KYBERNETIKA — VOLUME 8 (1972), NUMBER 3

Several Approaches to Pulse-Width-Modulated Regulator Synthesis via Quasilinearization

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Pulse-width modulation, one of the possible connections of the digital computer evaluating the variables defined at the discrete time moments to the plant with the variables defined on time continuum, is sometimes used because of the economical or technological constraints put on the actuators; for large class of problems the pulse-width modulation is used because of the advantages of the implementation and/or signal transmission. Three numerically oriented approaches to the pulse-width-modulated regulator synthesis are presented: discrete minimum principle, dynamic and linear programming, all using quasilinearization. Numerical results are presented in confirmation. The results suggest the suitability of the use of suboptimal regulators.

INTRODUCTION

Pulse-width modulation, one of the possible connections of the digital control computer evaluating the variables defined at the discrete time moments to the plant with the variables defined on time continuum, is sometimes used because of the economical or technological constraints put on the actuators; for large class of problems the pulse-width modulation is used because of the advantages of the implementation and/or signal transmission.

Authors' report [5] presents the bibliography of 101 works on pulse-width modulation: their spectrum involves applications in direct digital control, electrical and electronic engineering, air-conditioning, ergonomics, biocybernetics and astronautics. The contribution based on [5] presents the novel application of three different approaches for optimal or suboptimal regulator synthesis, all — to the various extent — based on quasilinearization chosen for its rapid convergence with good initial estimates. By quasilinearization QL we understand the mapping of the smooth nonlinear function $f(x)$ to the function linear (up to an additive term) in the new argument $x^{(N+1)}$

$$\text{QL} : f(x) \to f(x^{(N)}) + f_x^{(N)}(x^{(N+1)} - x^{(N)}) \tag{1}$$

where N is the iteration index and $f_x^{(N)}$ denotes the Fréchet derivative at the point $x^{(N)}$. Solution of the problem is sought sequentially for $N = 0, 1, 2, \ldots$ as the solution of simpler problems connected with the function $\mathrm{QL}\, f(x)$ which is linear in $x^{(N+1)}$.

In the following, the restriction will be made to a simple plant consisting sequentially of a pulse-width modulator, an ideal actuator, a static polynomial nonlinearity, and first order plant. It can be expected that this plant with the integrator at the beginning suffices to analyze only at the sampling moments, consequently as a discrete plant, and there will be no effect analyzed e.g. in [6], such that the plant with just negative eigenvalues has the limit cycle as a consequence of autonomous transient behaviour at the times between the end of an old and the beginning of a new width-modulated pulse. Justification of the synthesis only for discrete system can be verified by simulation.

The pulse-width modulator will be described by

$$v(t) = \begin{cases} M_1 \operatorname{sign} u_k & (kT < t < kT + \tau_k)\,, \\ 0 & (kT + \tau_k < t < (k+1)\,T)\,, \end{cases} \tag{2}$$

$$\tau_k = \begin{cases} M_2 |u_k| & (\tau_k < T)\,, \\ T & (\tau_k > T)\,, \end{cases} \tag{3}$$

where $(.)_k$ denotes $(.)\,(kT)$, for $k = 0, 1, \ldots;\ M_1, M_2, T > 0$. Width-modulated pulses are of a width τ_k variable from 0 to T and a height either M_1, or $-M_1$. The continuous plant will be described by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} M_3\, v(t) \\ \lambda x_2(t) + M_4 \sum_{l=0}^{L} c_l x_1^l(t) \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{4}$$

which can be – using (2), (3) – transformed analytically and/or numerically to the discrete plant

$$x_{k+1} = f(x_k, u_k)\,, \quad x_0 = \alpha \quad (k = 0, 1, \ldots) \tag{5}$$

where $f : E^2 \times E^1 \to E^2$. The criterion will be

$$I_q = \begin{cases} \sqrt[q]{\left(\sum_{j=0}^{J-1} \beta_1 |e_{1,j+1}|^q + \beta_2 |e_{2,j+1}|^q + \gamma |u_j|^q \right)} & (1 \leqq q < \infty)\,, \\ \max\limits_{j=0,\ldots,J-1} \{\beta_1 |e_{1,j+1}|, \beta_2 |e_{2,j+1}|, \gamma |u_j|\} & (q = \infty) \end{cases} \tag{6}$$

with $\beta_1, \beta_2, \gamma \geqq 0$, $\beta_1 \cdot \beta_2 \neq 0$, where e_k denotes the regulator error $x_k - \omega$, ω being the required state.

