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Linear Nonstationary System with Discrete-Time Input

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The work is concerned with a linear continuous system the parameters of which vary with time and an input is discrete-time. Two forms of a system discrete-time description and their mutual relations are investigated.

In linear stationary sampled-data systems a continuous part is usually described by its transfer function in the modified discrete Laplace transform or Z-transform. There are possibilities to use the corresponding difference equation either in the scalar or vector-matrix form ([1] and [2]). We can use several methods for the transformation of a continuous differential description into an ε-parameter difference form for stationary systems ([1], [2] and [3]).

Nonstationary systems can be practically investigated in a time domain only. We derive here the discrete-time form of state equations and the corresponding scalar difference equation of a linear nonstationary continuous-time system with discrete-time input. It is shown that both these forms can be obtained from a continuous-time description if the system transition matrix is known.

I. SYSTEM STATE EQUATIONS

Let us consider a single-input, single-output deterministic linear system described by two vector-matrix equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{b}(t)\,u(t)\,, \tag{1a}$$

$$y(t) = \mathbf{c}(t)\,\mathbf{x}(t) \tag{1b}$$

where a system input and output are denoted by $u(t)$ and $y(t)$ respectively,

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_s(t) \end{bmatrix} \tag{2}$$

is an $(s \times 1)$ vector of state variables $x_i(t)$, $\boldsymbol{A}(t)$ is an $(s \times s)$ matrix, $\boldsymbol{b}(t)$ an $(s \times 1)$ column vector and $\mathbf{c}(t)$ an $(1 \times s)$ row vector.

Solving the equation (1a) we obtain [4]

$$\mathbf{x}(t) = \boldsymbol{\Phi}(t, t_0)\, \mathbf{x}(t_0) + \int_{t_0}^{t} \boldsymbol{\Phi}(t, \tau)\, \boldsymbol{b}(\tau)\, u(\tau)\, \mathrm{d}\tau \tag{3}$$

where a system transition (fundamental) matrix $\boldsymbol{\Phi}(t, t_0)$ is the solution of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{\Phi}(t, t_0) - \boldsymbol{A}(t)\, \boldsymbol{\Phi}(t, t_0) = \boldsymbol{0} \tag{4}$$

under condition

$$\boldsymbol{\Phi}(t_0, t_0) = \boldsymbol{I}\,.$$

$\boldsymbol{\Phi}(t, t_0)$ posesses the properties:

1. The elements of $\boldsymbol{\Phi}(t, t_0)$ are continuous functions of both t and t_0;
2.

$$\boldsymbol{\Phi}(t, t) = \boldsymbol{I}\,; \tag{5}$$

3.

$$\boldsymbol{\Phi}(t, \tau)\, \boldsymbol{\Phi}(\tau, t_0) = \boldsymbol{\Phi}(t, t_0)\,. \tag{6}$$

Now assume that a discrete-time input signal is applied to the system described by (1). Let the constant time interval between two neighbouring input values (sampling period) be $T = 1$ here for simplicity without loss of generality, i.e., $t = n + \varepsilon$ where n ranges over the integers and ε is a continuous parameter, $0 \leqq \varepsilon \leqq 1$.

A dynamical behaviour of the continuous system (1) with a discrete-time input can be described in the discrete-time state space form as

$$\mathbf{x}(n + 1, \varepsilon) = \boldsymbol{F}(n, \varepsilon)\, \mathbf{x}(n, 0) + \boldsymbol{h}(n + 1, \varepsilon)\, u(n + 1)\,, \tag{7a}$$

$$y(n, \varepsilon) = \mathbf{c}(n, \varepsilon)\, \mathbf{x}(n, \varepsilon) \tag{7b}$$

where

$$\boldsymbol{F}(n, \varepsilon) = \boldsymbol{\Phi}(n + 1, \varepsilon; n)\,, \tag{9}$$

$$\boldsymbol{h}(n, \varepsilon) = \boldsymbol{E}(n, \varepsilon)\, \boldsymbol{b}(n, 0) \tag{10}$$

