explain (i) what is to be described (what are the basic notions of the theory) and (ii) why or how it should be described, i.e. I shall sketch the motivation of our study.

(i) Our first question reads: *What sentences can be stated by a research worker and what is their meaning*? (By the way, this, question is treated in the so-called extensional semantic, which could be said to be a classical part of mathematical logic; but our treatment will be a little unusual.) Our second question reads: *what is the aim, matter of interest, intention or orientation of the research worker*? Hence, we shall pay attention to the fact that the research worker holds some statements as important (relevant) and some not, that he tries to find the meaning of some sentences. i.e. the research worker solves problems.

(ii) The main motivation of our considerations is not the interest to find and appropriate general philosophical explication of the family of notions sketched above (even if the question of philosophical adequacy cannot be fully ignored). We are led by the observation that if we succeed in formalizing the mentioned notions in a way understandable to the *computer* then the computer can automatically perform a certain (not unimportant) part of the process of scientific research. Our definitions are adequate at least in some cases, maybe in many; *whenever* they are adequate, the computer can help considerably.

The present theory is an attempt to state a theoretical extract and find a generalization of papers on the GUHA method (written by myself and others, see [6]–[11]). We shall formulate the main idea of that method; but let us first recall the distinction between *observational and theoretical terms and sentences* (see [11])*, which will be useful for our purpose. Observational terms and sentences must be *decidable* (effectively evaluable or verifiable); not theoretical ones. Theoretical sentences are related with observational ones by means of some *correspondence rules*.

Using this we may (re)formulate the main idea of the GUHA method as the *aim to obtain automatically all the important observational statements concerning the given observational data.*** The computer can be delegated both the systematical *formulation* and the *evaluation* of observational sentences, i.e. we let the computer know (besides others) observational data, the syntax of the observational language and evaluating procedures***. The computer provides a systematic list of all the statements important from the point of view of the given problem and of the given data. Particular versions of the GUHA method differ by various particular definitions of accepted observational data, sentences and their meanings and by criteria of relevance.

In the present paper, we do not propose any new version of the GUHA method (particular versions serve as examples) but we shall try to formulate notions relevant for each possible version in a form as general as possible; we shall also investigate the relations of these notions.

Our programme could be perhaps classified as an epistemological our heuristical study, but with the following reservations: As far as epistemology is concerned, one has to think of "epistemology without a knowing subject" (see Popper [5]†). And

* Carnap writes b.o. the following: In discussions on the methodology of science, it is customary and useful to divide the language of science into two parts, the observation language and theoretical language. The observation language uses terms designating observable properties and relations for the description of observable things or events. The theoretical language, on the other hand, contains terms which may refer to unobservable events, unobservable aspects or features of events, e.g. to micro-particles (...) in physics, to drives and potentials (...) in psychology, etc. (Note that Carnap uses the word "observational" in the compound term "observational vocabulary".)

** In the papers on the GUHA method, one speaks of automatically obtaining all the important hypotheses from given experimental material; I consider the present formulation to be more exact and more general. For, first, observational data need not be of experimental nature; and, secondly, all the existing forms of the GUHA method find in fact certain observational statements true (in some sense) for observed objects. The hypothesis then is that the same or similar statements hold in general and one has stressed the fact that GUHA offers hypotheses that must be submitted to further verification. Thus one is led to some correspondence rules.

*** The computer could also be told some correspondence rules enabling it to formulate some theoretical statements; this possibility is not discussed in the present paper. The formulation of theoretical statements on the basis of observational ones remains the task of the research worker.

† Popper claims that there are "two different senses of knowledge or of thought: knowledge or thought in the subjective sense, consisting of a state of mind or of consciousness or a dispositon

when one speaks on heuristics in connection with computers, one always supposes – as far as I know – that the computer should *imitate* the behavior of a solver in a problem situation; however, here we do not want simulation but *replacement*. This means that the task of the computer is not to look for a solution but to find one; it should not imitate the research worker's more or less casual asking of questions and finding of answers but replace it by a systematic list.

I have not succeeded in finding a comprehensive name for the presented theory; but our guiding idea – *Automatic Listing of Important Observational Statements* – enables us to form a readable abbreviations ALIOS. I offer this abbreviation as a temporary name for the new theory.

The paper is divided into two parts. In §§ 1 – 6 we take the notion of a sentence and its meaning for granted (by the definition of a semantic system from § 1) and we do not provide sentences with any structure. We define problems and their solutions and consider various notions of worth of a solution. In §§ 7 – 10 we study the structure of sentences and the ways in which sentences take meanings. The functor calculi defined there generalize the predicate calculus; they enable us to illustrate notions from the first part in detail and also to derive some theoretical consequences. (The concluding § 11 contains some remarks.)

I gave a short communication on the ALIOS theory at the IV International Congress for Logic, Methodology and Philosophy of Science (Bucharest 1971, see [11]). Various preliminary versions of this paper were presented in the seminar of applications of mathematical logic at the Faculty of Mathematics and Physics of the Charles University, Prague. I thank the members of this seminar for their patience. I also wish to express my gratitude to my colleagues Doc. Dr. J. Bečvář, Dr. I. M. Havel, Dr. T. Havránek, Dr. J. Polívka and Dr. Z. Renc for their numerous stimulating remarks on the Czech version of the present paper and to Mrs. C. Vondrejs for her kind help with the translation into English.

## Part I — Problems and solutions

### 1. BASIC NOTIONS

We adopt Frege's theory of names and their meanings (see [2] – Introduction). In this theory, *sentences* are particular names – names with particular meanings. One usually considers two possible meanings of sentences, namely *truth* and *falsehood*. In many-valued logical systems one considers more abstract truth values. Here we assume that sentences take values from a set of *abstract values*; these values may, but need not be considered truth values. (See examples in § 8.) We use sentences to speak *about* something; the meaning (value) of a sentence is dependent on this

to behave or to react, and knowledge in an objective sense, consisting of problems, theories and arguments as such."

"something". For the present, we are not interested in how this dependence is managed; we only keep in mind that the meaning is a function of sentences and, in addition, of some non-linguistical entities, which will be called models. We are led to the following definition:

**1.1. Definition.** A *semantical system* is a quadruple $\mathfrak{S} = \langle Sent, \mathfrak{M}, V, Val\rangle$, where $Sent$, $\mathfrak{M}$, $V$ are non-empty sets and $Val$ is a mapping of the cartesian product $Sent \times \mathfrak{M}$ into $V$. We write $\|\varphi\|_M$ instead of $Val(\varphi, M)$ for $\varphi \in Sent$ and $M \in \mathfrak{M}$; $\|\varphi\|_M$ is read "*the value of the sentence* $\varphi$ *in the model* $M$". If $V_0 \subseteq V$ and $M \in \mathfrak{M}$ then $Tr_{V_0}(M) = \{\varphi \in Sent;\ \|\varphi\|_M \in V_0\}$ is the set of all the sentences $V_0$-*true in* $M$. A sentence $\varphi$ is a $V_0$-*tautology* (in $\mathfrak{S}$) if $\|\varphi\|_M \in V_0$ for each $M \in \mathfrak{M}$. Sentences $\varphi$, $\psi$ are $V_0$-*equivalent* (in $\mathfrak{S}$) if $\|\varphi\|_M \in V_0 \Leftrightarrow \|\psi\|_M \in V_0$ for each $M \in \mathfrak{M}$; they are *strongly equivalent* (in $\mathfrak{S}$) if $\|\varphi\|_M = \|\psi\|_M$ for each $M \in \mathfrak{M}$.

**1.2. Discussion.** We shall explain the relation of the notions just defined to the considerations of § 0; we shall further formulate intuitive assumptions that should be satisfied when a semantical system is to be called observational and when automatic listing of important observational statements is to be meaningful; finally, we are led to some new notions.