The discrete minimum principle transforms the simultaneous seeking for the input (manipulated variable) values at all control steps to the sequential seeking for inputs at the particular control steps. E.g., in the formulation of [2] such an end state x_J^* dependent on the input sequence $u_0 \ldots u_{J-1}$ is sought that

$$\tilde{c}'x_J^* = \min_{x_J} \tilde{c}'x_J \tag{7}$$

($(\cdot)'$ denotes the transposition of $(\cdot)$) with constraints (5). Costate vector p_k, which can be interpreted as a normal to a tangent hyperplane to a set of attainable states at the points of the optimal states x_k^*, is defined by

$$p_k = f'_{x_k} p_{k+1}\,, \quad p_J = \tilde{c} \quad (k = J - 1, \ldots, 0)\,, \tag{8}$$

and scalar hamiltonian which is to be minimized by the current input

$$H(u_{k-1}) = p_k' x_k \tag{9}$$

To satisfy the formulation (7), in addition to (5), the third component of state is appended to (5). From (6) it is obtained for $q = 2$:

$$x_{3,k+1} = f_3(x_{1,k};\, x_{2,k};\, x_{3,k};\, u_k) = x_{3,k} + e_k'\beta e_k + \gamma u_k^2; \quad x_{3,0} = 0 \tag{10}$$

where $\beta = \operatorname{diag}[\beta_1\ \beta_2]$. The extended state function $f : E^3 \times E^1 \to E^3$. To complete (7) it is set: $\tilde{c} = [0\ 0\ 1]'$. For synthesis, for the fixed initial state α and estimated inputs $u_0 \ldots u_{J-1}$, the forward equations (5) were solved with the appended equation (10), and afterwards the backward equations (8) with $p_k \in E^3$ were solved. For hamiltonian minimization with respect of current input was used quasilinearization (1) of the equation $H_u = 0$. With regard to the input limitation:

$$u^{(N+1)} = \begin{cases} \varphi(u^{(N)}) & (u_{\min} \leqq \varphi(u^{(N)}) \leqq u_{\max})\,, \\ u_{\min} & (\varphi(u^{(N)}) < u_{\min})\,, \\ u_{\max} & (\varphi(u^{(N)}) > u_{\max})\,, \end{cases} \tag{11}$$

where

$$\varphi(u^{(N)}) = u^{(N)} - H_u^{(N)}/H_{uu}^{(N)}\,; \quad H_{uu}^{(N)} > 0\,, \quad -u_{\min} = u_{\max} = T/M_2\,.$$

Fig. 1 presents the convergence of the synthetized inputs in the dependence on their initial estimates for $T = -\lambda = M_1 = \ldots = M_4 = c_1 = L = 1$, $J = 2$, $c_0 = \omega_1 = \omega_2 = 0$, $\alpha_1 = \alpha_2 = 0{\cdot}1$. In the case of divergence – which is the only in Fig. 1 – the golden rule instead of quasilinearization was used for minimization of $H(u)$ with $u \in [u_{\min}, u_{\max}]$. The points u_L, u_R $(u_{\min} < u_L < u_R < u_{\max})$ divided the initial interval and $H(u_L)$, $H(u_R)$ was evaluated. For $H(u_L) \leqq H(u_R)$ the minimum is located in the contracted interval $[u_{\min}, u_R]$ and analogically for $H(u_L) > H(u_R)$ in the contracted interval $[u_L, u_{\max}]$. One of the possible choices is the division of the interval by golden rule 1 : $[(1 + \sqrt{5})/2]$ and continuation in this division also in the following sequentially contracted intervals. As a difference from quasilinearization for which it suffices for convergence e.g. as much as existence of continuous, positive, convex H_{uu}, [8],