if

$$\boldsymbol{E}(n, \varepsilon) = \boldsymbol{\Phi}(n, \varepsilon; n) \tag{11}$$

with the property

$$\boldsymbol{E}(n, 0) = \boldsymbol{I}\,. \tag{12}$$

The equation (7a) for $\varepsilon = 0$ has the form

$$\mathbf{x}(n + 1, 0) = \boldsymbol{F}(n, 0)\, \mathbf{x}(n, 0) + \boldsymbol{b}(n + 1, 0)\, u(n + 1) \tag{8a}$$

and the equation (7b) can also be written as

(8b) $$y(n, \varepsilon) = \gamma(n, \varepsilon)\, \mathbf{x}(n, 0)$$

where

(13) $$\gamma(n, \varepsilon) = \mathbf{c}(n, \varepsilon)\, \mathbf{E}(n, \varepsilon)\,.$$

Then equivalent state equations (8) can be used instead of the equations (7).

Note that the state equations (7) have the form

(14a) $$\mathbf{x}(nT + T, \varepsilon T) = \mathbf{F}(nT, \varepsilon T)\, \mathbf{x}(nT, 0) + \mathbf{h}(nT + T, \varepsilon T)\, u(nT + T)\,,$$

(14b) $$y(nT, \varepsilon T) = \mathbf{c}(nT, \varepsilon T)\, \mathbf{x}(nT, \varepsilon T)$$

if $T \neq 1$.

Proof. A discrete-time input signal can be represented in continuous-time domain by a modulated Dirac function sequence

(15) $$u^*(t) = u(t) \sum_{k=0}^{\infty} \delta(t - k^+)\,.$$

Substituting the relation (15) into the equation (3) we get for $t_0 = k_0$

(16a) $$\mathbf{x}(t) = \boldsymbol{\Phi}(t, k_0)\, \mathbf{x}(k_0^-) + \sum_{k=k_0}^{n} \boldsymbol{\Phi}(t, k)\, \mathbf{b}(k)\, u(k) =$$

(16b) $$= \boldsymbol{\Phi}(t, k_0)\, \mathbf{x}(k_0^+) + \sum_{k=k_0+1}^{n} \boldsymbol{\Phi}(t, k)\, \mathbf{b}(k)\, u(k)$$

where $t - 1 < n < t$.

For $k_0 = n$ we have

(17a) $$\mathbf{x}(t) = \boldsymbol{\Phi}(t, n)\, \mathbf{x}(n^-) + \boldsymbol{\Phi}(t, n)\, \mathbf{b}(n)\, u(n) =$$

(17b) $$= \boldsymbol{\Phi}(t, n)\, \mathbf{x}(n^+)\,.$$

Using the discrete version of time we can write

$$\mathbf{x}(t) = \mathbf{x}(n, \varepsilon)\,,$$

$$\mathbf{x}(k^+) = \mathbf{x}(k, 0)$$

and

$$\mathbf{x}(k^-) = \mathbf{x}(k - 1, 1)\,.$$

Then the equation (17b) can be written as

(18) $$\mathbf{x}(n, \varepsilon) = \boldsymbol{\Phi}(n, \varepsilon; n)\, \mathbf{x}(n, 0)$$

and the equation (16b) for $k_0 = n$, $t = n + 1 + \varepsilon$

$$(19)\quad \mathbf{x}(n + 1, \varepsilon) = \boldsymbol{\Phi}(n + 1, \varepsilon; n)\, \mathbf{x}(n, 0) + \boldsymbol{\Phi}(n + 1, \varepsilon; n + 1)\, \mathbf{b}(n + 1)\, u(n + 1).$$

Obviously the equations (19) and (7a) are identical if the designations (9), (10) and (11) are used.

For $\varepsilon = 0$ the eq. (19) has the form

$$\mathbf{x}(n + 1, 0) = \boldsymbol{\Phi}(n + 1, 0; n)\, \mathbf{x}(n, 0) + \mathbf{b}(n + 1)\, u(n + 1)$$

and the validity of (8a) is proved.