(1) We imagine that the research worker has a *semantical system* at his disposal; i.e., he is able to express an arbitrary sentence $\varphi \in Sent$ and having *observational data* $M \in \mathfrak{M}$ he is able to determine $\|\varphi\|_M$. If one wants to call $\mathfrak{S}$ an *observational semantical system* then it is necessary that the function $Val$ is calculable (in a sense). We further assume that the research worker has a non-empty set $V_0 \subseteq V$ of *designated values*; having expressed $\varphi$ he wants to know whether $\|\varphi\|_M \in V_0$ or not. A sentence $\varphi$ can be $V_0$-*asserted* ($V_0$-*stated*), i.e. having expressed (pronounced) $\varphi$ one wants to say that the $M$-value of $\varphi$ is in $V_0$. (This is a natural generalization of the classical case where $V = \{true, false\}$; a sentence is asserted if having expressed it one wants to say that it is true.)*

(2) *Intuitive assumptions on the size of Sent.* It is *quite big* (a human being would not be able to pronounce all of its elements in a reasonable time); sometimes we allow *Sent* to be infinite and sometimes (thinking of a computer) we assume that it is *not too big* (the computer can pass through it or generate it in a reasonable time; but we do not assume that the computer can generate e.g. all the subsets of *Sent* in a reasonable time). We further assume that, given $\varphi$ and $M$ the computer *calculates* $\|\varphi\|_M$ *quickly*. (For examples see the GUHA-papers and also Part II of the present paper.)

* Imagine a situation (in a research centre) in which "the correlation coefficient of the quantities $F_1$, $F_2$" is a $V_0$-asserted sentence (a $V_0$-statement) where $V_0 = \langle 0{\cdot}9, 1\rangle$. Evidently, instead of $V_0$-asserting this one can assert ($\{true\}$-assert) the sentence "the correlation coefficient of $F_1$, $F_2$ lies in the interval $\langle 0{\cdot}9, 1\rangle$", but, at any rate, one need not do it.

(3) A $V_0$-true sentence need not be important from the point of view of the data $\boldsymbol{M}$, e.g. if we know it to be a $V_0$-tautology. We assume that the research worker (or the computer) has a (big) set $F \subseteq Sent$ of *relevant questions*. $F$ is usually defined in a syntactical way; but, for the present, we do not take any account of this. (We continue to assume that he has observational data $\boldsymbol{M} \in \mathfrak{M}$.) His aim is (at a certain stage of his research) to know $F \cap Tr_{V_0}(\boldsymbol{M})$, i.e. all the relevant questions that can be converted into $V_0$-statements (about $\boldsymbol{M}$). The set $F \cap Tr_{V_0}(\boldsymbol{M})$ is to be presented in a reasonable (economical) way; the mere list would be too long and therefore of little use. One can make use of the fact that some sentences are *immediate consequences* of others; knowing that $\varphi_1, \ldots, \varphi_n \in Tr_{V_0}(\boldsymbol{M})$ one sees at glance that also a certain $\varphi$ is in $Tr_{V_0}(\boldsymbol{M})$. So we look for a set $X \subseteq Tr_{V_0}(\boldsymbol{M})$ such that each $\varphi \in F \cap Tr_{V_0}(\boldsymbol{M})$ either belongs to $X$ or is an immediate consequence of some sentences in $X$. Such an $X$ will be called a *solution* of our problem. 

(4) We are obliged to spell out what determines a *problem*, whose solution has just been described. Given a fixed semantical system $\mathfrak{S}$, a problem is given (a) by a set $V_0$ of designated values, (b) by a set $F$ of relevant questions and (c) by a notion of immediate consequence (i.e. by what we admit to see at glance). We shall investigate relations of immediate consequence, both syntactically (without any respect to the values) and semantically (with respect to the values; this leads to the notion of $V_0$-soundness).

(5) It follows from our intuitive assumptions that the research worker not using a device like a computer can find only a (small) subset $X$ of $Tr_{V_0}(\boldsymbol{M})$ and has (sees) sentences immediately following from $X$. But he does not know whether he has (sees) the whole of $F \cap Tr_{V_0}(\boldsymbol{M})$, i.e. whether $X$ is a solution. This is why the *computer* is indispensable for finding solutions.

We now formulate exact (formal) definitions.

**1.3. Definition.** Let *Sent* be a non-empty set. A *relation of immediate consequence* (i.c.) on *Sent* is an arbitrary set $IC \subseteq Sent \times \mathfrak{P}_{fin}(Sent)$ (we use $\mathfrak{P}_{fin}(Sent)$ to denote the set of all finite subsets of *Sent*). If $IC$ is a relation of i.c. on *Sent* then the pair $\boldsymbol{L} = \langle Sent, IC \rangle$ is called a *syntactical system*. We write $\varphi ICe$ instead of $\langle \varphi, e \rangle \in IC$ and read this "*$\varphi$ is an immediate consequence of e*". Let $X \subseteq Sent$ and let $\varphi \in Sent$. $\varphi$ is said to *immediately follow from* $X$ (denotation: $\varphi \in IC(X)$) if either $\varphi \in X$ or there is a finite $e \subseteq X$ such that $\varphi ICe$. A sequence $\varphi_1, \ldots, \varphi_n$ of sentences is a *proof* from $X$ in $\boldsymbol{L}$ if for each $i = 1, \ldots, n$ either $\varphi_i \in X$ or $\varphi_i$ immediately follows from $\{\varphi_1, \ldots, \varphi_{i-1}\}$. A sentence $\varphi$ is *provable* from $X$ if it is a member of some proof from $X$.

**1.4. Definition.** Let $\mathfrak{S} = \langle Sent, \mathfrak{M}, V, Val \rangle$ be a semantical system, let $IC$ be a relation of i.c. on *Sent* and let $V_0 \subseteq V$. $IC$ is said to be $V_0$-*sound* w.r.t. $\mathfrak{S}$ if, for each $\varphi \in Sent$, $e \in \mathfrak{P}_{fin}(Sent)$ and $\boldsymbol{M} \in \mathfrak{M}$, $\varphi ICe$ and $(\forall \varphi \in e)(\|\psi\|_{\boldsymbol{M}} \in V_0)$ implies $\|\varphi\|_{\boldsymbol{M}} \in V_0$.

**1.5. Definition.** Let $\mathfrak{S} = \langle Sent, \mathfrak{M}, V, Val \rangle$ be a semantical system. An $\mathfrak{S}$-*problem* is a triple $\boldsymbol{P} = \langle F, V_0, IC \rangle$ where $\emptyset \neq F \subseteq Sent$, $\emptyset \neq V_0 \subseteq V$ and $IC$ is a relation of i.c. on *Sent* $V_0$-sound w.r.t. $\mathfrak{S}$. Let, in addition, $\boldsymbol{M} \in \mathfrak{M}$; a *solution* of $\boldsymbol{P}$ in $\boldsymbol{M}$ is an arbitrary $X \subseteq Tr_{V_0}(\boldsymbol{M})$ such that $F \cap Tr_{V_0}(\boldsymbol{M}) \subseteq IC(X)$.

**1.6. Discussion.** We shall answer some questions concerning the adequacy of the notions just defined with respect to observational semantical systems and to automatic listing of important observational statements.

(1) Let a semantical system $\mathfrak{S}$ and an $\mathfrak{S}$-problem $\boldsymbol{P}$ be given. If $\varphi ICe$, *do we really see it at glance?* Or do we see $\{\varphi;\ \varphi ICe\}$? Of course, this does not follow from the mathematical definitions and must be separately guaranteed in each particular case. But we realize that our motivation does not allow us to replace immediate consequence in the definition of a solution by provability; given $\varphi$ and $e$, one can in reasonable cases hardly suppose that we can see at glance whether $\varphi$ is provable from $e$.

(2) Is it reasonable to suppose $IC$ to be $V_0$-sound? (The research worker perhaps only *believes* $IC$ to be $V_0$-sound.) The assumption of $V_0$-soundness is indispensable for further theory. The reader should imagine that the nature of $IC$ is logical and mathematical rather than empirical. On the other hand, note that one can force $IC$ to be $V_0$-sound by diminishing $\mathfrak{M}$; simply omit each $\boldsymbol{M}$ such that, for some $\varphi$ and $e$, we have $\varphi ICe$, $e \subseteq Tr_{V_0}(\boldsymbol{M})$ and $\varphi \neq Tr_{V_0}(\boldsymbol{M})$. Provided the resulting set is non-empty, we obtain a new semantical system for which the given $IC$ in $V_0$-sound. This leads us to the following question:

(3) *What is* $\mathfrak{M}$? Our definition of a semantical system concerned two aspects; wo concentrate sometimes on the former, sometimes on the latter. We think of $\mathfrak{M}$ as

- the set of possible families of observational data,
- the set of possible computer inputs.

(4) What does each *programme for automatic listing of important observational data* (each GUHA-method, if you want) look like? The programme presupposes a fixed observational semantical system $\mathfrak{S}$. The input consists of (a) a particular model $\boldsymbol{M}$ (observational data) and (b) a parameter $p$ determining a problem $\boldsymbol{P}(p) = \langle F(p), V_0(p), IC(p) \rangle$. Hence, given $p$, the programme understands what the problem is.

