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Appendix to the Article "On Generalized Linear Discrete Inversion Filters"

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An expression for the mean square error of the inversion filter is derived and some consequences are shown.

1. INTRODUCTION

Let

(1) $$B_j(z) = b_{j0} + b_{j1}z^{-1} + \ldots + b_{jh}z^{-h}$$
$$(j = 0, 1, \ldots, m)$$

be Z-transforms of discrete signals, the signal with $j = 0$ being wanted and the other disturbing. The signals are weighted by weights w_j, where $w_0 = 1$, $w_j \geqq 0$ for $j = 1, 2, \ldots, m$.

Let us define

(2) $$B(z) = b_0 + b_1 z^{-1} + \ldots + b_s z^{-s}, \quad (s \leqq h)$$

with the aid of the relation

(3) $$|B(z)|^2 = \sum_{j=0}^{m} w_j |B_j(z)|^2, \quad z = \exp i\omega.$$

According to the known Fejér-Riesz theorem $B(z)$ exists and is unique if we choose its roots only on and outside the unit circle C_1.

Suppose $B(z)$ to have all roots only outside of C_1. Then, $B(z)\, B(z^{-1})\, z^s$ is a reciprocal polynomial with roots $\zeta_1, \ldots, \zeta_s$ outside C_1 and $\zeta_{s+1}, \ldots, \zeta_{2s}$ inside C_1 (the reciprocal values of the roots $\zeta_1, \ldots, \zeta_s$) and

(4) $$B(z)\, B(z^{-1})\, z^s = \frac{b_0 b_s}{(-1)^s \zeta_1 \ldots \zeta_s} (1 - \zeta_1 z) \ldots (1 - \zeta_s z)(z - \zeta_1) \ldots (z - \zeta_s).$$

Define

$$q_0 + q_1 z + \ldots + q_s z^s = \frac{b_0 b_s}{(-1)^s \zeta_1 \ldots \zeta_s} (z - \zeta_1) \ldots (z - \zeta_s) . \tag{5}$$

Put

$$C_j(z) = A(z) B_j(z) . \tag{6}$$

In [2] there has been shown that $A(z)$ fulfilling

$$\frac{1}{2\pi i} \left\{ w_0 \int_{C_1} |z^{-T} - A(z) B_0(z)|^2 . |dz| + \sum_{j=1}^{m} w_j . \int_{C_1} |A(z) B_j(z)|^2 . |dz| \right\} = \min \tag{7}$$

is given by the expression

$$A(z) = \frac{z^{s-T}(r_0 + r_1 z + \ldots + r_T z^T)}{(1 - \zeta_1 z) \ldots (1 - \zeta_s z)} , \tag{8}$$

where r_j ($j = 0, 1, \ldots, T$) result from the system of equations

$$\begin{aligned} q_0 r_0 &= b_{00} , \\ q_1 r_0 + q_0 r_1 &= b_{01} , \\ &\vdots \\ q_T r_0 + q_{T-1} r_1 + \ldots + q_0 r_T &= b_{0T} , \end{aligned} \tag{9}$$

where $q_j = 0$ for $j > s$, $b_{0j} = 0$ for $j > h$.

In [1], [2] has been shown that the minimum in (7) is $1 - c_{0T}$, where c_{0T} is the coefficient of z^{-T} in the development of (6), and also that $0 < c_{0T} \leqq 1$.

2. AN EXPRESSION FOR c_{0T}

From the system (9) it is seen that $r_0, \ldots, r_T$ are coefficients of the development

$$r_0 + r_1 z^{-1} + \ldots = \frac{b_{00} + b_{01} z^{-1} + \ldots + b_{0h} z^{-h}}{q_0 + q_1 z^{-1} + \ldots + q_s z^{-s}} . \tag{10}$$