The system output is given simply by the eq. (1b) as

$$(7\text{b})\qquad y(n, \varepsilon) = \mathbf{c}(n, \varepsilon)\, \mathbf{x}(n, \varepsilon)$$

or with respect to (18)

$$y(n, \varepsilon) = \mathbf{c}(n, \varepsilon)\, \boldsymbol{\Phi}(n, \varepsilon; n)\, \mathbf{x}(n, 0).$$

Using the relations (11) and (13) the validity of (8b) is evident.

II. SCALAR DIFFERENCE EQUATION

The results presented in the previous chapter make possible to determine a convenient, correct form of a scalar difference equation between a system input and output.

This difference equation can be written in two forms:

a)

$$(20)\qquad y(n + s, \varepsilon) + \sum_{i=0}^{s-1} \alpha_i(n, \varepsilon)\, y(n + i, \varepsilon) = \sum_{j=1}^{s} \beta_j(n, \varepsilon)\, u(n + j).$$

The coefficients $\alpha_i(n, \varepsilon)$ are the elements of the $(1 \times s)$ row vector

$$(21)\qquad \boldsymbol{\alpha}(n, \varepsilon) = [\alpha_0(n, \varepsilon)\, \alpha_1(n, \varepsilon) \dots \alpha_{s-1}(n, \varepsilon)] = \\ = -\gamma(n + s, \varepsilon)\, \boldsymbol{\Phi}(n + s, 0; n)\, \mathbf{Q}^{-1}(n, \varepsilon)$$

and $b_j(n, \varepsilon)$ are the elements of the $(1 \times s)$ row vector

$$(22)\qquad \boldsymbol{\beta}(n, \varepsilon) = [\beta_1(n, \varepsilon)\, \beta_2(n, \varepsilon) \dots \beta_s(n, \varepsilon)] = \\ = \gamma(n + s, \varepsilon)\, \boldsymbol{\Phi}(n + s, 0; n)\, \{\mathbf{B}(n, 0) - \mathbf{Q}^{-1}(n, \varepsilon)\, \mathbf{R}(n, \varepsilon)\}$$

where the $(s \times s)$ matrices $\mathbf{Q}(n, \varepsilon)$ and $\mathbf{B}(n, 0)$ are given by

$$(23)\qquad \mathbf{Q}(n, \varepsilon) = \begin{bmatrix} \gamma(n, \varepsilon) \\ \gamma(n + 1, \varepsilon)\, \boldsymbol{\Phi}(n + 1, 0; n) \\ \vdots \\ \gamma(n + s - 1, \varepsilon)\, \mathbf{Q}(n + s - 1, 0; n) \end{bmatrix}$$

and

$$\mathbf{B}(n, 0) = [\boldsymbol{\Phi}^{-1}(n + 1, 0; n)\, \boldsymbol{b}(n + 1, 0);\ \boldsymbol{\Phi}^{-1}(n + 2, 0; n)\, \boldsymbol{b}(n + 2, 0)\ ; \ldots \\ \ldots\ \boldsymbol{\Phi}^{-1}(n + s, 0; n)\, \boldsymbol{b}(n + s, 0)] \tag{24}$$

respectively, and the $(s \times s)$ matrix $\mathbf{R}(n, \varepsilon)$ has the structure

$$\mathbf{R}(n, \varepsilon) = \begin{bmatrix} 0 & 0 & 0 & \ldots 0 & 0 \\ r_{11}(n, \varepsilon) & 0 & 0 & \ldots 0 & 0 \\ r_{21}(n, \varepsilon) & r_{22}(n, \varepsilon) & 0 & \ldots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{s-1,1}(n, \varepsilon) & r_{s-1,2}(n, \varepsilon) & r_{s-1,3}(n, \varepsilon) & \ldots\ r_{s-1,s-1}(n, \varepsilon) & 0 \end{bmatrix} \tag{25}$$

where

$$r_{ij}(n, \varepsilon) = \gamma(n + i, \varepsilon)\, \boldsymbol{\Phi}(n + i, 0; n + j)\, \boldsymbol{b}(n + j, 0)\,, \\ i, j = 1, 2, \ldots, s - 1\,. \tag{26}$$

