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Linear Optimal Control System with Incomplete Information about State of System

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Optimal control problem of linear systems (continuous and discrete) with respect to a quadratic performance criterion is discussed. The additional constraint of making the use of only the measuralbe state variables is imposed. This problem is solved using observer, dynamic controler or suboptimal proporcional feedback derived only from measurable state variables.

In the recent literature many articles can be found dealing with problems of optimal control of linear systems using performance criterion formed by the integral of quadratic form of state variables of the system and of control.

The reason for this interest lies in the fact that so formulated problem of optimal control results in linear feedback control derived from all state variables of the system and this is very convenient for realization. Moreover, this form of performance criterion has direct physical basis — demand for the optimal control with minimal deviation and limited control energy.

Practically it is often impossible to measure actual values of all state variables of the system. This paper is devoted to this problem and it is given a survey of possible approaches to the solution of it. The problem is solved both for continuous and discrete systems. Problem of optimal tracking of reference input is discussed and also that of optimal control in case of partially determined feedback.

1. INTRODUCTION

Be given a linear multiparameter and generally time variable continuous system described in the state space by the equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\,, \qquad \mathbf{y} = \mathbf{C}\mathbf{x} \tag{1}$$

where $\mathbf{x}$ is state vector of the system (dimension n),

$\mathbf{y}$ is output vector (dimension m),
$\mathbf{u}$ is controlled vector (dimension r),
$\mathbf{A}$ is matrix of the system (dimension $n \times n$),
$\mathbf{B}$ is matrix of the control (dimension $n \times r$),
$\mathbf{C}$ is matrix of the output (dimension $m \times n$).

Note. Vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$ and matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are generally functions of time. For simplicity we drop arguments of these functions.

The discrete system is described in the state space by difference equations

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}_k \mathbf{x}_k + \mathbf{B}_k \mathbf{u}_k\,, \\ \mathbf{y}_k &= \mathbf{C}_k \mathbf{x}_k \end{aligned} \tag{2}$$

where $\mathbf{x}_k$, $\mathbf{y}_k$, $\mathbf{u}_k$ are state vectors, output and input, respectively, at the k-th sampling instant,

$\mathbf{A}_k$, $\mathbf{B}_k$, $\mathbf{C}_k$ are matrices of system, control and output at the k-th sampling instant.

We have the so called incomplete information about the system state, i.e. not all n state variables, but only m outputs can be measured where $m \leqq n$, supposing that outputs are linear independent (matrix $\mathbf{C}$ is of rank m).

As it is known and briefly mentioned in section 2, the optimal control is formed by linear combination of all state variables. If not all state variables of the system are known, there are in substance two approaches to solution of this problem (apart from direct modelling of inaccesible state variables by integrating and differentiating circuits, what is sensible to random distrubances):

a) artifitially increasing order of the system by inserting suitable dynamic system and then using algoritms for optimal control with complete information about the system. Here belong the method of reconstruction of the state variables by means of the "observer" (this method is discussed in Section 3) and optimal control by "dynamic controller" (discussed in Section 4).

b) controlling the system in suboptimal way. We restrict our attention on the control derived as linear combination of only measurable state variables and from this set of controls we compute optimal control minimizing the functional modified according to this restriction, or to other suitable function, for instance to difference or ratio of suboptimal and optimal (in case of full information) value of performance criterion.

In this article we shall mostly discuss a special case when according to point b) we minimize (by proportional feedback) the expected value of the performance criterion computed for a set of n linear independent unit initial state vectors. The resulting feedback system is then optimal "in the average" for various initial states of the system. This problem is solved by means of the so called matrix minimum principle [2] where state vector is characterized by state transition matrix, independent on initial value of system state. Section 5 is devoted to the solution of this problem.

2. OPTIMAL CONTROL WITH COMPLETE INFORMATION ABOUT THE STATE OF THE SYSTEM

Without proofs we will give a survey of the results of optimal control in case of complete information about the system state.

Results mentioned here were used by assembling programs for digital computer for computation coefficients of optimal control feedback. Here the solution both for continuous and discrete systems is mentioned. Programs for digital computer were assembled only for stationary systems [19].

2.1. Continuous System – Control of the State

Continuous system is described by equation (1.1). Performence criterion has the form

$$J = \tfrac{1}{2}\mathbf{x}_T'\mathbf{S}\mathbf{x}_T + \tfrac{1}{2}\int_0^T (\mathbf{x}'\mathbf{Q}\mathbf{x} + \mathbf{u}'\mathbf{R}\mathbf{u})\,\mathrm{d}t \tag{1}$$

where $\mathbf{S}$ is constant positiv semidefinite symmetric matrix $(n \times n)$,

$\mathbf{Q}$ is symmetric positive semidefinite $(n \times n)$ matrix which elements are functions of time,
$\mathbf{R}$ is symmetric positive definite $(r \times r)$ matrix elements of which are also functions of time,
T is time of control,
$\mathbf{x}_T$, $\mathbf{u}_T$ are value of vectors at time T.

Optimal control is [1, 18]

$$\mathbf{u}^* = -\mathbf{R}^{-1}\mathbf{B}'\mathbf{K}\mathbf{x} \tag{2}$$

where $\mathbf{K}$ is symmetric positive definite $(n \times n)$ matrix, which is the solution of the Riccati equation

$$\dot{\mathbf{K}} + \mathbf{A}'\mathbf{K} + \mathbf{K}\mathbf{A} + \mathbf{Q} - \mathbf{K}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\mathbf{K} = \mathbf{O}\,; \quad \mathbf{K}_T = \mathbf{S}\,. \tag{3}$$

For $T \to \infty$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{R}$, $\mathbf{Q}$ = const and $\mathbf{S} = \mathbf{O}$ is the matrix

$$\mathbf{K} = \hat{\mathbf{K}} = \text{const} \tag{4}$$

and $\hat{\mathbf{K}}$ is the solution of nonlinear algebraic equation

$$\mathbf{K}\mathbf{A} + \mathbf{A}'\mathbf{K} + \mathbf{Q} - \mathbf{K}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\mathbf{K} = \mathbf{O}\,. \tag{5}$$

2.2. Continuous System – Output Control

Controlled system is also described by equation (1.1). Performance criterion is

$$J = \tfrac{1}{2}y_T' S y_T + \tfrac{1}{2}\int_0^T (y'Qy + u'Ru)\,dt\,. \tag{6}$$

Optimal control is also determined by linear feedback – equation (2). Matrix K is determined by (3) or (5) where instead of matrices S and Q we write matrices $C'SC$ and $C'QC$.

2.3. Continuous System – Tracking Problem

Continuous system is described by equation (1.1). Performance criterion has the form

$$J = \tfrac{1}{2}(z - y)_T' S(z - y) + \tfrac{1}{2}\int_0^T [(z - y)' Q(z - y) + u'Ru]\,dt \tag{7}$$

where z is vector of desired value, which is function of the time defined in the interval of control, $0 \leqq t \leqq T$.

Optimal control is [1]

$$u^* = R^{-1}B'(g - Kx) \tag{8}$$

where matrix K is again the solution of Riccati equation

$$\dot{K} = -A'K - KA - C'QC + KBR^{-1}B'K\,, \quad K_T = C'SC\,, \tag{9}$$

and vector g is solution of the equation

$$\dot{g} = -(A - BR^{-1}B'K)'\,g - C'Qz\,, \quad g_T = C'Sz_T\,. \tag{10}$$

Problem of desired value optimal tracking can be transformed into the problem of the state control provided that it is possible to express vector of desired value as solution of the differential equation [19, 8]

$$\dot{z} = Hz\,. \tag{11}$$

Let us define extended vector

$$w = \begin{bmatrix} x \\ \hdashline z \end{bmatrix}; \tag{12}$$

then the following relations are valid

$$\dot{w} = \hat{A}w + \hat{B}u\,, \tag{13}$$

$$y = \hat{C}w\,, \tag{14}$$

where matrices are

(15) $$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{H} \end{bmatrix}; \quad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ \mathbf{O} \end{bmatrix}; \quad \hat{\mathbf{C}} = [\mathbf{C} \; \mathbf{0}] .$$

Performance criterion suppose to be

(16) $$\hat{J} = \tfrac{1}{2}\mathbf{w}'_T\hat{\mathbf{S}}\mathbf{w}_T + \tfrac{1}{2}\int_0^T (\mathbf{w}'\hat{\mathbf{Q}}\mathbf{w} + \mathbf{u}'\hat{\mathbf{R}}\mathbf{u})\,\mathrm{d}t .$$

Comparing elements of submatrices with functional (7) we obtain relations for matrices $\hat{\mathbf{S}}$, $\hat{\mathbf{Q}}$, $\hat{\mathbf{R}}$:

(17) $$\hat{\mathbf{S}} = \begin{bmatrix} \mathbf{C'SC} & -\mathbf{C'S} \\ -\mathbf{SC} & \mathbf{S} \end{bmatrix}; \quad \hat{\mathbf{Q}} = \begin{bmatrix} \mathbf{C'QC} & -\mathbf{C'Q} \\ -\mathbf{QC} & \mathbf{Q} \end{bmatrix}; \quad \hat{\mathbf{R}} = \mathbf{R} .$$

Performance criterion (16) can be minimalized in an analogous way by relations (1)–(4). The order of matrix $\mathbf{K}$ increases then to $(n + m) \times (n + m)$.