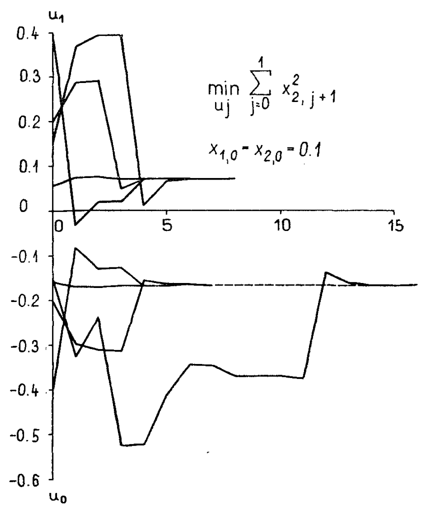

Fig. 1. Dependence of the convergence on the initial estimates of the synthetized optimal inputs (horizontal axis — number of iterations).

for golden rule it suffices only existence of the unique, sharp minimum of $H(u)$ at $[u_{\min}, u_{\max}]$. Convergence of golden rule is relatively slow to be suitable with respect to good initial estimates.

For implementation of the regulator the knowledge of just the first input u_0 under improper $(J \to \infty)$ criterion (6) is required. To find the number of control steps J for which $u_0(J)$ approximates $u_0(\infty)$ the function $J \to u_0(J)$ was investigated. No significant dependence was found. With increasing J the convergence was aggravated.

The dependence of inputs and final errors on the sampling period T for the fixed control time $J \,.\, T = 3$ was investigated, see Fig. 2 $(-\lambda = M_1 = \ldots = M_4 = c_2 = 1$, $c_0 = 0{\cdot}1$, $c_1 = \gamma = 0$, $L = 2$, $\alpha_1 = 0{\cdot}9$, $\alpha_2 = 0{\cdot}91$, $\omega_1 = 0{\cdot}5$, $\omega_2 = 0{\cdot}35)$. For shortening sampling period T, the inputs suggest the convergence to the bang-bang control. Low sensitivity of control quality to variations in sampling period is remarkable.

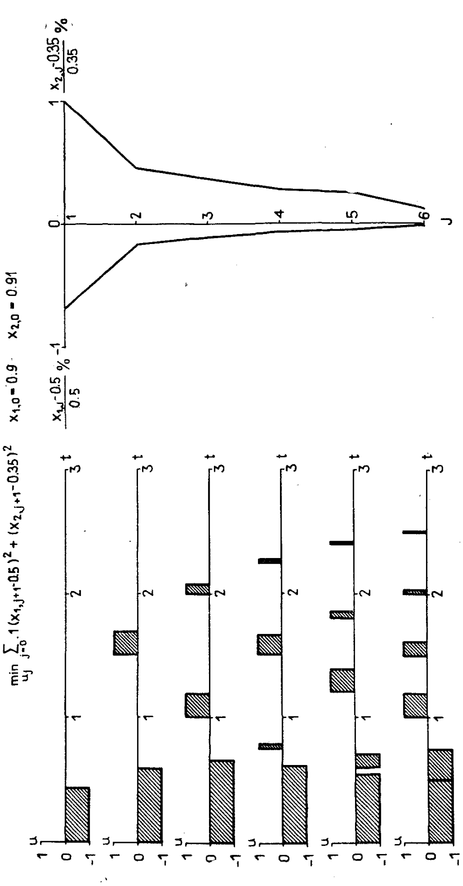

Fig. 2. Dependence on the sampling period of the optimal inputs and final control errors.

PWM REGULATOR SYNTHESIS VIA DYNAMIC PROGRAMMING AND QUASILINEARIZATION

Using the optimality principle of dynamic programming to the dynamical system (5) and the criterion (6) ($q = 2$), the recursive equation is obtained for evolution of minimum criterion

$$(12)\quad I^*_{J-k}(x_k) = \begin{cases} e_k'\beta e_k + \min\limits_{u_k} [\gamma u_k^2 + I^*_{J-k-1}(x_{k+1})] & (k = J-1, \ldots, 1), \\ \min\limits_{u_0} [\gamma u_0^2 + I^*_{J-1}(x_1)] & (k = 0) \end{cases}$$

with the initial condition

$$(13)\quad I^*_0(x_J) = e_J'\beta e_J$$

where $I^*_{J-k}(x)$ is minimum over $u_k \ldots u_{J-1}$ of the recursive criterion (compare with (6), (10)):

$$(14)\quad I_{J-k}(x_k) = \begin{cases} I_{J-k-1}(x_{k+1}) + e_k'\beta e_k + \gamma u_k^2 & (k = J-1 \ldots 1), \\ I_{J-1}(x_1) + \gamma u_0^2 & (k = 0). \end{cases}$$