The output is a solution $X(\boldsymbol{M}, p)$ of $P(p)$ in $\boldsymbol{M}$.

(5) *What statements are important*? Solutions are sets of sentences and as such can be compared according to various criteria; in particular, we can define various conditions for solutions to be optimal. Each criterion can be represented by a (partial) quasi-ordering $\prec$ of the set *Sent* (see § 3 for definition, if necessary); given $\boldsymbol{M}$ and $p$, the programme should find a solution as good (small) w.r.t. $\prec$ as possible. (Since the computer cannot pass through various solutions but must construct one solution (and also since $\prec$ need not have any least element) one can find the best solution only

in some particular cases.) Given an $\mathfrak{S}$, $\{P(p), p \text{ parameter}\}$ and the programme determining $X(M, p)$, we say that a sentence $\varphi$ is an important statement on $M$ if $\varphi \in X(M, p)$. (In particular cases this definition can be made independent of the programme.) The chosen criterion $\prec$ and the quality of the programme determine to what extent this notion of importance is adequate. We see that we are forced to compare solutions, e.g. w.r.t. inclusion or w.r.t. cardinality.

## 2. PROPERTIES OF SYNTACTICAL SYSTEMS; INDEPENDENCE

We shall first consider the possibility of changing a relation of immediate consequence without changing the corresponding operation associating with each set $X$ the set of all sentences that follow immediately from $X$. (Note that $\varphi ICX$ is not the same as $\varphi \in IC(X)$!) (We shall see e.g. in 5.4 that it is desirable to make the relation of i.c. as small as possible.) Then we shall be interested in $\subseteq$-minimal and card-minimal sets of sentences having various properties*. Such a property is e.g. – given a relation of i.c. – "to be a solution of an $\mathfrak{S}$-problem $P$" or – more generally and more syntactically – "to be $Y$-sufficient" ($Y$ a set of sentences, see below). We shall introduce two notions of independence for sets of sentences and establish their relation to one another and to the notions of minimality among $Y$-sufficient sets ($Y$ fixed). Finally we summarize consequences of the established facts for solutions of $\mathfrak{P}$-problems (Theorem 2.30).

**2.1. Definition.** A pair $\langle \varphi, e\rangle$ where $e = \{\varphi_1, \ldots, \varphi_n\}$ will be sometimes denoted by $\frac{\varphi_1, \ldots, \varphi_n}{\varphi}$ (cf. the usual way of expressing deduction rules). Hence $\varphi ICe$ means the same as $\frac{\varphi_1, \ldots, \varphi_n}{\varphi} \in IC$.

**2.2. Definition.** Let $IC_1$, $IC_2$ be relations of i.c. on a set *Sent*. $IC_1$ and $IC_2$ are said to be *equivalent* if $IC_1(X) = IC_2(X)$ for each $X \subseteq Sent$ (i.e. if the same sentences immediately follow from $X$ in the sense of $IC_1$ as in the sense of $IC_2$.)

In the sequel we assume that an arbitrary syntactical system $L = \langle Sent, IC\rangle$ is fixed.

**2.3. Theorem.** There is a $\subseteq$-largest and a $\subseteq$-least relation of i.c. equivalent to $IC$.

Proof. Put $\varphi\overline{IC}e \Leftrightarrow \varphi \in \vee (\exists e_0 \subseteq e)(\varphi ICe_0)$; then $IC \subseteq \overline{IC}$ and hence $IC(X) \subseteq \overline{IC}(X)$ for each $X$. If $\varphi \in \overline{IC}(X)$ then $\varphi \in X \vee (\exists e \subseteq X)(\varphi\overline{IC}e)$, hence $\varphi \in X \vee$

* $X$ is a $\subseteq$-*minimal* set with a property $\mathscr{P}$ if $X$ has the property $\mathscr{P}$ and no proper subset of $X$ has the property $\mathscr{P}$; $X$ is a *card-minimal* set with the property $\mathscr{P}$ if $X$ has the property $\mathscr{P}$ and no set of smaller cardinality has the property $\mathscr{P}$.

$\vee\ (\exists e \subseteq X)(\exists e_0 \subseteq e)(\varphi ICe_0)$, hence $\varphi \in X \vee (\exists e_0 \subseteq X)(\varphi ICe_0)$ and consequently $\varphi \in IC(X)$. So we have $\overline{IC}(X) = IC(X)$ for each $X$. Suppose that $IC_1 \nsubseteq \overline{IC}$, i.e. there are $\varphi, e$ such that $\varphi IC_1 e$ but $\varphi \notin e$ and there is no $e_0 \subseteq e$ such that $\varphi ICe_0$. Consequently, $\varphi \in IC_1(e) - IC(e)$. We see that $\overline{IC}$ is the $\subseteq$-largest relation of i.c. equivalent to $IC$.

Now put $\varphi IC^0 e \Leftrightarrow \varphi ICe \;\&\; \varphi \notin e \;\&\; \neg\,(\exists e_0 \subset e)(\varphi ICe_0)$; then one verifies easily that $IC^0 \subseteq IC$ and that $IC^0$ is equivalent to $IC$. Suppose that $IC^0 \nsubseteq IC_1$, i.e. there are $\varphi$, $e$ such that $\varphi IC^0 e$ and not $\varphi IC_1 e$. Then either there is a $e_0 \subseteq e$ such that $\varphi IC_1 e_0$, hence $\varphi \in IC_1(e_0)$ but $\varphi \notin IC^0(e_0) = IC(e_0)$, or there is no such $e_0$ and then $\varphi \in IC(e)$ but $\varphi \notin IC_1(e)$. We see that $IC^0$ is the $\subseteq$-least relation of i.c. equivalent to $IC$.

**2.4. Denotation.** $\overline{IC}$ denotes the $\subseteq$-largest relation of i.c. equivalent to $IC$ and $IC^0$ denotes the $\subseteq$-least relation of i.c. equivalent to $IC$.

**2.5. Definition.** $IC$ is *regular* if

$$(\forall \varphi \in Sent)(\forall e \in \mathfrak{P}_{fin}(Sent))\,[(\varphi \in e \vee (\exists e_0 \subset e)(\varphi ICe_0)) \Rightarrow \varphi ICe]\,;$$

$IC$ is *prime* if

$$(\forall \varphi \in Sent)(\forall e \in \mathfrak{P}_{fin}(Sent))\,[(\varphi \in e \vee (\exists e_0 \subset e)(\varphi ICe_0)) \Rightarrow \neg\,(\varphi ICe)]\,.$$

**2.6. Lemma.** (1) The following are equivalent: (a) $IC$ is regular, (b) $IC = \overline{IC}$, (c) $IC = \overline{IC}_1$ for some relation $IC_1$ of i.c. on *Sent*. (2) Also the following are equivalent: (a) $IC$ is prime, (b) $IC = IC^0$, (c) $IC = (IC_1)^0$ for some relation $IC_1$ of i.c. on *Sent*.

**2.7. Definition.** Let $F \subseteq Sent$. The *restriction* of $IC$ to $F$ is the relation $IC \cap (F \times \mathfrak{P}_{fin}(F))$; it is denoted by $IC \restriction F$.

**2.8. Lemma.** (1) $IC \restriction F$ is a relation of i.c. on $F$. (2) The following holds for each sequence $s$: $s$ is an $IC$-proof from $X$ containing only elements of $F$ iff $s$ is an $(IC \restriction F)$-proof from $X$.

**2.9. Definition.** $IC$ is *transitive* if

$$(\forall X) \subseteq Sent)\,(IC(IC(X)) = IC(X))\,.$$

**2.10. Theorem.** $IC$ is transitive iff the following holds for each $\varphi \in Sent$ and $X \subseteq Sent$: $\varphi$ is provable from $X \Leftrightarrow (\varphi \in X \vee (\exists e \subseteq X)(\varphi IC(e))$.

The proof is easy and can be left to the reader.

**2.11. Definition.** Let $X, Y \subseteq Sent$. $X$ is *Y-sufficient* if $Y \subseteq IC(X)$. (When necessary, one says that $X$ is $Y$-sufficient w.r.t. $IC$.)

**2.12. Remark.** Let $\mathfrak{S} = \langle Sent, \mathfrak{M}, V, Val \rangle$ be a semantical system, let $\boldsymbol{P} = \langle F, V_0, IC \rangle$ be an $\mathfrak{S}$-problem and let $\boldsymbol{M} \in \mathfrak{M}$ be a particular model. Then $X \subseteq Tr_{V_0}(\boldsymbol{M})$ is a solution of $\boldsymbol{P}$ iff $X$ is $(F \cap Tr_{V_0}(\boldsymbol{M}))$-sufficient.