Obviously the relation

$$\boldsymbol{\beta}(n, \varepsilon) = \gamma(n + s, \varepsilon)\, \boldsymbol{\Phi}(n + s, 0; n)\, \mathbf{B}(n, 0) + \boldsymbol{\alpha}(n, \varepsilon)\, \mathbf{R}(n, \varepsilon) \tag{27}$$

is valid between $\alpha_i(n, \varepsilon)$ and $\beta_j(n, \varepsilon)$.

b) A system difference equation can be written in the second form as

$$y(n + s, \varepsilon) + \sum_{i=0}^{s-1} \mu_i(n, \varepsilon)\, y(n + i, 0) = \sum_{j=1}^{s} \nu_j(n, \varepsilon)\, u(n + j) \tag{28}$$

where the coefficients $\mu_i(n, \varepsilon)$ are given by the row vector

$$\boldsymbol{\mu}(n, \varepsilon) = [\mu_0(n, \varepsilon)\, \mu_1(n, \varepsilon) \ldots \mu_{s-1}(n, \varepsilon)] = \\ = -\gamma(n + s, \varepsilon)\, \boldsymbol{\Phi}(n + s, 0; n)\, \mathbf{Q}^{-1}(n, 0) \tag{29}$$

and $\nu_j(n, \varepsilon)$ are the elements of the row vector

$$\boldsymbol{\nu}(n, \varepsilon) = [\nu_1(n, \varepsilon)\, \nu_2(n, \varepsilon) \ldots \nu_s(n, \varepsilon)] = \\ = \gamma(n + s, \varepsilon)\, \boldsymbol{\Phi}(n + s, 0; n)\, \{\mathbf{B}(n, 0) - \mathbf{Q}^{-1}(n, 0)\, \mathbf{R}(n, 0)\}\,. \tag{30}$$

Here

$$\mathbf{Q}(n, 0) = \mathbf{Q}(n, \varepsilon)_{\varepsilon=0} = \begin{bmatrix} \mathbf{c}(n, 0) \\ \mathbf{c}(n + 1, 0)\, \boldsymbol{\Phi}(n + 1, 0; n) \\ \vdots \\ \mathbf{c}(n + s - 1)\, \boldsymbol{\Phi}(n + s - 1, 0; n) \end{bmatrix}, \tag{31}$$

$\mathbf{B}(n, 0)$ is given by (24) and $\mathbf{R}(n, 0)$ of the structure (25) has the elements

$$r_{ij}(n, 0) = \mathbf{c}(n + i, 0)\, \boldsymbol{\Phi}(n + i, 0; n + j)\, \boldsymbol{b}(n + j)\,. \tag{32}$$

Obviously

$$v(n, \varepsilon) = \gamma(n + s, \varepsilon)\, \Phi(n + s, 0; n)\, \mathbf{B}(n, 0) + \mu(n, \varepsilon)\, \mathbf{R}(n, 0)\,. \tag{33}$$

The following relations can be used for the mutual transformation of the forms (a) and (b):

$$\alpha(n, \varepsilon)\, \mathbf{Q}(n, \varepsilon) = \mu(n, \varepsilon)\, \mathbf{Q}(n, 0)\,, \tag{34}$$

$$\beta(n, \varepsilon) \{\mathbf{B}(n, 0) - \mathbf{Q}^{-1}(n, \varepsilon)\, \mathbf{R}(n, \varepsilon)\}^{-1} = v(n, \varepsilon) \{\mathbf{B}(n, 0) - \mathbf{Q}^{-1}(n, 0)\, \mathbf{R}(n, 0)\}^{-1}\,, \tag{35}$$