The equation (12) can be solved analytically under the conditions of the linear dependence of the state on the input

$$(15)\quad x_{k+1} = g(x_k) + hu_k$$

and the quadratic optimal recursive criterion

$$(16)\quad I^*_{J-k} = e_k'Q_k e_k \quad (k = J-1, \ldots, 0),$$

where pos. def. $Q_k = \text{diag}[\tilde{q}_{1,k}\, \tilde{q}_{2,k}]$. To fulfil the conditions (15), (16) quasilinearization of the state equation (5) was used. Solving $I^{(N)}_{u_k} = 0$, the iteration of input was obtained

$$(17)\quad u_k^{*(N+1)} = -\frac{h^{(N)\prime}Q_k^{(N)}(g^{(N)} - \omega)}{\gamma + h^{(N)\prime}Q^{(N)}h^{(N)}} \quad (k = J-1, \ldots, 0)$$

which was again as in (11) modified because of u_k limited to $[u_{\min}, u_{\max}]$. The optimal recursive criterion is afterwards

$$(18)\quad I^*_{J-k}(x_k) = \begin{cases} [g(x_k) + hu_k^* - \omega]'\, Q_k[g(x_k) + hu_k^* - \omega] + \\ + e_k'\beta e_k + \gamma u_k^{*2} & (k = J-1 \ldots 1), \\ [g(x_k) + hu_k^* - \omega]'\, Q_k[g(x_k) + hu_k^* - \omega] + \\ + \gamma u_k^{*2} & (k = 0). \end{cases}$$

The values of optimal recursive criterion at four state grid points between which the next state evaluated from the last iteration of input laid were known from (13) or from the previous iteration. For $\tilde{q}_1$, $\tilde{q}_2$ four linear equations

$$(19)\quad I^*({}^i x) = {}^i e_1^2 \tilde{q}_1 + {}^i e_2^2 \tilde{q}_2 \quad (i = 1, \ldots, 4)$$

were obtained and solved using least squares method. Computation algorithm:

(i) Evaluation of optimal recursive criterion I_0^* at all state grid points according to (13). (ii) Evaluation of initial estimate of input at all state grid points as the input driving only integrator at one step. (iii) Selection of a new state grid point like in reading by the lines; at the end decreasing k. (iv) Evaluation of the new state using (5) for x_{k+1}. (v) Evaluation of $\tilde{q}_1, \tilde{q}_2$ from (19). (vi) Linearization of the state equations in the neighbourhood of the last input iteration. (vii) Evaluation of the new input iteration from (17). (viii) If required accuracy of u_k^* is not achieved, return to (iv). (ix) Evaluation of optimal recursive criterion for given state grid point and optimal input from (18), return to (iii).

Fig. 3 presents the contour lines of optimal input and optimal criterion for selected

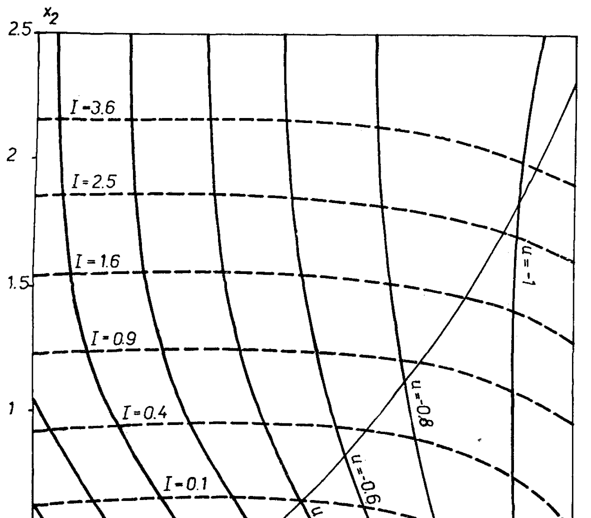

Fig. 3. Contour lines of optimal input and optimal criterion throughout the selected state region.

state region ($-\lambda = T = M_1 = \ldots = M_4 = c_2 = \beta_2 = 1$, $c_0 = \beta_1 = 0{\cdot}1$, $c_1 = \gamma = 0$, $L = 2$, $J = 3$). At the upper part of the state region quasilinearization led to divergence and golden rule was used beeing relatively slow but reliable.