The coefficient c_{0T} is given by

$$c_{0T} = \frac{1}{2\pi i} \int_{C_1} B_0(z) A(z) z^T \frac{dz}{z} , \tag{11}$$

 that is, the zero order term of the development

$$(12) \qquad B_0(z)\,A(z)\,z^T = \frac{b_{00} + \ldots + b_{0h}z^{-h}}{(z^{-1} - \zeta_1)\ldots(z^{-1} - \zeta_s)}(r_0 + \ldots + r_T z^T) =$$

$$= \frac{b_{00} + \ldots + b_{0h}z^{-h}}{\frac{(-1)^s \zeta_1 \ldots \zeta_s}{b_0 b_s}(q_0 + \ldots + q_s z^{-s})}(r_0 + \ldots + r_T z^T) =$$

$$= \frac{b_0 b_s}{(-1)^s \zeta_1 \ldots \zeta_s}(r_0 + r_1 z^{-1} + \ldots)(r_0 + \ldots + r_T z^T).$$

Thus

$$(13) \qquad c_{0T} = \frac{b_0 b_s}{(-1)^s \zeta_1 \ldots \zeta_s}(r_0^2 + r_1^2 + \ldots + r_T^2).$$

3. SOME SPECIAL CASES

There is seen from (13) that c_{0T} is a nondecreasing function of T, thus the minimum of (7), being $1 - c_{0T}$, is a nonincreasing function of T. Since $c_{0T} \leqq 1$, there exists the limit of C_{0T} for $T \to \infty$. From (13) with the aid of (12), (4), (5) and the Parseval indetity, there is

(14)

$$\lim_{T\to\infty} c_{0T} = \frac{1}{2\pi i}\int_{C_1} \frac{(b_{00} + \ldots + b_{0h}z^{-h})(b_{00} + \ldots + b_{0h}z^h)}{z^{-s}(1 - \zeta_1 z)\ldots(1 - \zeta_s z)(z - \zeta_1)\ldots(z - \zeta_s)\frac{b_0 b_s}{(-1)^s \zeta_1 \ldots \zeta_s}} \cdot \frac{dz}{z} =$$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{B_0(z)\,B_0(z^{-1})}{B(z)\,B(z^{-1})} \cdot \frac{dz}{z}.$$

In the case of "pure" inversion, that is $w_0 = 1$, $w_j = 0$ for $j = 1, \ldots, m$, there is $B_0(z) = B(z)$, thus $\lim_{T\to\infty} c_{0T} = 1$.

Suppose further the "pure" inversion and $T = 0$. Then from (9), (5), (13) one gets

$$(15) \qquad r_0 = b_0/q_0 = 1/b_h$$

and since

$$(16) \qquad b_h/b_0 = (-1)^h z_1 \ldots z_h,$$

there is

$$(17) \qquad c_0 = \frac{1}{(-1)^h \zeta_1 \ldots \zeta_h} \cdot \frac{b_0}{b_h} = \frac{1}{\zeta_1 \ldots \zeta_h z_1 \ldots z_h} = \frac{\zeta_1^* \ldots \zeta_h^*}{z_1 \ldots z_h},$$

where $z_1, \ldots, z_h$ are the roots of the polynomial $z^h B(z)$ and

$$(18) \qquad \zeta_j^* = \begin{cases} z_j & \text{for } |z_j| < 1, \\ z_j^{-1} & \text{for } |z_j| > 1, \end{cases} \qquad (j = 1, \ldots, h).$$

This result has been derived in [1] with the unnecessary restriction that all roots z_j are simple.

The restriction $|z_j| \neq 1$ in (18) is substantial since we know that no stable filter of the form (8) exists if some roots ζ_j lie on C_1.

4. CONCLUDING REMARKS

From (9), (5), and (13) one sees that for c_{0T} expressions in the symmetric functions of the roots ζ_j can be derived, but they will be substantially more complicated than (17).

Furthermore, it is seen from (9), (10) that the sequence $\{r_j\}$ with $j > h$ is solution of a homogeneous linear difference equation with characteristic roots $\zeta_{s+1}, \ldots, \zeta_{2s}$ lying inside C_1.

The initial conditions result from (9) or (10). This result may be useful in connection with (14) for computing (13) if T is substantially greater than h, especially if a "dominant" root ζ_j exists.

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REFERENCES

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VÝTAH

Doplněk k článku „O zobecněných lineárních diskrétních inverzních filtrech“

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V článku se odvozuje výraz pro střední kvadratickou chybu inverzního filtru a ukazují se některé důsledky.

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