$$\alpha_i(n, 0) = \mu_i(n, 0)\,; \quad i = 0, 1, \ldots, s - 1 \tag{36}$$

and

$$\beta_j(n, 0) = v_j(n, 0)\,; \quad j = 1, 2, \ldots, s\,. \tag{37}$$

Proof. a) Gradually applying the operator

$$V^i f(n, \varepsilon) = f(n + i, \varepsilon)\,; \quad i = 0, 1, \ldots, s\,, \tag{38}$$

to the eq. (8b) and using the eq. (8a) we have

$$\begin{aligned} y(n, \varepsilon) &= \gamma(n, \varepsilon)\, \mathbf{x}(n, 0)\,, \\ y(n + 1, \varepsilon) &= \gamma(n + 1, \varepsilon) \left[\Phi(n + 1, 0; n)\, \mathbf{x}(n, 0) + \mathbf{b}(n + 1, 0)\, u(n + 1)\right], \\ &\;\vdots \\ y(n + i, \varepsilon) &= \gamma(n + i, \varepsilon) \left[\Phi(n + i, 0; n)\, \mathbf{x}(n, 0) + \right. \\ &\quad \left. + \sum_{j=1}^{i} \Phi(n + i, 0; n + j)\, \mathbf{b}(n + j, 0)\, u(n + j)\right], \\ &\;\vdots \\ y(n + s - 1, \varepsilon) &= \gamma(n + s - 1, \varepsilon) \left[\Phi(n + s - 1, 0; n)\, \mathbf{x}(n, 0) + \right. \\ &\quad \left. + \sum_{j=1}^{s-1} \Phi(n + s - 1, 0; n + j)\, \mathbf{b}(n + j, 0)\, u(n + j)\right] \end{aligned} \tag{39}$$

and

$$\begin{aligned} y(n + s, \varepsilon) &= \gamma(n + s, \varepsilon) \left[\Phi(n + s, 0; n)\, \mathbf{x}(n, 0) + \right. \\ &\quad \left. + \sum_{j=1}^{s} \Phi(n + s, 0; n + j)\, \mathbf{b}(n + j, 0)\, u(n + j)\right]. \end{aligned} \tag{40}$$

Introducing the $(s \times 1)$ vectors

$$\mathbf{y}(n, \varepsilon) = \begin{bmatrix} y(n, \varepsilon) \\ y(n + 1, \varepsilon) \\ \vdots \\ y(n + s - 1, \varepsilon) \end{bmatrix} \tag{41}$$

 and

$$\mathbf{u}(n+1) = \begin{bmatrix} u(n+1) \\ u(n+2) \\ \vdots \\ u(n+s) \end{bmatrix} \tag{42}$$

the set of equations (39) can be written in the vector-matrix form

$$\mathbf{y}(n, \varepsilon) = \mathbf{Q}(n, \varepsilon)\, \mathbf{x}(n, 0) + \mathbf{R}(n, \varepsilon)\, \mathbf{u}(n+1) \tag{43}$$

where the $(s \times s)$ matrices $\mathbf{Q}(n, \varepsilon)$ and $\mathbf{R}(n, \varepsilon)$ are given by (23) and (25) with (26), respectively.

Solving the eq. (43) for $\mathbf{x}(n, 0)$ we get

$$\mathbf{x}(n, 0) = \mathbf{Q}^{-1}(n, \varepsilon)\, [\mathbf{y}(n, \varepsilon) - \mathbf{R}(n, \varepsilon)\, \mathbf{u}(n+1)] \tag{44}$$

provided $\mathbf{Q}(n, \varepsilon)$ is nonsingular.

Substituting now the relation (44) into the equation (40) we can write

$$\begin{aligned} y(n+s, \varepsilon) &= \gamma(n+s, \varepsilon)\, \Phi(n+s, 0; n)\, \mathbf{Q}^{-1}(n, \varepsilon)\, [\mathbf{y}(n, \varepsilon) - \mathbf{R}(n, \varepsilon)\, \mathbf{u}(n+1)] + \\ &\quad + \mathbf{r}_s(n, \varepsilon)\, \mathbf{u}(n+1) \end{aligned} \tag{45}$$

where the $(1 \times s)$ row vector is given by

$$\mathbf{r}_s(n, \varepsilon) = \gamma(n+s, \varepsilon)\, \Phi(n+s, 0; n)\, \mathbf{B}(n, 0) \tag{46}$$

if $\mathbf{B}(n, 0)$ has the form (24).