Note. In special case matrix $\mathbf{S}$ can satisfy algebraic equation (5), i.e.

(18) $$\mathbf{SA} + \mathbf{A'S} + \mathbf{Q} - \mathbf{SBR}^{-1}\mathbf{B'S} = \mathbf{O}$$

holds for control of the state, or

(19) $$\mathbf{C'SCA} + \mathbf{A'C'SC} + \mathbf{Q} - \mathbf{C'SCBR}^{-1}\mathbf{B'C'SC} + \mathbf{C'QC} = \mathbf{O}$$

for output control. Then regarding a boundary condition for matrix $\mathbf{K}$, matrix $\mathbf{K}$ is constant $\mathbf{K} = \mathbf{S}$ even for $T \neq \infty$.

2.3. Discrete System – Control of State

We are looking for such sequence of control vectors $\mathbf{u}_0, \ldots, \mathbf{u}_{N-1}$ which for given discrete system minimalize performance criterion

(20) $$J = \mathbf{x}'_N\mathbf{S}\mathbf{x}_N + \sum_{k=0}^{N-1} (\mathbf{x}'_k\mathbf{Q}\mathbf{x}_k + \mathbf{u}'_k\mathbf{R}\mathbf{u}_k)\, h$$

where N is number of steps of control

h is the time interval of one step (time of one sampling periode),
$T = hN$ is time of control,
$\mathbf{S}$, $\mathbf{Q}$ are constant symmetric positive semidefinite matrices,
$\mathbf{R}$ is constant symmetric positive definite matrix.

Let us define cost function V_N as minimal value of functional (20) (for optimal control) in dependence on initial value x_0

(21) $$V_N(\mathbf{x}_0) = J^*(\mathbf{x}_0) = \min_{u_0,\dots,u_{N-1}} J\,;$$

then for special n where $1 \leqq n \leqq N$ the following equation is valid ($\mathbf{S} = \mathbf{O}$)

(22) $$V_n(\mathbf{x}_0) = \min_{u_0} \left[(\mathbf{x}_0'\mathbf{Q}\mathbf{x}_0 + \mathbf{u}_0'\mathbf{R}\mathbf{u}_0)\,\mathbf{h} + \mathbf{V}_{n-1}(\mathbf{x}_1)\right].$$

Setting

(23) $$V_n(\mathbf{x}_0) = \mathbf{x}_0'\mathbf{G}\mathbf{x}_0$$

then by minimalization of $V_n(\mathbf{x}_0)$ follows the control law

(24) $$\mathbf{u}_0 = -\mathbf{D}_n\mathbf{x}_0$$

where

(25) $$\mathbf{D}_n = [h\mathbf{R} + \mathbf{B}'\mathbf{G}_{n-1}\mathbf{B}]^{-1}\,\mathbf{B}\mathbf{G}_{n-1}\mathbf{A}$$

and for $\mathbf{G}_n$ is valid discrete form of Riccati equation

(26) $$\mathbf{G}_n = (\mathbf{A} + \mathbf{D}_n'\mathbf{R}\mathbf{D}_n)\,h + (\mathbf{A} - \mathbf{B}\mathbf{D}_n)'\,\mathbf{G}(\mathbf{A} - \mathbf{B}\mathbf{D}_n)$$

with boundary value for $n = 1$

$$\mathbf{D}_1 = \mathbf{O}\,; \quad \mathbf{G}_1 = \mathbf{Q}h\,.$$

For optimal output control relations (24)–(27) are also valid, only instead of matrices $\mathbf{S}$ and $\mathbf{Q}$ the matrices $\mathbf{C}'\mathbf{S}\mathbf{C}$ and $\mathbf{C}'\mathbf{Q}\mathbf{C}$ are to be written.

3. OPTIMAL CONTROL WITH INCOMPLETE INFORMATION ABOUT THE STATE OF THE SYSTEM BY MEANS OF SYSTEM STATE RECONSTRUCTION

It can be shown that n-dimensional state vectors of the system (1.1) and (1.2) can be obtained using knowledge of m-dimensional output $\mathbf{y}$, provided that system is completely observable, by means of linear dynamic system of dimension $(n - m)$ inserted in cascade. According to Luenberger this dynamic system is called observer [11, 12].

The input of the observer is both output $\mathbf{y}$ of the system and control $\mathbf{u}$ of the system. Output of the observer is n-dimensional vector $\hat{\mathbf{x}}$ which in ideal case equals to the state vector $\mathbf{x}$ of the system (see fig. 1).

Let us suppose that the state vector of the observer is $\mathbf{z}$, then state equations of the observer are

(1) $$\dot{\mathbf{z}} = \mathbf{D}\mathbf{z} + \mathbf{E}\mathbf{y} + \mathbf{G}\mathbf{u},$$

(2) $$\hat{\mathbf{x}} = \mathbf{H}\mathbf{y} + \mathbf{M}\mathbf{z}$$

where matrices **D**, **E**, **G**, **H**, **M** have dimension $(n - m) \times (n - m)$, $(n - m) \times m$, $(n - m) \times r$, $n \times m$, $n \times (n - m)$, respectively.

Fig. 1.

We require the output of the observer to be the same as the state vector of the system, i.e.

(3) $$\hat{\mathbf{x}} = \mathbf{x}.$$

The following theorem can be easily proved:

Theorem. *Let the original system be completely observable. Let for the system* (1) *and* (2)

(4) a) $$\mathbf{G} = \mathbf{T} . \mathbf{B},$$

(5) b) $$\mathbf{T}\mathbf{A} - \mathbf{D}\mathbf{T} = \mathbf{E}\mathbf{C}$$

hold where **T** *is matrix of the dimension* $(n - m) \times n$ *and of the rank* m; $\mathbf{T} =$ const.,

(6) c) $$\mathbf{z}_0 = \mathbf{T}\mathbf{x}_0,$$

(7) d) $$\lambda_i - \mu_j \neq 0$$

where λ_i *are eigenvalues of matrix* **A**, $i = 1, \ldots, n$,
μ_j *are eigenvalues of matrix* **D**, $j = 1, \ldots, (n - m)$.

From condition d) *it follows that there exists only one solution* **T** *of the equation* (5).

e) *There exists inverse matrix* $\begin{bmatrix} \mathbf{C} \\ \hdashline \mathbf{T} \end{bmatrix}^{-1}$ of the dimension $n \times n$.

Then

(8) $$\mathbf{z} = \mathbf{T}\mathbf{x},$$

(9) $$\mathbf{H}\mathbf{C} + \mathbf{M}\mathbf{T} = \mathbf{I}.$$

where **I** is unit matrix.

Algorithm for optimal control is then the same as in the problem with full information about state of system

(10) $$u^* = -R^{-1}B'K\hat{x} = -F_0\hat{x} = -F_0 \quad (Hy + Mz)\,.$$

The closed control system is then discribed by equation

(11) $$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A - BFH_0C & -BF_0M \\ EC - TBF_0HC & D - TBF_0M \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}$$

and its block diagram is shown in fig. 2.