**2.13. Lemma.** (1) $X$ is $IC(X)$-sufficient. (2) If $X$ is $\subseteq$-minimal $Y$-sufficient and if $Y \subseteq Z \subseteq IC(X)$ then $X$ is $\subseteq$-minimal $Z$-sufficient.

**2.14. Definition.** $X$ is *weakly independent* if

$$(\forall \varphi \in X)\,(IC(X - \{\varphi\}) \neq IC(X))\,;$$

$X$ is *strongly independent* if

$$(\forall \varphi \in X)\,(\varphi \notin IC(X - \{\varphi\})\,.$$

**2.15. Remark.** (1) Evidently, $X$ is weakly independent iff $X$ is $\subseteq$-minimal $IC(X)$-sufficient; $X$ is strongly independent iff $X$ is $\subseteq$-minimal $X$-sufficient. (2) If $IC_1$, $IC_2$ are equivalent relations of i.c. then the notion of weak independence w.r.t. $IC_1$ coincides with the notion of weak independence w.r.t. $IC_2$; similarly for strong independence, sufficiency etc.

**2.16. Lemma.** If $X$ is strongly independent then $X$ is weakly independent.

**2.17. Remark.** The last inplication cannot be converted: Let $Sent = \{1, 2, 3\}$, $IC = \{\frac{1}{2}, \frac{2}{3}\}$ (schematically: $1 \to 2 \to 3$), $X = \{1, 2\}$. Then $X$ is weakly independent but not strongly independent.

**2.18. Lemma.** If $X$ is $\subseteq$-minimal $Y$-sufficient then $X$ is weakly independent.

Proof. By Lemma 2.13 (2), $X$ is $\subseteq$-minimal $IC(X)$-sufficient; the lemma follows by the remark 2.15 (1).

**2.19. Lemma.** If *Sent* is finite then each $Y$-sufficient set $X$ contains an $X_0 \subseteq X$ which is $\subseteq$-minimal $Y$-sufficient (and hence weakly independent).

**2.20. Remark.** (1) One can easily show that the assumption that *Sent* is finite is necessary. (2) A $Y$-sufficient set need not contain a strongly independent $Y$-sufficient subset: see the example in 2.17.

**2.21. Lemma.** Each subset of a strongly independent set is strongly independent.

**2.22. Remark.** The analogous statement concerning weak independence is not valid. Let $Sent = \{1, 2, 3, 4, 5\}$ and let $IC = \{\frac{1}{4}, \frac{1}{2}, \frac{2,3}{5}\}$. Then $\{1, 2, 3\}$ is weakly independent but $\{1, 2\}$ is not.

**2.23. Lemma.** Let *Sent* be finite. The following are equivalent:

(i) For each $Y$, each $Y$-sufficient set contains a strongly independent $Y$-sufficient subset.

(ii) For each $X$, $X$ is strongly independent iff $X$ weakly independent.

Proof. (ii) $\Rightarrow$ (i) by Lemma 2.19. We prove (i) $\Rightarrow$ (ii). If $X$ is weakly independent then $X$ is $\subseteq$-minimal $IC(X)$-sufficient; if $X_0 \subseteq X$ is strongly independent and $IC(X)$-sufficient (it exists by (i)) then we have $X_0 = X$ and $X$ is strongly independent.

**2.24. Lemma.** If $IC$ is transitive then, for each $X$, $X$ is strongly independent iff $Y$ is weakly independent.

Proof. If $X$ is not strongly independent then there is a $\varphi \in X$ such that $\varphi \in IC(X - \{\varphi\})$, hence $IC(X - \{\varphi\}) = IC(IC(X - \{\varphi\})) \supseteq IC(X)$, hence $IC(X) = IC(X - \{\varphi\})$ and $X$ is not weakly independent.

**2.25. Remark.** Note that a card-minimal $Y$-sufficient set need not be strongly independent even if strongly independent $Y$-sufficient sets exist. Let $Sent = Y = \{1, 2, 3, 4\}$, $IC = \{\frac{1}{2}, \frac{2}{3}, \frac{2}{4}\}$. Then $\{1, 2\}$ is a card-minimal $Y$-sufficient set which is not strongly independent and $\{1, 3, 4\}$ is strongly independent and $Y$-sufficient. If *Sent* is finite then of course each card-minimal $Y$-sufficient set is $\subseteq$-minimal $Y$-sufficient and therefore weakly independent.

We now recall semantical systems, problems and their solutions. Let $\mathfrak{S} = \langle Sent, \mathfrak{M}, V, Val \rangle$ be a semantical system, let $\boldsymbol{P} = \langle F, V_0, IC \rangle$ be an $\mathfrak{S}$-problem and let $M \in \mathfrak{M}$.

**2.27. Definition.** Under the denotation just introduced, a solution $X$ is *direct* if $X \subseteq F$.

A direct solution contains only some relevant questions with good values. But it is reasonable to consider also indirect solutions containing also some "*auxiliary questions*" (with good values) taken from a set $Q \subseteq Sent$ (of auxiliary questions). Theoretically, we can restrict ourselves to some "standard cases". Looking for a direct solution we may assume that $F = Sent$; and looking for an indirect solution with auxiliary questions taken from $Q$ we may assume $F \cap Q = Sent$. We define:

**2.28. Definition.** Let $Sent_0 \subseteq Sent$, $\emptyset \neq Sent_0$. (a) If $\mathfrak{S} = \langle Sent, \mathfrak{M}, V, Val \rangle$ is a semantical system we put

$$\mathfrak{S} \upharpoonright Sent_0 = \langle Sent_0, \mathfrak{M}, V, Val \upharpoonright (Sent_0 \times \mathfrak{M}) \rangle .$$

(b) If in addition $\boldsymbol{P} = \langle F, V_0, IC \rangle$ is an $\mathfrak{S}$-problem we put

$$\boldsymbol{P} \upharpoonright Sent_0 = \langle F \cap Sent_0, V_0, IC \upharpoonright Sent_0 \rangle .$$

(Obviously, $\mathfrak{S} \upharpoonright Sent_0$ is a semantical system and $P \upharpoonright Sent_0$ is an $(\mathfrak{S} \upharpoonright Sent_0)$-problem.) 

**2.29. Theorem.** Let $\mathfrak{S} = \langle Sent, M, V, Val \rangle$ be a semantical system, let $\boldsymbol{P} = \langle F, V_0, IC \rangle$ be an $\mathfrak{S}$-problem, let $\mathfrak{M} \in M$ and let $X \subseteq Sent$. (1) $X$ is a direct solution of the $\mathfrak{S}$-problem $\boldsymbol{P}$ in $\boldsymbol{M}$ iff $X$ is a solution of the $(\mathfrak{S} \upharpoonright F)$-problem $\boldsymbol{P} \upharpoonright F$ in $\boldsymbol{M}$. (2) $X$ is a solution of the $\mathfrak{S}$-problem $\boldsymbol{P}$ in $\boldsymbol{M}$ and $X \subseteq F \cap Q$ iff $X$ is a solution of the $(\mathfrak{S} \upharpoonright (F \cup Q))$-problem $\boldsymbol{P} \upharpoonright (F \cap Q)$ in $\boldsymbol{M}$.

Proof. Obvious, cf. 2.8.

**2.30. Theorem.** ($\mathfrak{S}$, $\boldsymbol{P}$, $Sent$, $F$, $\boldsymbol{M}$ have their usual meanings.) (1) If $Sent$ is finite then each solution of $\boldsymbol{P}$ in $\boldsymbol{M}$ contains a $\subseteq$-minimal solution. (b) If $X$ is an $\subseteq$-minimal solution of $\boldsymbol{P}$ in $\boldsymbol{M}$ then $X$ is weakly independent. (3) If $F = Sent$ then a solution of $\boldsymbol{P}$ in $\boldsymbol{M}$ is $\subseteq$-minimal iff it is weakly independent. (4) If $X$ is a strongly independent direct solution then it is $\subseteq$-minimal.