With respect to (46) the equation (45) can be written as

$$\begin{aligned} & y(n+s, \varepsilon) - \gamma(n+s, \varepsilon)\, \Phi(n+s, 0; n)\, \mathbf{Q}^{-1}(n, \varepsilon)\, \mathbf{y}(n, \varepsilon) = \\ &\quad = \gamma(n+s, \varepsilon)\, \Phi(n+s, 0; n)\, \{\mathbf{B}(n, 0) - \mathbf{Q}^{-1}(n, \varepsilon)\, \mathbf{R}(n, \varepsilon)\}\, \mathbf{u}(n+1) \end{aligned} \tag{47}$$

and the validity of the relations (20), (21) and (22) is evident.

b) Writing the equation (43) for $\varepsilon = 0$ we have

$$\mathbf{y}(n, 0) = \mathbf{Q}(n, 0)\, \mathbf{x}(n, 0) + \mathbf{R}(n, 0)\, \mathbf{u}(n+1) \tag{48}$$

where $\mathbf{Q}(n, 0)$ is given by (31) and $\mathbf{R}(n, 0)$ given by (25) has now the elements (32).

Substituting the solution $\mathbf{x}(n, 0)$ from the equation (48) into the equation (40) we get

$$\begin{aligned} y(n+s, \varepsilon) &= \gamma(n+s, \varepsilon)\, \Phi(n+s, 0; n)\, \mathbf{Q}^{-1}(n, 0)\, [\mathbf{y}(n, 0) - \mathbf{R}(n, 0)\, \mathbf{u}(n+1)] + \\ &\quad + \mathbf{r}_s(n, \varepsilon)\, \mathbf{u}(n+1)\,. \end{aligned} \tag{49}$$

Using (24)

$$(50)\quad y(n+s,\varepsilon) - \gamma(n+s,\varepsilon)\,\boldsymbol{\Phi}(n+s,0;n)\,\mathbf{Q}^{-1}(n,0)\,\mathbf{y}(n,0) = \\ = \gamma(n+s,\varepsilon)\,\boldsymbol{\Phi}(n+s,0;n)\,\{\mathbf{B}(n,0) - \mathbf{Q}^{-1}(n,0)\,\mathbf{R}(n,0)\}\,\boldsymbol{u}(n+1)$$

and the validity of (28), (29) and (30) is proved.

The relations (34)–(37) results directly comparing the equations (20) and (28).

We can see that the coefficients $\mu_i(n,\varepsilon)$, $v_j(n,\varepsilon)$ can be obtained in general by simpler way than $\alpha_i(n,\varepsilon)$, $\beta_j(n,\varepsilon)$.

III. SIMPLIFIED CASES

1. Normalized canonical form

The above relations are much simpler if the system (1) can be described on definite time interval in normalized canonical form [5, pp. 317] as

$$(51a)\qquad \dot{\mathbf{x}}(t) = \mathbf{b}(t)\,u(t)\,,$$

$$(51b)\qquad y(t) = \mathbf{c}(t)\,\mathbf{x}(t)\,.$$

Then according to (4)

$$\boldsymbol{\Phi}(t, t_0) = \mathbf{I}$$

and the discrete-time version of state equations has the form

$$(52)\qquad \mathbf{x}(n+1,\varepsilon) = \mathbf{x}(n+1,0) = \mathbf{x}(n,0) + \mathbf{b}(n+1,0)\,u(n+1)\,, \\ y(n,\varepsilon) = \mathbf{c}(n,\varepsilon)\,\mathbf{x}(n,0)\,.$$