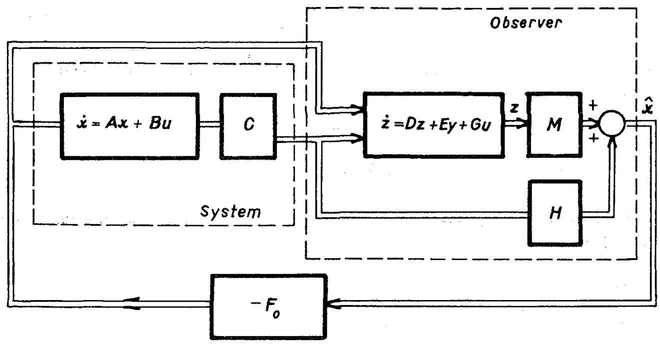

Fig. 2.

In the case of full information about the system state the optimal feedback system would be

(12) $$\dot{x} = (A - BF_0)\,x = A_0 x\,.$$

Even when observer itself is stable we are interested wheater it does not affect the stability of the system as a whole. It can be prooved [8] that eigenvalues of the system matrix (11) are all eigenvalues of the matrix A_0 or eigenvalues of the matrix D.

Stability of the observer is therefore sufficient condition for stability of the resulting system.

The observer design can be made for instance in such a way:

1. We choose matrices D and E. The absolute values of eigenvalues of matrix D are chosen as large as possible. The difference between vector z and Tx will then decreas rapidly.

2. From equation (5) we can compute matrix T.

3. From equation (4) compute matrix $\boldsymbol{G}$.

4. Compute inverse matrix $\begin{bmatrix} \boldsymbol{C} \\ \hdashline \boldsymbol{T} \end{bmatrix}^{-1}$.

5. By partioning of the matrix $\begin{bmatrix} \boldsymbol{C} \\ \hdashline \boldsymbol{T} \end{bmatrix}^{-1}$ matrices $\boldsymbol{M}$, $\boldsymbol{H}$ can be determined. From (9) follows

$$[\boldsymbol{H} \mid \boldsymbol{M}] = \begin{bmatrix} \boldsymbol{C} \\ \hdashline \boldsymbol{T} \end{bmatrix}^{-1}.$$

From condition (6) it is obvious that relation (8) must be valid also at initial instant. In most cases it cannot be fulfilled in praxis, because the initial state $\boldsymbol{x}_0$ of the system is in the best case determined only statistically (we are dealing with incomplete information about the system state).

So we are interested in absolute value of optimal performance criterion increase, defined by relation

$$V = J^* = \tfrac{1}{2} \int_0^\infty (\boldsymbol{x}'\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^{*\prime}\boldsymbol{R}\boldsymbol{u}^*)\, \mathrm{d}t \,. \tag{13}$$

Let us consider for simplicity an invariant system and infinite control time. For $\hat{\boldsymbol{x}} = \boldsymbol{x}$, V reaches the minimal value

$$V_1 = V_{\min} = \tfrac{1}{2}\hat{\boldsymbol{x}}_0'\boldsymbol{K}\hat{\boldsymbol{x}}_0 = \tfrac{1}{2}\boldsymbol{x}_0'\boldsymbol{K}\boldsymbol{x}_0 \,. \tag{14}$$

When relation (6) is not valid, then $\hat{\boldsymbol{x}} \neq \boldsymbol{x}$ and V increases its value to

$$V_2 = V_1 + \Delta V \,. \tag{15}$$

It can be proved that

$$\Delta V = \boldsymbol{\delta}'\boldsymbol{P}\boldsymbol{\delta} \tag{16}$$

where

$$\boldsymbol{\delta} = \boldsymbol{z}_0 - \boldsymbol{T}\boldsymbol{x}_0 \neq \boldsymbol{O} \tag{17}$$

is deviation vector of the initial state $\boldsymbol{z}_0$ from correct value. Matrix $\boldsymbol{P}$ is solution of the equation

$$\boldsymbol{D}'\boldsymbol{P} + \boldsymbol{P}\boldsymbol{D} = -\boldsymbol{M}'\boldsymbol{F}_0'\boldsymbol{R}\boldsymbol{F}_0\boldsymbol{M} \,. \tag{18}$$

4. OPTIMAL CONTROL WITH INCOMPLETE INFORMATION ABOUT STATE OF THE SYSTEM BY DYNAMIC CONTROLLER

In control praxis the information about the system state are mostly incomplete and so very often the proportional controller is not sufficient.

In case that we do not use observer and in spite of it we insist on the condition of strict optimality we can consider a dynamic controller [14].

Let us add in cascade to our system

$$(1) \qquad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$(2) \qquad \mathbf{y} = \mathbf{C}\mathbf{x}$$

the other dynamic system (dynamic controller) of the dimension rp, where p is not determined at present. The dynamic controller is described by equations

$$(3) \qquad \dot{\mathbf{z}} = \mathbf{D}\mathbf{z} + \mathbf{E}\mathbf{w}\,,$$

$$(4) \qquad \mathbf{u} = \mathbf{G}\mathbf{z}$$

where $\mathbf{z}$ is state vector of dynamic controller (dimension rp),
$\mathbf{w}$ is input vector of dynamic controller (dimension r),
$\mathbf{u}$ is output vector of dynamic controller (dimension r) and it is in the same time the input vector of the system,
$\mathbf{D}$ is matrix of controller (dimension $rp \times rp$),
$\mathbf{E}$ is $(rp \times r)$ matrix,
$\mathbf{G}$ is $(r \times rp)$ matrix.

Composed system consisting of controlled system and controller in cascade is described by equation

$$(5) \qquad \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{BG} \\ \mathbf{O} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} + \begin{bmatrix} \mathbf{O} \\ \mathbf{E} \end{bmatrix} \mathbf{w}$$

or

$$(6) \qquad \dot{\mathbf{v}} = \hat{\mathbf{A}}\mathbf{v} + \hat{\mathbf{B}}\mathbf{w}$$

Vector $\mathbf{w}$ is now control of overall system and $\mathbf{v}$ is state vector.

When the overall system (5) is controlable, then considering performance criterion

$$(7) \qquad \hat{J} = \tfrac{1}{2}(\mathbf{v}_T'\hat{\mathbf{S}}\mathbf{v}_T) + \tfrac{1}{2}\int_0^T (\mathbf{v}'\hat{\mathbf{Q}}\mathbf{v} + \mathbf{x}'\hat{\mathbf{R}}\mathbf{w})\,\mathrm{d}t$$

the optimal control is

$$(8) \qquad \mathbf{w} = -\mathbf{F}\mathbf{v} = -\mathbf{F}_0\mathbf{x} - \mathbf{F}_1\mathbf{z}$$

where

$$(9) \qquad \mathbf{F} = \hat{\mathbf{R}}^{-1}\mathbf{B}'\mathbf{K} = [\mathbf{F}_0 \,\vdots\, \mathbf{F}_1]$$

and $\mathbf{K}$ is solution of corresponding Riccati equation. Block diagram of the optimal system is in fig. 3.

For the present we still require complete information about the system state.

Establishing of dynamic controller enables us to transform the block diagram from fig. 3 in an other block diagram where there is sufficient to know only the output $\mathbf{y}$.

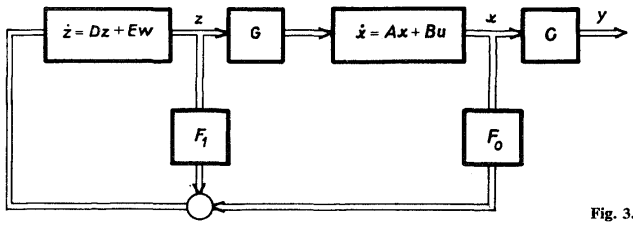

Fig. 3.

Desired transformation is then

$$\bar{\mathbf{z}} = \mathbf{z} + \mathbf{T}\mathbf{x} \tag{10}$$

where $\mathbf{T}$ is an $(rp \times n)$ matrix, elements of which are generally time functions. For invariant system and infinite control time $\mathbf{T}$ is a constant matrix.