Proof. (1) follows from 2.19, (2) follows from 2.18. (3) Suppose that $F = Sent$ and $X$ is a (direct) weakly independent solution. Hence $IC(X) \supseteq Tr_{V_0}(\boldsymbol{M})$, i.e. $IC(X) = Tr_{V_0}(\boldsymbol{M})$ and $IC(X - \{\varphi\}) \neq Tr_{V_0}(\boldsymbol{M})$ for each $\varphi$. (4) If $\varphi \in X \subseteq F \cap Tr_{V_0}(\boldsymbol{M})$ and if $X$ is strongly independent then $\varphi \notin IC(X - \{\varphi\})$, hence $X - \{\varphi\}$ is not a solution.

**2.31. Remark.** The inplication in (2) cannot be generally converted; a weakly independent solution need not be $\subseteq$-minimal since it is possible that there is a $\varphi \in X$ such that $X - \{\varphi\}$ is a solution and the difference $IC(X) - IC(X - \{\varphi\})$ consists only from some sentences not in $F$. Similarly, a strongly independent solution can be diminished by omitting an arbitrary $\varphi \notin F$, provided $X - \{\varphi\}$ is $(F \cap Tr_{V_0}(\boldsymbol{M}))$ – sufficient.

## 3. QUASIORDERINGS

In the present section we recall and introduce some notions concerning quasiordered sets and mention their properties used in the sequel. Since this section is auxiliary from our point of view we shall give no proofs. But the reader can supply the proofs with ease.

**3.1. Definition.** A *quasiordering* on a set $A$ is any reflexive and transitive relation on $A$. (I.e., $R \subseteq A^2$ is a quasiordering on $A$ iff $(\forall x \in A)(\langle x, x \rangle \in R)$ and $(\forall x, y, z \in A) . ((\langle x, y \rangle \in R \,\&\, \langle y, z \rangle \in R) \Rightarrow \langle x, z \rangle \in R)$.) We write $x \leqq_R y$ instead of $\langle x, y \rangle \in R$, $x <_R y$ instead of $\langle x, y \rangle \in R \,\&\, \langle y, x \rangle \notin R$ and $x \equiv_R y$ instead of $\langle x, y \rangle \in R \,\&\, \langle y, x \rangle \in R$. A *quasiordered set* is a pair $\langle A, R \rangle$ where $R$ is a quasiordering on $A$.

Let $R$ be a quasiordering on $A$. $R$ is an *ordering* if $(\forall x, y \in A)(x \equiv_R y \Rightarrow y = y)$

(antisymmetry). $R$ is a *linear quasiordering* if $(\forall x, y \in A)(x \leqq_R y \vee y \leqq_R x)$ (linearity). $R$ is a *linear ordering* if it is a liner quasiordering and an ordering. $R$ is an *equivalence* if $(\forall x, y \in A)(x \leqq_R y \Rightarrow x \equiv_R y)$ (symmetry).

Let $R$ be a quasiordering on $A$ and let $x \in A$. The *R-segment* determined by $x$ is the set $Seg_R(x) = \{y \in A; y \leqq_R x\}$. A set $X \subseteq A$ is an $R$-segment if there is an $x \in A$ such that $X = Seg_R(x)$. $\boldsymbol{Seg}_R$ denotes the set of all the $R$-segments.

**3.2. Lemma.** $R$ is linear iff $\boldsymbol{Seg}_R$ is linearly ordered by inclusion.

**3.3. Definition.** Let $A$ be a set. A *monotone covering* of $A$ is an arbitrary system of non-empty subsets of $A$ linearly ordered by the includion whose union is $A$.

**3.4. Remark.** If $A$ is finite then a system $G$ of nonempty subsets of $A$ linearly ordered by the inclusion is a monotone covering of $A$ iff $A \in G$.

**3.5. Theorem.** Let $A$ be a finite set. (1) if $R$ is a linear quasiordering on $A$ then $\boldsymbol{Seg}_R$ is a monotone covering of $A$. (2) If $G$ is a monotone covering of $A$ then there is a uniquely determined linear quasiordering $R$ on $A$ such that $G = \boldsymbol{Seg}_R$.

**3.6. Definition.** Let $R$ be a quasiordering on $A$. The relation $E_R$ defined by the condition $\langle x, y\rangle \in E_R \Leftrightarrow x \equiv_R y$ is (obviously an equivalence and is) called the *canonical equivalence* on $\langle A, R\rangle$.

**3.7. Definition.** Let $R$ be a quasiordering on $A$. Put $x_R = \{y \in A; y \equiv_R x\}$, $A/R = \{x_R; x \in A\}$ and for $u, v \in A/R$ define

$$\langle u, v\rangle \in \overline{R} \Leftrightarrow (\exists x, y \in A)(u = x_R \,\&\, v = y_R \,\&\, x \leqq_R y)\,.$$

**3.8. Theorem.** If $\langle A, R\rangle$ is a quasiordered set then $\langle A/R, \overline{R}\rangle$ is an ordered set.

**3.9. Definition.** Let $R$, $S$ be quasiorderings on $A$. $S$ *extends* $R$ if $R \subseteq S$. $S$ extends $R$ *conservatively* if $S$ extends $R$ and $E_R = E_S$. (Then we have $\overline{R} \subseteq \overline{S}$.)

**3.10. Theorem.** Every quasiordering can be extended conservatively to a linear quasiordering.
(The proof is easy under the assumption that $A$ is finite. For infinite sets one needs some additional axioms of Set Theory, e.g. the axiom of choice – the theorem is equivalent to the so-called Order Extension Principle.)

**3.11. Definition.** Let $R$ be a quasiordering on $A$ and let $S$ be a linear ordering on $A$. $S$ is *coherent with* $R$ if

(a) $$(\forall x, y \in A)(x <_R y \Rightarrow x <_S y)\,,$$

(b) $$(\forall x, y, z \in A)((x \equiv_R y \,\&\, x \leqq_S z \leqq_S y) \Rightarrow x \equiv_R y \equiv_R z)$$ 

(i.e., each $x_R$ is an interval in $S$).

**3.12. Theorem.** Let $A$ be finite, let $G$ be a monotone covering of $A$ and let $S$ be a linear ordering on $A$ . $S$ is coherent with the linear quasiordering determined by $G$ iff each $X \in G$ is an $S$-segment.

**3.13. Theorem.** For each quasiordering $R$ on $A$ there is a linear ordering $S$ on $A$ coherent with $R$.

(Hint: Associate with each $u \in A/R$ a linear ordering $S_u$ on $u$. Let $Q$ be a linear quasiordering extending $R$ conservatively. One takes for $S$ the direct sum of the ordered sets $\langle u, S\rangle$ using the ordered set $\langle A/Q, \overline{Q}\rangle$. See the remark following 3.10.)

## 4. LINEARLY ORDERED SYNTACTICAL SYSTEMS

We shall now pay attention to the fact that one often needs a linear ordering of the set of sentences, e.g. when the computer is to generate sentences in certain order or since one wants to have a fixed list of all the sentences in order to be able to decide quickly whether a given sentence is in a given sublist (e.g. in a solution). We are led to the notion of a linearly ordered syntactical system (briefly, l.o. syntactical system); for l.o. syntactical systems we define a notion of an increasingly independent set and find conditions under which this notion coincides with the notion of a strongly independent set. Finally we introduce a notion of a l.o. $\mathfrak{S}$-problem; consequences of our considerations for l.o. $S$-problems are summarized in Theorem 4.9.

**4.1. Definition.** A *l.o. syntactical system* is a triple $\langle Sent, IC, S\rangle$ where $\langle Sent, IC\rangle$ is a syntactical system and $S$ is a linear ordering of *Sent*.

**4.2. Definition.** Let $\langle Sent, IC, S\rangle$ be a l.o. syntactical system. A set $X \subseteq Sent$ is *increasingly independent* if there is no $\varphi \in X$ that follows immediately from the preceding elements of $X$, i.e. if $(\forall \varphi \in X)(\varphi \notin IC((X - \{\varphi\}) \cap Seg_S(\varphi))$.

**4.3. Lemma.** (1) Any subset of an increasingly independent set is increasingly independent. (2) If $X$ is strongly independent then $X$ is increasingly independent.

**4.4. Theorem.** Let *Sent* be finite. For each $Y \subseteq Sent$ there is a $\subseteq$-minimal $X \subseteq Y$ such that $X$ is increasingly independent and $Y$-sufficient.

Proof. $S$ orders $Y$ into a finite sequence $\varphi_1, \ldots, \varphi_n$. We define $X_0$ inductively as follows: $\varphi_1 \in X_0$; for $i > 1$ let $\varphi_i \in X_0 \Leftrightarrow \varphi_i \notin IC(X_0 \cap \{\varphi_1, \ldots, \varphi_{i-1}\}$. Clearly, $X_0$ is increasingly independent and $Y$-sufficient. The theorem follows by 4.3 (1).

