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On the Weak Borel Property of Methods of Summation

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For summation methods, the connections of the Toeplitz regularity conditions and the weak Borel property (a special case of the weak law of large numbers) are investigated.

1. INTRODUCTION

Let $j = 0, \pm 1, \pm 2, \ldots$ Let in this introduction $\{x_j\}$ be a stationary white sequence and $\{\zeta_j\}$ a sequence defined by

$$\zeta_j = x_j - x_{j-1} . \tag{1}$$

Let $m \geqq 1$ be a natural number. The relation

$$x_j^* = a_{m0}\zeta_j + a_{m1}\zeta_{j-1} + \ldots + a_{mm}\zeta_{j-m} \tag{2}$$

will be called inversion of (1), if the a_{mn} $(n = 0, 1, \ldots, m)$ fulfill the postulate

$$\mathbf{E}\{(x_j - x_j^*)^2\} = \min , \tag{3}$$

where $\mathbf{E}$ denotes the mean value.

For $m \to \infty$, a sequence of relations (2) results, representing in essence the Cesàro $\mathscr{C}_1$ summing method, as Frisch (see the references in [1]) has been shown.

Denoting

$$\begin{aligned} t_{m0} &= 1 - a_{m0} , \\ t_{mn} &= a_{m(n-1)} - a_{mn} \end{aligned} \tag{4}$$

for

$$n = 1, 2, \ldots , \quad m + 1 ,$$

the problem can be reformulated [1] to find the t_{mn} so that

$$(5) \qquad \sum_{n=0}^{m+1} t_{mn}^2 = \min$$

with the supplementary condition

$$(6) \qquad \sum_{n=0}^{m+1} t_{mn} = 1 .$$

The t_{mn} $(n = 0, 1, \ldots, m + 1)$ of the solution represent the m-th row of the transform matrix of $\mathscr{C}_1$.

Relaxing (5) to

$$(7) \qquad \lim_{m \to \infty} \sum_{n=0}^{m+1} t_{mn}^2 = 0$$

and, eventually, (6) to

$$(8) \qquad \lim_{m \to \infty} \sum_{n=0}^{m+1} t_{mn} = 1 ,$$

other regular summing methods can be found fulfilling (7), (8), thus representing approximate inversion of (1).

Note that a transform $\mathscr{T}$ with the matrix

$$(9) \qquad \mathbf{T} = \begin{pmatrix} t_{10}, t_{11}, \ldots \\ t_{20}, t_{21}, \ldots \\ \ldots \end{pmatrix}$$

is regular if and only if the known Toeplitz conditions are fulfilled:

$$(10)\ \text{a)} \qquad \sum_{n=0}^{\infty} |t_{mn}| < K < \infty$$

where K is independent on m,

$$\text{b)} \qquad \lim_{m \to \infty} t_{mn} = 0$$

for every n,

$$\text{c)} \qquad \lim_{m \to \infty} \sum_{n=0}^{\infty} t_{mn} = 1 .$$

Hill, Garreau and Lorentz (see the references in [1], [2]) have been shown the connection of the so-called Borel property of $\mathscr{T}$ with (7). Regular $\mathscr{T}$ have been concerned although some sort of independence of regularity and the Borel property has been shown on examples. In what follows we will investigate the problem in some detail for the so-called weak Borel property.

With respect to (5), (6), only row-finite matrices (9) will be concerned (that is such containing in each row only a finite number of terms different from zero) in all cases in which complications could arise with concerning infinite sums of random variables.

2. PROBLEM FORMULATION

Let $\{x_n\}$ be now a sequence of 0's and 1's with infinite number of 1's. We connect the binary number $0, x_1, x_2, \ldots$ in the interval $(0, 1\rangle$ with $\{x_n\}$. Introducing the usual Lebesgue measure μ on $\langle 0, 1\rangle$, one says that $\mathcal{T}$ possess the weak Borel property if for arbitrary $\varepsilon > 0$

$$\lim_{m\to\infty} \mu\{|\sum_{n=0}^{n_m} t_{mn}x_n - \tfrac{1}{2}| < \varepsilon\} = 1\,. \tag{11}$$

We remark that the introduced measure is identical with the probability on the space of sequences $\{x_n\}$ of independent random variables, each of which can assume only the values 0, 1 with the probabilities $\frac{1}{2}$, $\frac{1}{2}$.