Equations (3) and (4) have then the form

$$\dot{\bar{\mathbf{z}}} = (\mathbf{D} - \mathbf{E}\mathbf{F}_1)\,\bar{\mathbf{z}} + [\dot{\mathbf{T}} + \mathbf{T}\mathbf{A} - (\mathbf{D} - \mathbf{E}\mathbf{F}_1)\,\mathbf{T} - \mathbf{E}\mathbf{F}_0]\,\mathbf{x} + \mathbf{T}\mathbf{B}\mathbf{u}\,, \tag{11}$$

$$\mathbf{u} = \mathbf{G}\bar{\mathbf{z}} - \mathbf{G}\mathbf{T}\mathbf{x}\,. \tag{12}$$

We are looking for such a matrix $\mathbf{T}$ for which all terms containing $\mathbf{x}$ depend only on $\mathbf{y}$. Then relations

$$\dot{\mathbf{T}} + \mathbf{T}\mathbf{A} - (\mathbf{D} - \mathbf{E}\mathbf{F}_1)\,\mathbf{T} - \mathbf{E}\mathbf{F}_0 = \mathbf{M}\mathbf{C}\,, \tag{13}$$

$$\mathbf{G}\mathbf{T} = \mathbf{N}\mathbf{C} \tag{14}$$

must be valid. If there exist such $\mathbf{T}$, $\mathbf{M}$, $\mathbf{N}$ (in general functions of time) then we can write

$$\dot{\bar{\mathbf{z}}} = \bar{\mathbf{D}}\bar{\mathbf{z}} + \bar{\mathbf{M}}\mathbf{y}\,, \tag{15}$$

$$\mathbf{u} = \mathbf{G}\bar{\mathbf{z}} - \mathbf{N}\mathbf{y}\,, \tag{16}$$

where

$$\bar{\mathbf{D}} = \mathbf{D} - \mathbf{E}\mathbf{F}_1 + \mathbf{T}\mathbf{B}\mathbf{G}\,, \tag{17}$$

$$\bar{\mathbf{M}} = \mathbf{M} - \mathbf{T}\mathbf{B}\mathbf{N}\,. \tag{18}$$

Resulting transformed block diagram is in fig. 4.

The following theorem is valid:

Theorem. *Let* $\mathbf{A}$, $\mathbf{C}$ *and* $\mathbf{D} - \mathbf{EF}$, $\mathbf{G}$ *be completely observable. Let order of the system be rp where*

$$rp \geqq r(\nu - 1) \tag{19}$$

and ν *is observability index of the system* $\mathbf{A}$, $\mathbf{C}$.

Then there exist and can be found matrices $\mathbf{T}$, $\mathbf{M}$, $\mathbf{N}$ *satisfying equations* (13) *and* (14).

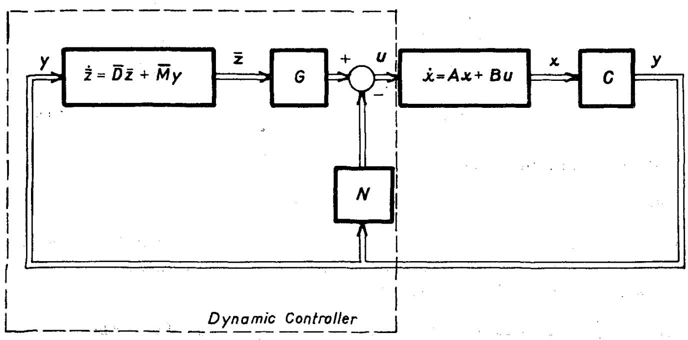

Fig. 4.

Procedure for design of the dynamic controller:

1. Choose matrices $\mathbf{D}$, $\mathbf{E}$, in the simplest possible form. Order of controller is chosen to $rp = r(\nu - 1)$. If we put for instance

$$\mathbf{z} = \begin{bmatrix} \mathbf{u} \\ \dot{\mathbf{u}} \\ \vdots \\ \mathbf{u}^{(p-1)} \end{bmatrix}; \quad \mathbf{w} = \mathbf{u}^{(p)} \tag{20}$$

then

$$\underset{(rp \times rp)}{\mathbf{D}} = \begin{bmatrix} \mathbf{O} & \mathbf{I} & \mathbf{O} & \dots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} & \dots & \mathbf{O} \\ \vdots & & & & \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{I} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} \end{bmatrix}; \quad \underset{(rp \times r)}{\mathbf{E}} = \begin{bmatrix} \mathbf{O} \\ \mathbf{O} \\ \vdots \\ \mathbf{O} \\ \mathbf{I} \end{bmatrix}; \tag{21}$$

$$\underset{(r \times rp)}{\mathbf{G}} = [\, \mathbf{I} \; \mathbf{O} \; \mathbf{O} \dots \mathbf{O} \,].$$

2. Solve Riccati equation for system (5) and performance criterion (7).
3. Compute matrix $\boldsymbol{F}$ and partition it by relation (9).
4. Compute matrices $\boldsymbol{T}$, $\boldsymbol{M}$, $\boldsymbol{N}$.

The problem has not unique solution; we look for the simplest form.

Methods of optimal control with incomplete information about the system state by means of state reconstruction by the observer or by means of dynamic controller result in the increase of order of the overall system. In the first case (observer) by $(n - m)$, in the second case (dynamic controller) by $r(v - 1)$ where

$$\frac{n}{m} - 1 \leqq v - 1 \leqq n - m . \tag{22}$$

If we do not consider other requirements, then from the point of view of minimal order of the overall system it is sometimes more conventient to use the observer, in other case the dynamic controller.

Consider the lowest (when using dynamic controller the most convenient) limit for v

$$v = \frac{n}{m}$$

and for n and r given (as an example), then we solve quadratic inequality for m

$$f_1(m) = r\left(\frac{n}{m} - 1\right) - (n - m) > 0 . \tag{23}$$

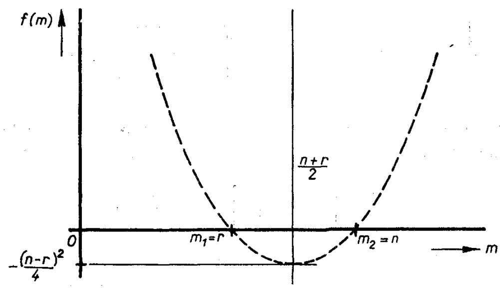

Fig. 5.

If $f_1(m) > 0$ then the order of the overall system with dynamic controller is greater then that of overall system with observer. By modification of the equation (23) we have

$$m f_1(m) = f(m) = m^2 - (n + r) m + rn > 0 . \tag{24}$$

 Graph of function $f(m)$ is in fig. 5.

From inequality (24) it follows:

a) for $m \in (m_1, m_2)$ (see fig. 5) the dynamic controller can be more convenient. Wheather it is actualy more convenient it depends on the index of observability ν.

b) for systems with one input and one output function $(m = r = 1)$ the order of the overall system is the same in both cases.

5. SUBOPTIMAL PROPORTIONAL CONTROLLER WITH RESPECT TO THE LEAST EXPECTED VALUE OF PERFORMANCE CRITERION

The suboptimal controller is always suitable when the complete set of n state variables is not available for feedback purposes and the use of an observer would be rather expensive. In such cases the so called "incomplete state feedback" or "output feedback", derived only from the measurable states, is introduced.

The suboptimal control law is then

$$u = -R^{-1}B'K \, . \, x = -F \, . \, y = -FC \, . \, x \tag{1}$$

where the matrix K is not a solution of the Riccati equation, but K has $n - m$ zero columns, corresponding to the unavailable $n - m$ state variables.

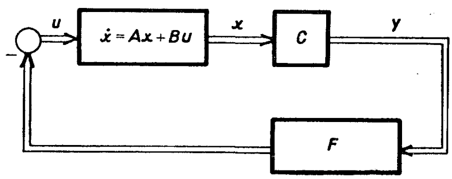

Fig. 6.

The necessary condition of the feedback system stability is that all the $n - m$ unavailable states must be stable. For example the 2-nd order plant with

$$A = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad C = [0 \ 1]$$

cannot be stabilized for any F, the unavailable state component x_1 beeing unstable.

The performance criterion depends both on the feedback matrix F and the initial state x_0. It will be shown that under certain conditions the dependence on x_0 can be eliminated. The whole problem will be transferred from the state vector space to the state transition matrix space, using the matrix minimum principle [2].

Let us consider the most general case – the tracking problem. The block diagram of the control loop is in fig. 6.

The performance criterion is

$$J = \tfrac{1}{2}\langle \mathbf{e}_T, \boldsymbol{S}\mathbf{e}_T\rangle + \tfrac{1}{2}\int_0^T [\langle \mathbf{e}, \boldsymbol{Q}\mathbf{e}\rangle + \langle \boldsymbol{u}, \boldsymbol{R}\boldsymbol{u}\rangle]\,\mathrm{d}t\,, \tag{2}$$

where $\mathbf{e}$ is the error, $\mathbf{e} = \mathbf{z} - \mathbf{y}$.