**4.5.** It is not true that each $Y$-sufficient subset $X$ of $Y$ contains an increasingly independent $Y$-sufficient set: Let $Y = Sent\ \{1, 2, 3\}$ and let $IC = \{\frac{1}{2}, \frac{2}{3}\}$. Then $\{1, 2\}$ is $\subseteq$-minimal $Y$-sufficient but does not contain any increasingly independent $Y$-sufficient set.

**4.6. Definition.** Let $IC$ be a relation of i.c. on *Sent* and let $e$ be a finite subset of *Sent*. $e$ is said to be *relevant for IC* if there is a $\varphi \in e$ such that $\varphi IC(e - \{\varphi\})$. Let $\langle Sent, IC, S\rangle$ be a l.o. syntactical system. $IC$ is said to be $S$-admissible if no set relevant for $IC$ is increasingly independent.

**4.7. Theorem.** Let $\langle Sent, IC, S\rangle$ be a l.o. syntactical system. $IC$ si $S$-admissible iff the following holds for each $X \subseteq Sent$:

(*) $X$ is increasingly independent $\Leftrightarrow X$ is strongly independent.

Proof. ($\Leftarrow$) Let $e$ be relevant for $IC$, then $e$ is not strongly independent and, by (*), $e$ is not increasingly independent. Hence $IC$ is $S$-admissible. ($\Rightarrow$) Let $IC$ be $S$-admissible. Suppose that $X$ is a set of sentences that is not strongly independent, i.e. there is a finite $e \subseteq X$ and a $\varphi \in e$ such that $\varphi IC(e - \{\varphi\})$. This $e$ is relevant for $IC$ and, by the $S$-admissibility of $IC$, $e$ is not increasingly independent. Hence $X$ is not increasingly independent either.

**4.8. Definition.** Let $\mathfrak{S}$ be a semantical system. A *l.o. $\mathfrak{S}$-problem* is a quadruple $\boldsymbol{P} = \langle F, V_0, IC, S\rangle$ where $\langle F, V_0, IC\rangle$ is an $\mathfrak{S}$-problem and $S$ is a linear ordering of *Sent*. (Consequently, $\langle Sent, IC, S\rangle$ is a l.o. syntactical system.)

**4.9. Theorem.** Let $\mathfrak{S} = \langle Sent, \mathfrak{M}, V, Val\rangle$ be a semantical system, let *Sent* be finite, let $\boldsymbol{P}$ be a l.o. $S$-problem and let $\boldsymbol{M} \in \mathfrak{M}$. (1) There is an increasingly independent direct solution of $\boldsymbol{P}$ in $\boldsymbol{M}$.* (2) If $IC$ is $S$-admissible then each increasingly independent solution of $\boldsymbol{P}$ in $\boldsymbol{M}$ is strongly independent.

Proof. The theorem follows by 4.4 and by 4.7.

## 5. PAIRS OF PROBLEMS; HIERARCHICAL PROBLEMS

Let $\mathfrak{S}$ be a semantical system, let $\boldsymbol{P}$ be an $\mathfrak{S}$-problem or a l.o. $\mathfrak{S}$-problem and let $\emptyset \neq Sent_0 \subseteq Sent$. Put $\mathfrak{S}_0 = S \restriction Sent_0$, $\boldsymbol{P}_0 = \boldsymbol{P} \restriction Sent_0$.** Our first question (question schema) reads: Under which conditions can a "good" solution of the (l.o.) $\mathfrak{S}_0$-problem $\boldsymbol{P}_0$ be extended to a "good" solution of the (l.o.) $\mathfrak{S}$-problem $\boldsymbol{P}$? We shall

* By a solution of a l.o. $\mathfrak{S}$-problem $\langle F, V_0, IC, S\rangle$ we mean a solution of the problem $\langle F, V_0, IC\rangle$.

** If $\boldsymbol{P} = \langle F, V_0, IC, S\rangle$ then $\boldsymbol{P} \restriction Sent_0$ evidently means $\langle F \cap Sent_0, V_0, IC \restriction Sent_0, S \cap \cap (Sent_0^2)\rangle$; $\boldsymbol{P} \restriction Sent_0$ is then a l.o. $(\mathfrak{S} \restriction Sent_0)$-problem.

give an answer for $\subseteq$-minimal and for increasingly independent solutions. The process of finding a solution of a partial problem and extending it to a solution of a greater problem can be iterated. This leads to the definition of a hierarchical problem. 

In the present section we assume *Sent* to be finite. Let $M \in \mathfrak{M}$ be given.

**5.1. Theorem.** Under the above conditions, suppose that the following holds:

$$(*) \qquad (\forall \varphi, e)\left((\varphi ICe \;\&\; \varphi \in Sent_0\right) \Rightarrow e \subseteq Sent_0) .$$

Then each $\subseteq$-minimal solution of the $\mathfrak{S}_0$-problem $P_0$ can be extended to a $\subseteq$-minimal solution of the $\mathfrak{S}$-problem $P$.

Proof. Let $X_0$ be a $\subseteq$-minimal solution of $P_0$. Let $X_1$ be a $\subseteq$-minimal subset of $Tr_{V_0}(M) \cap (Sent - Sent_0)$ such that $X_0 \cap X_1$ is $(F \cap Tr_{V_0}(M))$-sufficient. (Note that e.g. $X_0 \cup (Tr_{V_0}(M) \cap (Sent - Sent_0))$ is $(F \cap Tr_{V_0}(M))$-sufficient.) We prove that $X_0 \cup X_1$ is $\subseteq$-minimal $(F \cap Tr_{V_0}(M))$-sufficient. Suppose not. Then there is a $\varphi \in X_0 \cup X_1$ such that, for each $\psi \in F \cap Tr_{V_0}(M)$, $\psi \in IC((X_0 \cup X_1) - \{\varphi\})$. This $\varphi$ cannot be in $X_1$ by the definition of $X_1$, hence $\varphi \in X_0$. But $X_0$ is a $\subseteq$-minimal solution of $P_0$, i.e. there is a $\psi \in F \cap Tr_{V_0}(M) \cap Sent_0$ such that $\psi \notin IC(C_0 - \{\varphi\})$. Given such a $\psi$, there is a $e_0 - X_0 \subseteq \{\varphi\}$ and $e_1 - X_1$, $e_1$ non-empty, such that $\psi IC(e_0 \cup e_1)$. This contradicts $(*)$.

**5.2. Theorem.** Let now $P$ be a l.o. $\mathfrak{S}$-problem with a linear ordering $S$ and suppose that the following holds:

$$(\forall \varphi, \psi \in Sent)\left((\psi \in Sent_0 \;\&\; \varphi \leqq_S \psi) \Rightarrow \varphi \in Sent_0\right)$$

(i.e., $Sent_0$ is a segment). Then each increasingly independent solution of $P_0$ can be extended to an increasingly independent solution of $P$.

Proof. Let $X_0$ be an increasingly independent solution of $P_0$ and let $Sent - Sent_0 = \{\varphi_1, \ldots, \varphi_n\}$ (order in $S$). Define by induction:

$$\varphi_i \in X_1 \Leftrightarrow \varphi_i \in Tr_{V_0}(M) \;\&\; \varphi_i \notin IC(X_0 \cup (X_1 \cap \{\varphi_1, \ldots, \varphi_{i-1}\})) .$$

Evidently, $X_0 \cup X_1$ is a solution of $P$; it is increasingly independent since each sentence in $X_0$ $S$-precedes each sentence in $X_1$.

**5.3. Corollary.** If, moreover, $IC$ is $S$-admissible then each strongly independent solution of $P_0$ can be extended to a strongly independent solution of $P$. (See 4.7.)

**5.4. Definition.** Let $\mathfrak{S} = \langle Sent, \mathfrak{M}, V, Val \rangle$ be a semantical system. A *hierarchical $\mathfrak{S}$-problem* is a quadruple $P = \langle F, V_0, IC, H \rangle$ where $\langle F, V_0, IC \rangle$ is an $S$-problem (denoted by $P^0$) and $H$ is a monotone covering of *Sent*. A solution of $P$ in $M \in \mathfrak{M}$ is a system $\{X_h; h \in H\}$ such that $(\forall h, h' \in H)(h \leqq h' \Rightarrow X_h \leqq X_{h'})$ and that, for each $h \in H$, $X_h$ is a solution of the $(\mathfrak{S} \upharpoonright h)$-problem $P^0 \upharpoonright h$. ($X_{Sent}$ is then obviously a solution of $P^0$.)