The scalar difference equation has the coefficients given for the form (a) by

$$(53)\qquad \boldsymbol{\alpha}(n,\varepsilon) = -\mathbf{c}(n+s,\varepsilon)\,\mathbf{Q}^{-1}(n,\varepsilon)$$

and

$$(54)\qquad \boldsymbol{\beta}(n,\varepsilon) = \mathbf{c}(n+s,\varepsilon)\,\{\mathbf{B}(n,0) - \mathbf{Q}^{-1}(n,\varepsilon)\,\mathbf{R}(n,\varepsilon)\}$$

where

$$(55)\qquad \mathbf{Q}(n,\varepsilon) = \begin{bmatrix} \mathbf{c}(n,\varepsilon) \\ \mathbf{c}(n+1,\varepsilon) \\ \vdots \\ \mathbf{c}(n+s-1,\varepsilon) \end{bmatrix},$$

$$(56)\qquad \mathbf{B}(n,0) = [\mathbf{b}(n+1,0)\,\mathbf{b}(n+2,0)\ldots\mathbf{b}(n+s,0)]$$

and $\mathbf{R}(n, \varepsilon)$ has the elements

$$r_{ij}(n, \varepsilon) = \mathbf{c}(n + i, \varepsilon)\, \mathbf{b}(n + j, 0)\,. \tag{57}$$

The coefficients $\mu(n, \varepsilon)$ and $\nu(n, \varepsilon)$ of the form (b) are simplified by similar way.

2. Stationary system

If the parameters of the system (1) do not vary with time then

$$\Phi(t, t_0) = e^{A(t-t_0)} \tag{58}$$

and the state equations are given by

$$\begin{aligned} \mathbf{x}(n + 1, \varepsilon) &= e^{A(1+\varepsilon)}\mathbf{x}(n, 0) + e^{A\varepsilon}\mathbf{b}\, u(n + 1)\,, \\ y(n, \varepsilon) &= \mathbf{c}\, \mathbf{x}(n, \varepsilon) \end{aligned} \tag{59}$$

or

$$\begin{aligned} \mathbf{x}(n + 1, 0) &= e^{A}\mathbf{x}(n, 0) + \mathbf{b}\, u(n + 1)\,, \\ y(n, \varepsilon) &= \mathbf{c}e^{A\varepsilon}\, \mathbf{x}(n, 0)\,. \end{aligned} \tag{60}$$

To find the coefficients of scalar difference equation we determine

$$\mathbf{Q}(\varepsilon) = \begin{bmatrix} \mathbf{c} \\ \mathbf{c}e^{A} \\ \vdots \\ \mathbf{c}e^{A(s-1)} \end{bmatrix} e^{A\varepsilon}\,, \tag{61}$$

$$\mathbf{B} = \left[e^{-A}\mathbf{b};\, e^{-2A}\mathbf{b};\, \ldots;\, e^{-sA}\mathbf{b}\right] \tag{62}$$

and

$$r_{ij}(\varepsilon) = \mathbf{c}e^{A\varepsilon}e^{A(i-j)}\mathbf{b}\,. \tag{63}$$

Then

$$\boldsymbol{\alpha} = -\mathbf{c}e^{A\varepsilon}e^{As}\, \mathbf{Q}^{-1}(\varepsilon) = -\mathbf{c}e^{As} \begin{bmatrix} \mathbf{c} \\ \mathbf{c}e^{A} \\ \vdots \\ \mathbf{c}e^{A(s-1)} \end{bmatrix}^{-1} \tag{64}$$

and

$$\boldsymbol{\beta}(\varepsilon) = \mathbf{c}e^{A\varepsilon}\mathbf{B} + \boldsymbol{\alpha}\, \mathbf{R}(\varepsilon)\,. \tag{65}$$

By similar way can be obtained

$$\boldsymbol{\mu}(\varepsilon) = -\mathbf{c}e^{A\varepsilon}e^{As} \begin{bmatrix} \mathbf{c} \\ \mathbf{c}e^{A} \\ \vdots \\ \mathbf{c}e^{A(s-1)} \end{bmatrix}^{-1} \tag{66}$$

and

(67) $$\mathbf{v}(\varepsilon) = \mathbf{c}e^{\mathbf{A}\varepsilon}e^{\mathbf{A}s}\mathbf{B} + \mu(\varepsilon)\,\mathbf{R}\,.$$