Substituting for $\boldsymbol{u}$, $\mathbf{e}$ into the performance criterion and rearranging gives

$$J = \tfrac{1}{2}(\mathbf{z}_T'\boldsymbol{S}\mathbf{z}_T - 2\boldsymbol{x}_T'\boldsymbol{C}'\boldsymbol{S}\mathbf{z}_T + \boldsymbol{x}_T'\boldsymbol{C}'\boldsymbol{S}\boldsymbol{C}\boldsymbol{x}_T) + \tag{3}$$
$$+ \tfrac{1}{2}\int_0^T (\mathbf{z}'\boldsymbol{Q}\mathbf{z} - 2\boldsymbol{x}'\boldsymbol{C}'\boldsymbol{Q}\mathbf{z} + \boldsymbol{x}'\boldsymbol{C}'\boldsymbol{Q}\boldsymbol{C}\boldsymbol{x} + \boldsymbol{x}'\boldsymbol{C}'\boldsymbol{F}'\boldsymbol{R}\boldsymbol{F}\boldsymbol{C}\boldsymbol{x})\,\mathrm{d}t\,.$$

The equation for the closed loop system is

$$\dot{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F}\boldsymbol{C})\,\boldsymbol{x}\,; \quad \boldsymbol{x}(0) = \boldsymbol{x}_0\,, \tag{4}$$

with the solution

$$\boldsymbol{x} = \boldsymbol{\Phi}\,.\,\boldsymbol{x}_0\,, \tag{5}$$

where $\boldsymbol{\Phi}(t, 0)$ is the state transition matrix for the system, satisfying the equation

$$\dot{\boldsymbol{\Phi}} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F}\boldsymbol{C})\,\boldsymbol{\Phi} \tag{6}$$

with the initial condition

$$\boldsymbol{\Phi}(0, 0) = \boldsymbol{I}\,. \tag{7}$$

Substituting (5) into (3) yields

$$J = \tfrac{1}{2}(\mathbf{z}_T'\boldsymbol{S}\mathbf{z}_T - 2\mathbf{x}_0'\boldsymbol{\Phi}_T'\boldsymbol{C}'\boldsymbol{S}\mathbf{z}_T + \mathbf{x}_0'\boldsymbol{\Phi}_T'\boldsymbol{C}'\boldsymbol{\Phi}_T\mathbf{x}_0) +$$
$$+ \tfrac{1}{2}\int_0^T (\mathbf{z}'\boldsymbol{Q}\mathbf{z} - 2\mathbf{x}\boldsymbol{\Phi}'\boldsymbol{C}'\boldsymbol{Q}\mathbf{z} + \mathbf{x}_0'\boldsymbol{\Phi}'\boldsymbol{Q}\boldsymbol{C}\boldsymbol{\Phi}\mathbf{x}_0 + \mathbf{x}_0'\boldsymbol{\Phi}'\boldsymbol{C}'\boldsymbol{F}'\boldsymbol{R}\boldsymbol{F}\boldsymbol{C}\boldsymbol{\Phi}\mathbf{x}_0)\,\mathrm{d}t\,.$$

Let us assume the initial state $\mathbf{x}_0$ to be a random variable uniformly distributed on the surface of the n-dimensional unit sphere.

The expected value of the performance criterion for the set of n initial state vectors $\mathbf{x}_{0i}$ $(i = 1, 2, \ldots, n)$ is then

$$\hat{J} = \frac{1}{2n}(n\mathbf{z}_T'\boldsymbol{S}\mathbf{z}_T - 2\sum_1^n \mathbf{x}_{0i}'\boldsymbol{\Phi}_T'\boldsymbol{C}_T'\boldsymbol{S}\mathbf{z}_T + \boldsymbol{\Phi}_T'\boldsymbol{C}_T'\boldsymbol{S}\boldsymbol{\Phi}_T\sum_1^n \mathbf{x}_{0i}, \mathbf{x}_{0i}') + \tag{8}$$
$$+ \tfrac{1}{2}\int_0^T (n\mathbf{z}'\boldsymbol{Q}\mathbf{z} + \boldsymbol{\Phi}'(\boldsymbol{C}'\boldsymbol{Q}\boldsymbol{C} + \boldsymbol{C}'\boldsymbol{F}'\boldsymbol{R}\boldsymbol{F}\boldsymbol{C})\,\boldsymbol{\Phi}\sum_1^n \mathbf{x}_{0i}\mathbf{x}_{0i}' + 2\mathbf{z}\sum_1^n \mathbf{x}_{0i}'\boldsymbol{\Phi}\boldsymbol{C}'\boldsymbol{Q})\,\mathrm{d}t\,.$$

With the help of the following relations

$$\boldsymbol{u}'\boldsymbol{v} = \langle \boldsymbol{u}, \boldsymbol{v}\rangle = \mathrm{tr}\,[\boldsymbol{u}\,.\,\boldsymbol{v}']\,; \quad \mathrm{tr}\,[\boldsymbol{A}] = \mathrm{tr}\,[\boldsymbol{A}']\,; \quad \mathrm{tr}\,[\boldsymbol{A}\boldsymbol{B}] = \mathrm{tr}\,[\boldsymbol{B}\boldsymbol{A}]\,;$$

 (8) can be re writen as

$$(9) \qquad \hat{J} = \frac{1}{2n}\,\mathrm{tr}\Big[n\mathbf{z}_T\mathbf{z}_T'\boldsymbol{S} - 2\mathbf{z}_T\sum_1^n \mathbf{x}_{0i}'\boldsymbol{\Phi}_T'\boldsymbol{C}_T'\boldsymbol{S} + \boldsymbol{\Phi}_T'\boldsymbol{C}_T'\boldsymbol{S}\boldsymbol{C}_T\boldsymbol{\Phi}_T\sum_1^n \mathbf{x}_{0i}\mathbf{x}_{0i}'\Big] +$$

$$+ \frac{1}{2n}\,\mathrm{tr}\left[\int_0^T n\mathbf{z}\mathbf{z}'\mathbf{Q} + \boldsymbol{\Phi}'(\boldsymbol{C}'\boldsymbol{Q}\boldsymbol{C} + \boldsymbol{C}'\boldsymbol{F}'\boldsymbol{R}\boldsymbol{F}\boldsymbol{C})\,\boldsymbol{\Phi}\sum_1^n \mathbf{x}_{0i}\mathbf{x}_{0i}' + 2\mathbf{z}\sum_1^n \mathbf{x}_{0i}'\boldsymbol{\Phi}'\boldsymbol{C}'\mathbf{Q}\right]\mathrm{d}t\,.$$

If vectors $\mathbf{x}_{0i}$ are chosen in the following manner

$$\begin{aligned} \mathbf{x}_{01} &= [1\ 0\ 0\ldots 0]'\,, \\ \mathbf{x}_{02} &= [0\ 1\ 0\ldots 0]'\,, \\ &\vdots \\ \mathbf{x}_{0n} &= [0\ 0\ 0\ldots 1]'\,, \end{aligned}$$

then

$$\sum_1^n \mathbf{x}_{0i}' = [1\ 1\ 1\ldots 1]\,; \quad \sum_1^n \mathbf{x}_{0i}\mathbf{x}_{0i}' = \boldsymbol{I}\,.$$

Let us define

$$(10) \qquad \bar{\mathbf{Z}} \equiv n\,.\,\mathbf{z}\mathbf{z}'$$

($m \times m$ symmetrical matrix) and

$$(11) \qquad \tilde{\mathbf{Z}} \equiv \mathbf{z}\,.\sum_1^n \mathbf{x}_{0i}' = [\mathbf{z} \mid \mathbf{z} \mid \mathbf{z} \mid \ldots \mid \mathbf{z}]$$

($m \times n$ matrix).

Both $\bar{\mathbf{Z}}$ and $\tilde{\mathbf{Z}}$ are functions of the reference output $\mathbf{z}$, thanks to our rather peculiar selection of $\mathbf{x}_{0i}$.