**5.5. Discussion.** The definition of a hierarchical problem and of its solution is motivated by two things: (1) We imagine that the computer will *successively construct* the sets $X_h$ for increasing $h$; the programme will thus have the form of a cycle with the parameter $h$. If it is necessary (e.g. for some technical or financial reasons) to break the computation and if $h$ is the last processed value of the parameter then we have a solution of $\boldsymbol{P}^0 \upharpoonright h$. (If a programme has not this form then it is possible that we do not know anything before the computer stops or at least that we do not know for what subset of $F \cap Tr_{V_0}(\boldsymbol{M})$ our results are sufficient.) (2) The *interpretation of results* is also divided by a hierarchical solution into a series of subtasks, namely interpretations of various sets $X_h$ as solutions of problems $\boldsymbol{P}^0 \upharpoonright h$.

**5.6. Definition.** A solutoin $X = \{X_h;\ h \in H\}$ of a hierarchical problem $\boldsymbol{P}$ in $\boldsymbol{M} \in \mathfrak{M}$ is *locally weakly independent* if each $X_h$ is weakly independent. $X$ is *locally* $\subseteq$*-minimal* if each $X_k$ is a $\subseteq$-minimal solution of $\boldsymbol{P}^0 \upharpoonright h$ in $\boldsymbol{M}$. $X$ is *globally* $\subseteq$*-minimal* if $X_{Sent}$ is a $\subseteq$-minimal solution of $\boldsymbol{P}^0$. Analogously for locally and globally card-minimal.

**5.7. Definition.** A *linearly ordered hierarchical problem* is a quintuple $\boldsymbol{P} = \langle F, V_0, IC, H, S \rangle$ where $\langle F, V_0, IC, H \rangle$ is a hierarchical problem, $\langle F, V_0, IC, S \rangle$ is a l.o. problem and $S$ is coherent with the linear quasiordering given by $H$.

Hence each $h$ is a segment in $S$ (cf. 3.2). It is natural to suppose this; recall that we imagine $S$ as defining a fixed *list* of *Sent* and $H$ as defining *stages* in processing the problem; so our stages consist in processing some segments of *Sent*.

We shall use Theorems 5.1 and 5.2 for the formulation of some conditions sufficient for the existence of "locally" good solutions. We first define two quasiorderings determined by a relation of immediate consequence.

**5.8. Definition.** Let $\langle Sent, IC \rangle$ be a syntactical system. The quasiordering $R_{IC}$ *induced* by $IC$ is the least quasiordering containing the relation $Q_{IC}$ defined on *Sent* by the following condition:

$$\langle \varphi, \psi \rangle \in Q_{IC} \Leftrightarrow (\exists e)\,(\psi IC(e \cup \{\varphi\}))\,.$$

The quasiordering $R^w_{IC}$ *weakly induced* by $IC$ is the least quasiordering containing the relation $Q^w_{IC}$ defined on ***Sent*** by the following condition:

$$\langle \varphi, \psi \rangle \in Q^w_{IC} \Leftrightarrow (\exists e)\,(\psi IC(e \cup \{\varphi\})\ \&\ \neg\,(\varphi IC(e \cup \{\psi\})))\,.$$

We write $\varphi \leqq_{IC} \psi$ instead of $\varphi \leqq_{R_{IC}} \psi$ and $\varphi \leqq^w_{IC}$ instead of $\varphi \leqq_{R_{IC^w}} \psi$.

**5.9. Theorem.** Let $\mathfrak{S}$ be a semantical system and let $\boldsymbol{P} = \langle F, V_0, IC, H \rangle$ be a hierarchical $\mathfrak{S}$-problem. Denote by $R_H$ the quasiordering determined by $H$. If $R_{IC} \subseteq R_H$ and if $X$ is a solution of $\boldsymbol{P}^0 = \langle F, V_0, IC \rangle$ in a $\boldsymbol{M} \in \mathfrak{M}$ then $\{X \cap h;\ h \in H\}$ is a solution of $\boldsymbol{P}$ in $\boldsymbol{M}$.

Proof. Let $\varphi \in F \cap h \cap Tr_{V_0}(M)$. We show that $\varphi \in IC(X \cap h)$. Certainly, $\varphi \in IC(X)$, i.e. either $\varphi \in X$ and then $\varphi \in IC(X \cap h)$, or there is an $e \subseteq X$ such that $\varphi ICe$. If $e = \{\varphi_1, \ldots, \varphi_n\}$ then $\varphi_1, \ldots, \varphi_n \leqq_{IC} \varphi$, hence $\varphi_1, \ldots, \varphi_n \leqq_{R_H} \varphi$, hence $\varphi_1, \ldots, \varphi_h \in h$ and consequently $\varphi \in IC(X \cap h)$.

We are interested not only in existence of a locally good solution but also in the possibility of its successive construction. This means that, given a $h \in H$ and its immediate successor $h' \in H$ (i.e. $h \subset h'$ and there is no $h^* \in H$ such that $h \subset h^* \subset h'$), $X_{h'}$ should be definable from $X_h$, $\mathfrak{S} \upharpoonright h'$, $\boldsymbol{P} \upharpoonright h'$ an $\boldsymbol{M}$. The following theorem 5.10 together with Remark 5.11 show that under certain conditions this requirement can be satisfied.

**5.10. Theorem.** Let $\mathfrak{S}$ be a semantical system, let $\boldsymbol{P} = \langle F, V_0, IC, H\rangle$ be a hierarchical problem and let $R_{IC} \subseteq R_H$. Suppose that a system $\{X_h; h \in H\}$ of sets of sentences has the following properties: (1) For the least $h_0$ in $H$, $X_{h_0}$ is some $\subseteq$-minimal solution of the $(\mathfrak{S} \upharpoonright h_0)$-problem $\boldsymbol{P}^0 \upharpoonright h_0$. (2) For $h, h' \in H$, if $h'$ is the immediate successor of $h$ then $(X_{h'} - X_h) = Z$ is some $\subseteq$-minimal subset of $h' - h$ such that $X_h \cup Z$ is a solution of the $(\mathfrak{S} \upharpoonright h')$-problem $\boldsymbol{P}^0 \upharpoonright h'$. Then $\{X_h; h \in H\}$ is a locally $\subseteq$-minimal (and hence locally weakly independent) solution of the problem $\boldsymbol{P}$.

The proof is by induction on the elements of $H$ and follows easily from the proof of Theorem 5.1. One must only realize that if $h \in H$, $\varphi \in H$, $\varphi ICe$ and $\varphi = \{\varphi_1, \ldots \ldots, \varphi_n\}$ then $\varphi_1, \ldots, \varphi_n \leqq_{IC} \varphi$, i.e. $\varphi_1, \ldots, \varphi_n \leqq_{R_H} \varphi$ and consequently $\varphi_1, \ldots, \varphi_n \in k$.

**5.11. Remark.** The proof 5.1 also shows that the assumption $R_{IC} \subseteq R_H$ implies the existence of a solution satisfying conditions (1) and (2) of 5.10.

**5.12. Theorem.** Let $\mathfrak{S}$ be a semantical system, let $\boldsymbol{P} = \langle F, V_0, IC, H, S\rangle$ be a l.o. hierarchical problem and let $\boldsymbol{M} \in \mathfrak{M}$. Then there is a solution $\{X_h; h \in H\}$ of $\boldsymbol{P}$ which is increasingly independent (i.e. $X_{Sent}$ and therefore each $X_h$ is increasingly independent). If $IC$ is $S$-admissible then this solution is strongly independent.

The proof is by induction and uses Theorem 5.2 and the coherence of $S$ with $R_H$. (The appendix follows by Corollary 5.3.) Note that if $IC$ is $S$-admissible then $IC \upharpoonright \upharpoonright Sent_0$ is $(S \cap Sent_0^2)$-admissible.

**5.13. Lemma.** Let $\langle Sent, IC, S\rangle$ be a l.o. syntactical system. If $R_{IC}^w$ is an ordering and if $S$ extends $R_{IC}^w$ then $IC$ is $S$-admissible. (Consequently, if $R_{IC}$ is an ordering and if $S$-extends $R_{IC}$ then $IC$ is $S$-admissible.)