PWM REGULATOR SYNTHESIS VIA LINEAR PROGRAMMING AND QUASILINEARIZATION

Two previous methods were concerned with the numerical synthesis of the regulator as the tabulated function of the current state, state function, and criterion; now this function will be parametrized by the gain π. Limitation will be made to a linear regulator

(20) $$u_k = \pi' e_k \quad (k = 0, \ldots, J - 1)$$

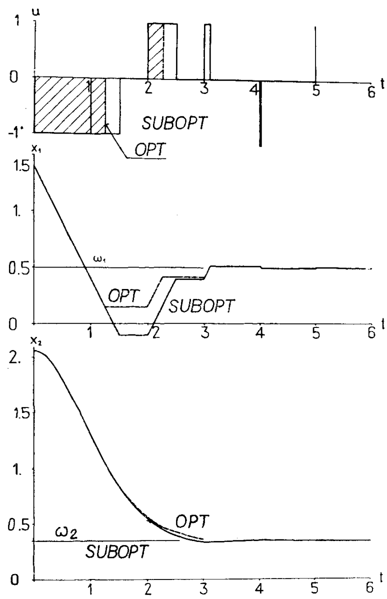

Fig. 4. Comparison of suboptimal and optimal solutions.

and the linear ($q = 1$) or the Tchebyshev ($q = \infty$) criterion, which will be minimized by the standard linear programming algorithm. Minimization of the expression $|\xi_1 \varepsilon_1| + \ldots + |\xi_k \varepsilon_k|$ for $q = 1$ or of the expression $\max \{|\xi_1 \varepsilon_1|, \ldots, |\xi_k \varepsilon_k|\}$ for $q = \infty$

with inequality constraints (in this case of the type $u_{\min} \leqq u_k \leqq u_{\max}$, $k = 0 \dots J - 1$) for linear forms of $\varepsilon_1, \dots, \varepsilon_k$ can be transformed to minimization of the expressions $\mu_1 + \dots + \mu_k$ $(q = 1)$ or μ $(q = \infty)$ by adjoining to the above mentioned constraints other inequalities: either $\pm\xi_1\varepsilon_1 \leqq \mu_1, \dots, \pm\xi_k\varepsilon_k \leqq \mu_k$ $(q = 1)$ or $\pm\xi_1\varepsilon_1 \leqq \leqq \mu, \dots, \pm\xi_k\varepsilon_k \leqq \mu$ $(q = \infty)$. To obtain the linear dependence of the error e_k on the synthetized parameter $\pi = [\pi_1\ \pi_2]'$, the state equation (5) will be quasilinearized and adjoined with the equation expressing the time invariance of parameter to be synthetized

(21) $$\pi_{k+1} = \pi_k \quad (\pi_0 = \pi) .$$

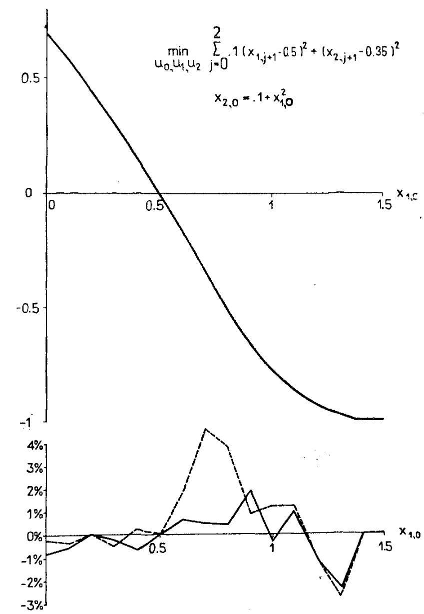

Fig. 5. Dependence of optimal input on the initial state resulting from the three approaches (full line — $(u_0^{DP} - u_0^{DMP})/u_0^{DMP}$; dash line — $(u_0^{LP} - u_0^{DMP})/u_0^{DMP}$).