We can write, substituting (10) and (11) into (9),

$$(12) \qquad \hat{J} = \frac{1}{2n}\,\mathrm{tr}\,[\boldsymbol{\Phi}_T'\boldsymbol{C}_T'\boldsymbol{S}\boldsymbol{C}_T\boldsymbol{\Phi}_T + \bar{\mathbf{Z}}_T\boldsymbol{S} - 2\tilde{\mathbf{Z}}_T\boldsymbol{\Phi}_T'\boldsymbol{C}_T'\boldsymbol{S}] +$$

$$+ \frac{1}{2n}\int_0^T \mathrm{tr}\,[\boldsymbol{\Phi}'(\boldsymbol{C}'\boldsymbol{Q}\boldsymbol{C} + \boldsymbol{C}'\boldsymbol{F}'\boldsymbol{R}\boldsymbol{F}\boldsymbol{C})\,\boldsymbol{\Phi} + \bar{\mathbf{Z}}\mathbf{Q} - 2\tilde{\mathbf{Z}}\boldsymbol{\Phi}'\boldsymbol{C}'\mathbf{Q}]\,\mathrm{d}t\,.$$

The modified performance criterion $\hat{J}$ must be minimized by $\boldsymbol{F}$ subject to the constraint imposed by the system (6). The matrix minimum principle will be used.

The Hamiltonian is

$$(13) \qquad H = \tfrac{1}{2}\,\mathrm{tr}\,[\boldsymbol{\Phi}'(\boldsymbol{C}'\boldsymbol{Q}\boldsymbol{C} + \boldsymbol{C}'\boldsymbol{F}'\boldsymbol{R}\boldsymbol{F}\boldsymbol{C})\,\boldsymbol{\Phi} + \bar{\mathbf{Z}}\mathbf{Q} - 2\tilde{\mathbf{Z}}\boldsymbol{\Phi}'\boldsymbol{C}'\mathbf{Q}] +$$

$$+ \mathrm{tr}\,[(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F}\boldsymbol{C})\,\boldsymbol{\Phi}\boldsymbol{P}']\,,$$

where the constant $1/n$ has been dropped for simplicity.

The matrix $\boldsymbol{P}$ is the solution of the adjoint system

(14) $$\dot{\boldsymbol{P}} = -\frac{\partial H}{\partial \boldsymbol{\Phi}} = -(\boldsymbol{C}'\boldsymbol{Q}\boldsymbol{C} + \boldsymbol{C}'\boldsymbol{F}'\boldsymbol{R}\boldsymbol{F}\boldsymbol{C})\,\boldsymbol{\Phi} + 2\boldsymbol{C}'\boldsymbol{Q}\tilde{\boldsymbol{Z}} - (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F}\boldsymbol{C})'\,\boldsymbol{P}\,.$$

The transversality condition is

(15) $$\boldsymbol{P}_T = \frac{\partial}{\partial \boldsymbol{\Phi}_T}\{\tfrac{1}{2}\,\mathrm{tr}\,[\boldsymbol{\Phi}_T'\boldsymbol{C}_T'\boldsymbol{S}\boldsymbol{C}_T\boldsymbol{\Phi}_T + \boldsymbol{Z}_T\boldsymbol{S} - 2\tilde{\boldsymbol{Z}}_T\boldsymbol{\Phi}_T'\boldsymbol{C}_T'\boldsymbol{S}]\} = \boldsymbol{C}_T'\boldsymbol{S}\boldsymbol{C}_T\boldsymbol{\Phi}_T - \boldsymbol{C}_T'\boldsymbol{S}\tilde{\boldsymbol{Z}}_T\,.$$

The necessary condition of optimality is

$$\left.\frac{\partial H}{\partial \boldsymbol{F}}\right|_{*} = \boldsymbol{R}\boldsymbol{F}^*\boldsymbol{C}\boldsymbol{\Phi}^*\boldsymbol{\Phi}^{*\prime}\boldsymbol{C}' - \boldsymbol{B}'\boldsymbol{P}^*\boldsymbol{\Phi}^{*\prime}\boldsymbol{C}' = \boldsymbol{O}\,,$$

which yields

$$\boldsymbol{F}^* = \boldsymbol{R}^{-1}\boldsymbol{B}'\boldsymbol{P}^*\boldsymbol{\Phi}^{*\prime}\boldsymbol{C}'(\boldsymbol{C}\boldsymbol{\Phi}^*\boldsymbol{\Phi}^{*\prime}\boldsymbol{C}')^{-1}\,.$$

Suppose that $\boldsymbol{P}$ can be writen as

$$\boldsymbol{P}^* = \boldsymbol{K}^*\boldsymbol{\Phi}^* - \boldsymbol{G}^*\,,$$

where the matrix $\boldsymbol{G}$ is connected somehow with the reference output $\boldsymbol{z}$. Differentiating of $\boldsymbol{P}^*$ gives

(16) $$\dot{\boldsymbol{P}}^* = \dot{\boldsymbol{K}}^*\boldsymbol{\Phi}^* + \boldsymbol{K}^*\dot{\boldsymbol{\Phi}}^* - \dot{\boldsymbol{G}}^* = \dot{\boldsymbol{K}}^*\boldsymbol{\Phi}^* + \boldsymbol{K}^*(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F}^*\boldsymbol{C})\,\boldsymbol{\Phi}^* - \dot{\boldsymbol{G}}^*\,.$$

Equalizing the equations (14) and (16) we get the differential equation for $\boldsymbol{K}^*$

(17) $$\dot{\boldsymbol{K}}^* = -\boldsymbol{K}^*(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F}^*\boldsymbol{C}) - (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F}^*\boldsymbol{C})'\,\boldsymbol{K}^* - \boldsymbol{C}'\boldsymbol{Q}\boldsymbol{C} - \boldsymbol{C}'\boldsymbol{F}^{*\prime}\boldsymbol{R}\boldsymbol{F}^*\boldsymbol{C}$$

with the boundary condition

(18) $$\boldsymbol{K}_T^* = \boldsymbol{K}^*(T) = \boldsymbol{C}'\boldsymbol{S}\boldsymbol{C}$$

and the differential equation for $\boldsymbol{G}$

(19) $$\dot{\boldsymbol{G}}^* = -(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F}^*\boldsymbol{C})'\,\boldsymbol{G}^* - \boldsymbol{C}'\boldsymbol{Q}\tilde{\boldsymbol{Z}}$$

with the boundary condition

(20) $$\boldsymbol{G}_T^* = \boldsymbol{G}^*(T) = \boldsymbol{C}'\boldsymbol{Q}\tilde{\boldsymbol{Z}}_T\,.$$

The optimal output feedback gain is then

(21) $$\boldsymbol{F}^* = \boldsymbol{R}^{-1}\boldsymbol{B}'(\boldsymbol{K}^*\boldsymbol{\Phi}^* - \boldsymbol{G}^*)\,\boldsymbol{\Phi}^{*\prime}\boldsymbol{C}'(\boldsymbol{C}\boldsymbol{\Phi}^*\boldsymbol{\Phi}^{*\prime}\boldsymbol{C}')^{-1}\,.$$

The optimal feedback gain can thus be divided into two parts (see fig. 7); one part is identical with the feedback gain designed for the state regulator problem (see [4]), the second part is fully determined by $\mathbf{z}$.

In such cases where the initial value $\mathbf{z}_0$ of the vector $\mathbf{z}$ can be considered to be a random variable, too, it will be shown that the tracking problem can be treated as an ordinary state controller problem.

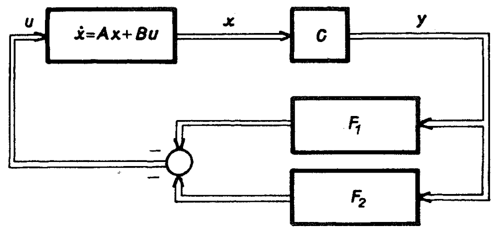

Fig. 7.