Proof. Let $e$ be relevant for $IC$, i.e. let $\varphi IC(e - \{\varphi\})$ and put $\psi = \max_S e$. Either $\varphi = \psi$ and then $e$ is not increasingly independent or $\varphi \neq \psi$ and $\psi IC(e - \{\psi\})$, i.e. $e$ is not increasingly independent either. The last possibility, $\varphi \neq \psi$ and $\neg[\psi IC(e - \{\psi\})]$ contradicts to the assumption since, by the definition of $R_{IC}^w$, we have $\psi \leqq_S \varphi$ which together with $\varphi \leqq_S \psi$ yields $\varphi = \psi$. (The appendix is obvious since $R_{IC}^w \subseteq R_{IC}$.)

**5.14. Remark.** Trying to satisfy the assumptions of Theorem 5.10 and Lemma 5.13 we want $R_{IC}$ or $R^w_{IC}$ to be as small as possible. And we know that notions as "solution", "weakly independent solution", "strongly independent solution" etc. do not change their meanings when $IC$ is replaced by another equivalent relation of i.c. This shows that it is advantageous to work with prime relations of i.c.

## 6. SIMPLE SYNTACTICAL SYSTEMS

We shall now consider a particular class of relations of i.c.; having a problem with a simple relation of i.c. (in the sense to be defined) one easily finds a card-minimal solution. (Note that the version of the GUHA-method described in [6] and [7] belongs to this case. See also § 9 (a) of the present paper.)

**6.1. Definition.** A syntactical system $\boldsymbol{L} = \langle Sent, IC \rangle$ is *simple* if there is a quasi-ordering $R$ such that

$$(\forall \varphi \in Sent)\,(\forall X \subseteq Sent)\,(\varphi \in IC(X) \Leftrightarrow (\exists \psi \in X)\,(\psi \leqq_R \varphi))\,.$$

**6.2. Lemma.** Suppose $IC$ to be a prime relation of i.c. $\langle Sent, IC \rangle$ is simple iff the following conditions (1) and (2) hold:

$$(1) \qquad (\forall \varphi \in Sent)\,(\forall e \in \mathfrak{P}_{fin}(Sent))\,(\varphi ICe \Rightarrow e \text{ has cardinality } 1)\,,$$

$$(2) \qquad (\forall \varphi, \psi \in Sent)\,(\varphi IC\{\psi\} \Leftrightarrow (\varphi \neq \psi \,\&\, \psi \leqq_{IC} \varphi)\,.$$

Proof. If $IC$ is prime and simple then (1) obviously holds; (2) would hold if we replaced $\leqq_{IC}$ by $\leqq_R$ ($R$ from Definition 6.1). We show that $\leqq_{IC}$ is the same as $\leqq_R$. If $\psi \leqq_R \varphi$ and $\psi \neq \varphi$ then $\varphi IC\{\psi\}$ and hence $\psi \leqq_{IC} \varphi$. If $\psi \leqq_{IC} \varphi$ and $\psi \neq \varphi$ then there are $\psi_1, \ldots, \psi_n$ and $e_1, \ldots, e_{n-1}$ such that $\psi = \psi_1$, $\varphi = \psi_n$ and $\psi_i \in e_i$ and $\psi_{i+1} IC\, e_i$ for $i = 1, \ldots, n-1$. Then $e_i = \{\psi_i\}$, $\psi_i \leqq_R \psi_{i+1}$ $(i = 1, \ldots, n-1)$ and hence $\psi \leqq_R \varphi$. The implication $[((1) \,\&\, (2)) \Rightarrow IC \text{ simple}]$ is obvious.

**6.3. Lemma.** If $\langle Sent, IC \rangle$ is simple then $IC$ is transitive.

**6.4. Definition.** Let $\boldsymbol{L} = \langle Sent, IC \rangle$ be a syntactical system. For $Y \subseteq Sent$ put $\overline{Y} = \{u \in Sent/R_{IC};\, Y \cap u \neq \emptyset\}$ and let $\overline{\overline{Y}}$ be the set of all the $\overline{R_{IC}}$-minimal elements of $\overline{Y}$. (Note that $\langle Sent/R_{IC}, \overline{R_{IC}} \rangle$ is an ordered set.)

**6.5. Theorem.** Let $Sent$ be finite and let $IC$ be a prime relation of i.c. on $Sent$. The system $\boldsymbol{L} = \langle Sent, IC \rangle$ is simple iff the following holds for each $X \subseteq Y \subseteq Sent$: $X$ is $Y$-sufficient iff $X \cap u$ is non-empty for each $u \in \overline{\overline{Y}}$.

Proof. ($\Rightarrow$) Let $\boldsymbol{L}$ be simple and suppose that $X$ is Y-sufficient. Let $u$ be minimal in $\overline{\overline{Y}}$ and let $\varphi \in Y \cap u$. Since $\varphi \in IC(X)$, there is a $\varphi \in X$ such that $\varphi IC\{\psi\}$, hence

$\psi \leqq_R \varphi$. Since $u$ is minimal in $\overline{\overline{Y}}$ we obtain $\psi \leqq_R \varphi$, hence $\psi \in u$ and consequently $X \cap u \neq 0$. Conversely, suppose $(\forall u \in \overline{\overline{Y}})(X \cap u \neq \emptyset)$ and let $\varphi \in Y$. Then there is a $v \in Sent/R_{IC}$ such that $\varphi \in v$; furthermore, there is a $u \in \overline{\overline{Y}}$ such that $u \leqq_{\overline{R_{IC}}} v$. Let $\psi \in X \cap u$. We have $\psi \leqq_{IC} \varphi$, i.e. $\varphi \in IC(X)$ and hence $X$ is $Y$-sufficient.

($\Leftarrow$) Assume the condition for sufficiency. We show $\boldsymbol{L}$ to be simple, i.e. we prove $\varphi \in IC(X) \Leftrightarrow (\exists \psi \in X)(\psi \leqq_{IC} \varphi)$. Let $\psi \leqq_{IC} \varphi$, then $\{\psi\}$ is $\{\varphi, \psi\}$-sufficient and hence $\varphi \in IC(\{\psi\}) \subseteq IC(X)$. Conversely, let $\varphi \in IC(X)$; then $X$ is $(X \cup \{\varphi\})$-sufficient and hence $X$ contains a sentence $\psi$ such that $\psi \leqq_{IC} \varphi$.

**6.6. Discussion.** Let *Sent* be finite and let *IC* be prime.

(1) If $\boldsymbol{L} = \langle Sent, IC \rangle$ is simple then the following holds for each $X \subseteq Y \subseteq Sent$: $X$ is $\subseteq$-minimal $Y$-sufficient iff it contains exactly one member of each $u \in \overline{\overline{Y}}$. Hence all the $\subseteq$-minimal $Y$-sufficient subsets of $Y$ have the same cardinality and therefore coincide with card-minimal $Y$-sufficient subsets of $Y$.

(2) If *IC* is simple and $R_{IC}$ is an ordering then, for each $Y \subseteq Sent$, there is a uniquely determined least (both $\subseteq$-least and card-least) $Y$-sufficient subset of $Y$; it contains precisely all the minimal elements of $Y$.

(3) It follows by 6.3, 2.18 and 2.24 that the described $\subseteq$-minimal $Y$-sufficient sets are strongly independent. If $S$ is a linear ordering of *Sent* coherent with $R_{IC}$ (i.e. $S$ extends $R_{IC}$, $R_{IC}$ being an ordering) then *IC* is S-admissible (prove!) and therefore increasingly independent means the same as strongly independent. So we have the following:

(4) If $\mathfrak{S}$ is a semantical system and if $\boldsymbol{P}$ is a l.o. $\mathfrak{S}$-problem such that $\langle Sent, IC \rangle$ is simple and $\mathfrak{S}$ is coherent with $R_{IC}$ then, for each $\boldsymbol{M} \in \mathfrak{M}$, finding an increasingly independent $(F \cap Tr_{V_0}(\boldsymbol{M}))$-sufficient subset of $(F \cap Tr_{V_0}(\boldsymbol{M}))$ (which is realizable by the computer from our point of view) one has a card-minimal direct solution of $\boldsymbol{P}$ in $\boldsymbol{M}$; this solution is strongly independent. Note that, by 5.9, this solution determines in a natural way a solution of each hierarchical problem which results by adding to $\boldsymbol{P}$ a hierarchy $H$ satisfying the condition $R_{IC} \leqq R_H$.

*Remark.* An example of a simple system will be considered in § 9.

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REFERENCES

The list of references will be presented in the part II.

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