CONCLUSIONS

Considering an application of discrete-time input to a linear nonstationary system given by continuous-time state equations

a) the state equations in the discrete-time form or the scalar difference equation can be obtained if system transition matrix $\boldsymbol{\Phi}(t, t_0)$ is known;

b) in spite of $\boldsymbol{\Phi}(t, t_0)$ cannot be obtained in general by analytical way the presented relations might be useful for numerical solution;

c) using the above relations the mutual transformation between vector-matrix and scalar discrete-time description is always possible.

EXAMPLE

A discrete-time signal is applied to the continuous system described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} e^t \\ 1 \end{bmatrix} u(t)\,,$$

$$y(t) = \begin{bmatrix} 1 & e^{-t} \end{bmatrix} \mathbf{x}(t)\,.$$

Obviously the matrix $\mathbf{A}$ is stationary in this case and the transition matrix can be obtained as

$$\boldsymbol{\Phi}(t, t_0) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-(t-t_0)} \end{bmatrix}.$$

Let us determine the discrete-time state equations and the corresponding scalar difference equation of this system.

1. Using the relations (9)–(13) we have

$$\mathbf{F}(\varepsilon) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-(1+\varepsilon)} \end{bmatrix},$$

$$\mathbf{E}(\varepsilon) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\varepsilon} \end{bmatrix},$$

$$\mathbf{h}(n, \varepsilon) = \begin{bmatrix} e^n \\ e^{-\varepsilon} \end{bmatrix}$$

and

$$\boldsymbol{\gamma}(n, \varepsilon) = \begin{bmatrix} 1 & e^{-(n+2\varepsilon)} \end{bmatrix}.$$

The state equations are given by (7) as

$$x(n + 1, \varepsilon) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-(1+\varepsilon)} \end{bmatrix} x(n, 0) + \begin{bmatrix} e^{n+1} \\ e^{-\varepsilon} \end{bmatrix} u(n + 1),$$

$$y(n, \varepsilon) = [1 \ \ e^{-(n+\varepsilon)}] \, x(n, \varepsilon)$$

or according to (8)

$$x(n + 1, 0) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-1} \end{bmatrix} x(n, 0) + \begin{bmatrix} e^{n+1} \\ 1 \end{bmatrix} u(n + 1),$$

$$y(n, \varepsilon) = [1 \ \ e^{-(n+2\varepsilon)}] \, x(n, 0).$$

2. Using (23)–(26) we have

$$Q(n, \varepsilon) = \begin{bmatrix} 1 & e^{-(n+2\varepsilon)} \\ 1 & e^{-(n+2+2\varepsilon)} \end{bmatrix},$$

$$B(n, 0) = \begin{bmatrix} e^{n+1} & e^{n+2} \\ e & e^2 \end{bmatrix}$$

and

$$R(n, \varepsilon) = \begin{bmatrix} 0 & 0 \\ e^{n+1} + e^{-(n+1+2\varepsilon)} & 0 \end{bmatrix}.$$

From the relation (21) we get

$$\alpha(n, \varepsilon) = \alpha = [e^{-2}; \ -(1 + e^{-2})]$$

and using (22) or (27)

$$\beta(n, \varepsilon) = [-e^{n-1} - e^{-(n+1+2\varepsilon)}; \ e^{n+2} + e^{-(n+2+2\varepsilon)}].$$

Using the relation (29) we have

$$\mu(\varepsilon) = \frac{1}{1 - e^{-2}} [e^{-2} - e^{-(4+2\varepsilon)}; \ e^{-(4+2\varepsilon)} - 1]$$

and from (30) or (33)

$$\nu(n, \varepsilon) = \left[\frac{e^{-(2+2\varepsilon)} - 1}{1 - e^{-2}} (e^{n-1} + e^{-(n+1)}); \ e^{n+2} + e^{-(n+2+2\varepsilon)}\right].$$

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