Vector $\mathbf{z}$ is supposed to be the solution of the differential equation

(22) $$\dot{\mathbf{z}} = \mathbf{H} . \mathbf{z}$$

with the initial condition

$$\mathbf{z}(0) = \mathbf{z}_0 .$$

Let us define the new state vector $\mathbf{w}$ as

(23) $$\mathbf{w} = \begin{bmatrix} \mathbf{x} \\ \hdashline \mathbf{z} \end{bmatrix} .$$

Equation for the original system and the equation (22) can be put together, using (23)

(24) $$\dot{\mathbf{w}} = \begin{bmatrix} \dot{\mathbf{x}} \\ \hdashline \dot{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hdashline \mathbf{z} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \hdashline \mathbf{O} \end{bmatrix} \mathbf{u} .$$

Let the output $\bar{\mathbf{y}}$ from the composed plant (24) be

$$\bar{\mathbf{y}} = \bar{\mathbf{C}} \mathbf{w} ,$$

where

$$\bar{\mathbf{C}} = [\mathbf{C} \mid -\mathbf{I}] .$$

It follows that

$$\bar{\mathbf{y}} = \mathbf{y} - \mathbf{z} = -\mathbf{e} .$$

The tracking problem for the original system is equivalent to the output regulator problem for the system (24), see fig. 8.

We have been considering the finite time problem with $T \neq \infty$ so far; the result is time-variant feedback. The case of infinite time $T = \infty$ was solved in [3] for state regulator problem. The optimal feedback matrix $\boldsymbol{F}$ is time-invariant (in case the controlled plant itself is time-invariant, of course) and is given by

$$F^* = R^{-1}B'K^*L^*C'(CL^*C')^{-1} \tag{25}$$

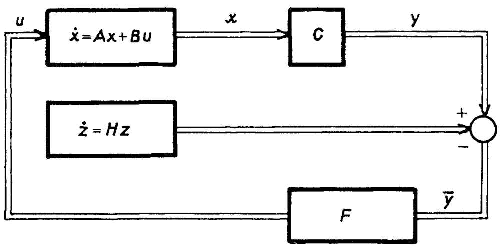

Fig. 8.

where K^*, L^* are positive semidefinite and positive definite solutions of

$$K^*A^* + A^{*\prime}K^* + Q + C'F^{*\prime}RF^*C = O \tag{26}$$

and

$$L^*A^{*\prime} + A^*L + I = O\,, \tag{27}$$

with

$$A^* = A - BF^*C\,. \tag{28}$$

There are very often additional constraints imposed on the feedback matrix $\boldsymbol{F}$. For instance, some of the matrix elements may be fixed and only the rest of them are adjustable. In that case the matrix $\boldsymbol{F}$ can be divided into two parts (submatrices): adjustable and non-adjustable.

There are two possibilities for the decomposition:

a) Let

$$F = [\overbrace{F_1}^{m_1} \;\vdots\; \overbrace{F_2}^{m-m_1}]\,.$$

In that case the matrix $\boldsymbol{C}$ may be decomposed in the following way:

$$C = \begin{bmatrix} C_1 \\ \cdots \\ C_2 \end{bmatrix} \begin{matrix} \}\, m_1 \\ \\ \}\, m - m_1\,. \end{matrix}$$

 We can write

$$u = -FC \,.\, x = -(F_1C_1 + F_2C_2) \,.\, x \,.$$

The Hamiltonian for the state controller problem is

$$H = \tfrac{1}{2}\,\mathrm{tr}\,\{\Phi'[Q + (C_1'F_1' + C_2'F_2')\,R(F_1C_1 + F_2C_2)]\,\Phi\} +$$
$$+ \mathrm{tr}\,\{[A - B(F_1C_1 + F_2C_2)]\,\Phi P'\} \,.$$

Suppose that F_1 is non-adjustable. With the same technique as applied at the beginning of this section it can easily be verified, that the optimal setting for F_2 is given by

(29) $$F_2 = (R^{-1}B'KLC_2' - F_1C_1)\,LC_2'(C_2LC_2')^{-1} \,.$$

On the other hand, if F_2 is constant and F_1 is to be adjusted, the optimal adjustment for F_1 is

(30) $$F_1 = (R^{-1}B'KLC_1' - F_2C_2)\,LC_1'(C_1LC_1')^{-1} \,.$$

The matrices K, L both in (29) and (30) are the solutions of (26) and (27).

This case can be solved much simplier: Suppose that $F_1 = \text{const}$; the problem is to find the optimal feedback matrix F_2 for the plant

$$\dot{x} = (A - BF_1C_1)\,x + Bu \,,$$
$$y = C_2x \,.$$

b) Let

$$F = \begin{bmatrix} F_1 \\ \hdashline F_2 \end{bmatrix} \begin{matrix} \}\, r_1 \\ \}\, r - r_1 \,. \end{matrix}$$

We shall decompose the matrix B in the following manner:

$$B = [\underbrace{B_1}_{r_1} \,\vdots\, \underbrace{B_2}_{r_2}] \,.$$

The overall feedback system is then described by

$$\dot{x} = (A - BFC)\,x = [A - (B_1F_1 + B_2F_2)\,C]\,x = (A - B_1F_1C)\,x - B_2F_2Cx \,.$$

Suppose that $F_1 = \text{const}$; the problem is to find the optimal feedback matrix F_2 for the plant given by

$$\dot{x} = (A - B_1F_1C)\,x + B_2u \,,$$
$$y = Cx \,.$$

7. THE DISCRETE SYSTEM WITH INCOMPLETE STATE INFORMATION. THE SUBOPTIMAL PROPORTIONAL CONTROLLER

We shall solve the state controller problem only. The extension of the method for the output controller or the tracking problem would be straightforward.

The linear discrete plant is given by

$$\mathbf{x}_{k+1} = \mathbf{A}_k\mathbf{x}_k + \mathbf{B}_k\mathbf{u}_k \,, \tag{1}$$

$$\mathbf{y}_k = \mathbf{C}_k\mathbf{x}_k \,, \tag{2}$$

the performance criterion to be minimized is

$$J_N = \sum_{k=0}^{N-1} (\mathbf{x}_k'\mathbf{Q}\mathbf{x}_k + \mathbf{u}_k'\mathbf{R}_k\mathbf{u}_k)\, h \,, \tag{3}$$

where N is the number of sampling intervals, h is the length of one sampling interval (sampling period).

The suboptimal control law $\mathbf{u}_k$ is expected to be of the form

$$\mathbf{u}_k = -\mathbf{F}_k\mathbf{C}_k\mathbf{x}_k = -\mathbf{F}_k\mathbf{y}_k \,,$$

where $\mathbf{F}_k$ is the unknown feedback matrix in the k-th sampling interval; $k = 1, 2, \ldots \ldots, N - 1$.

The functional (3) can be written as

$$J_N = \sum_{k=0}^{N-1} \mathbf{x}_k'(\mathbf{Q}_k + \mathbf{C}_k'\mathbf{F}_k'\mathbf{R}_k\mathbf{F}_k\mathbf{C}_k)\, \mathbf{x}_k h \,.$$

The state vector $\mathbf{x}_k$ can be expressed as

$$\mathbf{x}_k = \boldsymbol{\Phi}_k\mathbf{x}_0 \,,$$

where

$$\boldsymbol{\Phi}_k = \prod_{i=0}^{k-1} (\mathbf{A}_i - \mathbf{B}_i\mathbf{F}_i\mathbf{C}_i) \,; \quad \boldsymbol{\Phi}_{k+1} = (\mathbf{A}_k - \mathbf{B}_k\mathbf{F}_k\mathbf{C}_k)\, \boldsymbol{\Phi}_k \quad ; \quad \boldsymbol{\Phi}_0 = \mathbf{I}. \tag{4}$$

The constant h can be dropped for simplicity.

The expected value of the performance criterion (assuming the initial state $\mathbf{x}_0$ to be a random variable uniformly distributed on the unit sphere and using the relations previously developed in Section 5)

$$\hat{J}_N = \mathrm{tr}\left[\sum_{k=0}^{N-1} \boldsymbol{\Phi}_k'(\mathbf{Q}_k + \mathbf{C}_k'\mathbf{F}_k'\mathbf{R}_k\mathbf{F}_k\mathbf{C}_k)\, \boldsymbol{\Phi}_k\right]. \tag{5}$$

The problem of minimizing the functional (5) will be solved via dynamic programming. First, define the cost function

$$(6) \qquad V_N(\boldsymbol{\Phi}_0) = \hat{J}_N^* = \min_{F_0, F_1, \ldots F_{N-1}} \{\operatorname{tr} \sum_{k=0}^{N-1} \boldsymbol{\Phi}_k'(\mathbf{Q}_k + \mathbf{C}_k'\mathbf{F}_k'\mathbf{R}_k\mathbf{F}_k\mathbf{C}_k)\, \boldsymbol{\Phi}_k\},$$

which is the minimum value of the performance index with respect to all $\mathbf{F}_k$. Using the Bellman's principle of optimality gives

$$(7) \qquad V_N(\boldsymbol{\Phi}_0) = \min_{F_0} \{\operatorname{tr}[\boldsymbol{\Phi}_0'(\mathbf{Q}_0 + \mathbf{C}_0'\mathbf{F}_0'\mathbf{R}_0\mathbf{F}_0\mathbf{C}_0)\, \boldsymbol{\Phi}_0]\} + V_{N-1}(\boldsymbol{\Phi}_1).$$

(the Bellman's equation).