To these adjoint state equations the Cauchy type lemma will be applied: Cauchy problem for the linear difference equation

(22) $$z_{k+1} = A_k z_k + b_k , \quad z_0 = a \quad (A_k : E^n \to E^n)$$

has the unique solution

(23) $$z_k = [\tilde{z}_{1,k} \dots \tilde{z}_{n,k}]\, a + \tilde{z}_{n+1,k}$$

where $\tilde{z}_{\cdot,k}$ are the solutions of the auxiliary, on the initial state a independent Cauchy problems

$$(24) \qquad [\tilde{z}_{1,k+1} \ldots \tilde{z}_{n,k+1}] = A_k[\tilde{z}_{1,k} \ldots \tilde{z}_{n,k}]\,; \quad [\tilde{z}_{1,0} \ldots \tilde{z}_{n,0}] = I\,,$$

$$(25) \qquad \tilde{z}_{n+1,k+1} = A_k \tilde{z}_{n+1,k} + b_k\,, \quad \tilde{z}_{n+1,0} = 0\,.$$

Table 1.

Comparison of numerical synthesis via three approaches

approach / properties of the solution	discrete minimum principle and quasi-linearization	dynamic programming and quasi-linearization	linear programming and quasilinearization
solution speed	slightly significant differencies		
number of the state variables	easily extendable	extendable with severe restrictions	easily extendable
number of the inputs	extendable only after modification of the static minimization		easily extendable
change of criterion	easy		after modification of the static minimization
extra state constraints	after modification	the solution is easier to obtain	easy to introduce
computing the inputs thoughtout the state region	one point solution useful to the solution in next points	the solution is more easy	one point solution is of no use to the solution in next points

In the mentioned case the parameter π will be at each iteration determined to minimize the criterion (6) with the components derived from the iteration of error

$$(26) \qquad e_k^{(N+1)} = [\tilde{y}_{1,k}^{(N+1)}\ \tilde{y}_{2,k}^{(N+1)}]\, \pi^{(N+1)} + \tilde{y}_{3,k}^{(N+1)} - \omega\,,$$

where $\tilde{y}_{\cdot,k}^{(N+1)}$ are obtained from the solution of the difference equation (corresponding to (24), (25)):

$$\begin{bmatrix} y_{k+1}^{(N)} \\ \tilde{y}_{1,k+1}^{(N+1)} \\ \tilde{y}_{2,k+1}^{(N+1)} \\ \tilde{y}_{3,k+1}^{(N+1)} \end{bmatrix} = \begin{bmatrix} f^{(N)} \\ f_y^{(N)} \tilde{y}_{1,k}^{(N+1)} + f_\pi^{(N)} [1\ 0]' \\ f_y^{(N)} \tilde{y}_{2,k}^{(N+1)} + f_\pi^{(N)} [0\ 1]' \\ f^{(N)} + f_y^{(N)} \tilde{y}_{3,k}^{(N+1)} - f_\pi^{(N)} p^{(N)} - f_y^{(N)} y_k^{(N)} \end{bmatrix} \tag{27}$$

with zero initial conditions, where y denotes $x - \alpha$.

Fig. 4 presents the comparison of suboptimal and optimal solutions obtained via the discrete minimum principle $(-\lambda = T = c_2 = \beta_2 = 1,\ c_0 = \beta_1 = 0.1,\ c_1 = 0,\ L = 3,\ \omega_1 = 0{\cdot}5,\ \omega_2 = 0{\cdot}35)$.

CONCLUSIONS

Tab. 1 and Fig. 5 compare properties of the solution of the numerical synthesis of the pulse-width-modulated regulator according to three approaches. In all approaches the iterative solution was successful (though sometimes at the cost of good initial estimates and the use of golden rule) and the results were compatible. The results suggest the suitability of the suboptimal control.

(Received October 12, 1971.)

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Několik přístupů k syntéze regulátoru se šířkovou impulsní modulací užitím quasilinearizace

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Impulsní šířková modulace, jedno z možných spojení číslicového řídícího počítače vyčíslujícího proměnné definované v diskrétních časových okamžicích se soustavou s proměnnými definovanými na časovém kontinuu, je někdy užívána pro ekonomická či technická omezení akčních členů: pro velkou třídů problémů je impulsní šířková modulace užívána pro výhody implementace nebo přenosu signálů. Jsou předloženy tři numericky orientované přístupy k syntéze regulátoru se šířkovou impulsní modulací: diskrétní princip minima, dynamické a lineární programování — všechny užívající quasilinearizaci. Numerické výsledky jsou připojeny pro verifikaci. Výsledky ukazují na vhodnost užití suboptimálního regulátoru.

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