In Eq. (6) N means that there are N sampling intervals to the end of the control proces. Generally, in the $(N - n)$-th interval there are n intervals left. The corresponding value of the cost function is $V_n(\boldsymbol{\Phi}_{N-n})$.

Suppose $V_n(\boldsymbol{\Phi}_{N-n})$ can be written as

$$(8) \qquad V_n(\boldsymbol{\Phi}_{N-n}) = \operatorname{tr}[\boldsymbol{\Phi}_{N-n}'\mathbf{G}_n\boldsymbol{\Phi}_{N-n}],$$

where $\mathbf{G}_n$ is unknown positive definite matrix. It follows from the definition of the cost function that

$$\mathbf{G}_0 = \mathbf{O}.$$

Let the time invariant system be considered for the sake of simplicity, with

$$\mathbf{A}_k \equiv \mathbf{A}, \quad \mathbf{B}_k \equiv \mathbf{B}, \quad \mathbf{C}_k \equiv \mathbf{C}, \quad \mathbf{Q}_k \equiv \mathbf{Q}, \quad \mathbf{R}_k \equiv \mathbf{R}.$$

Using (4) we get

$$(9) \qquad V_{n-1}(\boldsymbol{\Phi}_{N-n+1}) = \operatorname{tr}[\boldsymbol{\Phi}_{N-n}'(\mathbf{A} - \mathbf{B}\mathbf{F}_{N-n}\mathbf{C})'\, \mathbf{G}_{n-1}(\mathbf{A} - \mathbf{B}\mathbf{F}_{N-n}\mathbf{C})\, \boldsymbol{\Phi}_{N-n}],$$

so that (8) can be written as

$$(10)$$
$$V_n(\boldsymbol{\Phi}_{N-n}) = \operatorname{tr}[\boldsymbol{\Phi}_{N-n}'\mathbf{G}_n\boldsymbol{\Phi}_{N-n}] = \min_{F_{N-n}} \{\operatorname{tr}[\boldsymbol{\Phi}_{N-n}(\mathbf{Q} + \mathbf{C}'\mathbf{F}_{N-n}'\mathbf{R}\mathbf{F}_{N-n}\mathbf{C})\, \boldsymbol{\Phi}_{N-n}] +$$
$$+ \operatorname{tr}[\boldsymbol{\Phi}_{N-n}'(\mathbf{A} - \mathbf{B}\mathbf{F}_{N-n}\mathbf{C})'\, \mathbf{G}_{n-1}(\mathbf{A} - \mathbf{B}\mathbf{F}_{N-n}\mathbf{C})\, \boldsymbol{\Phi}_{N-n}]\}.$$

The right side of (10) must be minimized by $\mathbf{F}_{N-n}$, therefore

$$(11) \qquad \frac{\partial}{\partial \mathbf{F}_{N-n}} \{\operatorname{tr}[\boldsymbol{\Phi}_{N-n}'\mathbf{G}_n\boldsymbol{\Phi}_{N-n}]\} = 2\mathbf{R}\mathbf{F}_{N-n}\mathbf{C}\boldsymbol{\Phi}_{N-n}\boldsymbol{\Phi}_{N-n}'\mathbf{C}' -$$
$$- 2\mathbf{B}'\mathbf{G}_{n-1}\mathbf{A}\boldsymbol{\Phi}_{N-n}\boldsymbol{\Phi}_{N-n}'\mathbf{C}' + 2\mathbf{B}'\mathbf{G}_{n-1}\mathbf{B}\mathbf{F}_{N-n}\mathbf{C}\boldsymbol{\Phi}_{N-n}\boldsymbol{\Phi}_{N-n}'\mathbf{C}' = 0.$$

Rearranging of (11) gives

$$(\mathbf{R} + \mathbf{B}'\mathbf{G}_{n-1}\mathbf{B})\, \mathbf{F}_{N-n}\mathbf{C} = \mathbf{B}'\mathbf{G}_{n-1}\mathbf{A}.$$

Therefore

$$(12) \qquad \mathbf{F}_{N-n} = (\mathbf{R} + \mathbf{B}'\mathbf{G}_{n-1}\mathbf{B})^{-1}\,\mathbf{B}'\mathbf{G}_{n-1}\mathbf{A}\mathbf{C}^{-1}\,.$$

It follows from (10) that

$$(13) \qquad \mathbf{G}_n = \mathbf{Q} + \mathbf{C}'\mathbf{F}'_{N-n}\mathbf{R}\mathbf{F}_{N-n}\mathbf{C} + (\mathbf{A} - \mathbf{B}\mathbf{F}_{N-n}\mathbf{C})'\,\mathbf{G}_{n-1}(\mathbf{A} - \mathbf{B}\mathbf{F}_{N-n}\mathbf{C})\,.$$

The relations for $n = 1$ must be derived separately. The corresponding value of the feedback matrix $\mathbf{F}_{N-1}$ must minimize – see (7)

$$\operatorname{tr}\left[\boldsymbol{\Phi}_{N-1}(\mathbf{Q} + \mathbf{C}'\mathbf{F}'_{N-n}\mathbf{R}\mathbf{F}_{N-1}\mathbf{C})\,\boldsymbol{\Phi}_{N-1}\right].$$

Therefore

$$(14) \qquad \mathbf{F}_{N-1} = \mathbf{O}\,.$$

From (13) it follows

$$(15) \qquad \mathbf{G}_1 = \mathbf{Q}\,.$$

7. CONCLUSION

In this paper the problem of optimal control of linear systems with respect to a quadratic performance criterion is discussed. The additional constraint of making the use of only the measurable state variables is imposed. Generally, the demand for the optimality results in the increase of the order of the system, because of the dynamics of the optimal feedback controller. In Section 3 the Luenberger's observer is applied. The second possible technique, the dynamic controller, is discussed in Section 4. Both methods are then compared as to the order of the overall system. Unfortunately, there does not exist the unique solution, a certain degree of freedom in how to choose some parameters still remains.

However, the main part of the paper is contained in Sections 5 and 6, where the synthesis of the suboptimal proportional (without dynamics) controller is presented. Such controller is "optimal" with respect to the expected value of the performance criterion, evaluated for the set of n rather speculatively selected vectors of initial states. This is very close to the assumption that the initial state $\mathbf{x}_0$ is a random variable uniformly distributed on the surface of the n-dimensional unit sphere. The problem has always unique solution. For obtaining the results, the matrix minimum principle has been used. In the case of time-invariant systems the solution can be obtained directly without minimum principle. This is shown in [3], where the authors have presented also the method for numerical computation in which solving nonlinear algebraic equations in each step of iteration is necessary. The authors of this paper are developing the new algorithm where only linear algebraic equations would be solved.

In the case of discrete systems the developed relations can be used directly for computation.

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Lineární optimální regulační obvody s neúplnou informací o stavu systému

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Práce se zabývá problémem optimálního řízení lineárního dynamického systému při kvadratickém kritériu jakosti regulace a při neúplné informaci o stavu systému. Požadavek optimálnosti vede v tomto případě ke zvýšení řádu výsledného systému, což je způsobeno dynamickým charakterem optimální zpětné vazby. Sem patří metoda pozorovatele a dynamického regulátoru. Hlavní pozornost je nicméně věnována syntéze suboptimálního proporcionálního regulátoru, který minimalizuje střední hodnotu kritéria jakosti regulace, za jistých předpokladů o statistickém rozložení vektoru počátečního stavu $\mathbf{x}_0$. Problém je tak převeden z prostoru stavových vektorů do oblasti stavových matic (matic přechodových funkcí, které nyní představují stav systému nezávisle na stavu počátečním). Při řešení bylo použito maticového principu minima, pro diskrétní systémy metody dynamického programování.

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