Thus the weak Borel property is the usual convergence in probability for sequences of linear combinations formed from such independent random variables according to (11).

In what follows, the known substitution

$$\xi_n = 2x_n - 1\,, \qquad x_n = \tfrac{1}{2}\xi_n + \tfrac{1}{2} \tag{12}$$

transforming the sequences $\{x_n\}$ of 0's and 1's in the sequences $\{\xi_n\}$ of -1's and $+1$'s, will be used.

3. SUMMING TRANSFORMS WITH A SPECIAL PROPERTY

Define for a transform $\mathcal{T}$ (eventually non-regular and row-infinite)

$$\tau_m = \max_n |t_{mn}| \tag{13}$$

if this maximum exists for every m.

Lemma 1. *Let for a transform $\mathcal{T}$ (10) a) hold (thus (13) is defined) and let*

$$\lim_{m\to\infty} \tau_m = 0\,. \tag{14}$$

Then

$$\lim_{m\to\infty} \sum_{n=0}^{\infty} t_{mn}^2 = 0\,. \tag{15}$$

Proof. From (10) a) and (13) one has

$$\sum_{n=0}^{\infty} t_{mn}^2 \leqq \tau_m \sum_{n=0}^{\infty} |t_{mn}| < \tau_m K\,, \tag{16}$$

from which the lemma is obvious.

Lemma 2. *For a transform $\mathcal{T}$, (14) follows from (15).*

Proof. If (14) is not valid, then there exists a $\delta > 0$ such that for a subsequence $\{m_j\}$ there is $\tau_{m_j} > \delta$ so that (15) cannot hold. This is a contradiction.

Corollary 1. *If $\mathcal{T}$ satisfies (10) a), then (14) and (15) are equivalent.*

Proof. Obvious from both preceding lemmas.
We remark that (14) is stronger than (10) b).

4. REGULARITY CONDITIONS AND THE WEAK BOREL PROPERTY

Theorem 1. *Let $\{\eta_m\}$ be a sequence of random variables. A necessary and sufficient condition for*

$$\lim_{m\to\infty} p\{|\eta_m| < \varepsilon\} = 1 \tag{17}$$

to hold for arbitrary $\varepsilon > 0$ is

$$\lim_{m\to\infty} \mathbf{E}\, \frac{\eta_m^2}{1 + \eta_m^2} = 0\,. \tag{18}$$

Proof. See [3], p. 169.

Let now $\{\xi_n\}$ be a sequence of independent random variables each of which can assume only the value $+1$, -1 with the probabilities $\frac{1}{2}$, $\frac{1}{2}$.

Let $\mathcal{T}$ be a transform with row-finite matrix and define

$$\eta_m = t_{m0}\xi_0 + \ldots + t_{mn_m}\xi_{n_m}\,. \tag{19}$$

Then

$$\mathbf{E}(\eta_m) = 0\,, \tag{20}$$

$$\mathbf{E}(\eta_m^2) = \sum_{n=0}^{n_m} t_{mn}^2\,. \tag{21}$$

Lemma 3. (15) *is a sufficient condition for* (17).

Proof. Since

$$\frac{\eta_m^2}{1+\eta_m^2} \leqq \eta_m^2, \tag{22}$$

there is

$$\mathrm{E}\frac{\eta_m^2}{1+\eta_m^2} \leqq \mathrm{E}(\eta_m^2) \tag{23}$$

and the lemma follows from (21), (23) and theorem 1.

Lemma 4. (17) *and* (10) a) *are sufficient conditions for* (15).

Proof. Since

$$|\eta_m| \leqq \sum_{n=0}^{n_m} |t_{mn}|, \tag{24}$$

there is with (10) a)

$$\frac{\eta_m^2}{1+\eta_m^2} \geqq \frac{\eta_m^2}{1+K^2}, \tag{25}$$

thus

$$\mathrm{E}(\eta_m^2) \leqq (1+K^2)\,\mathrm{E}\frac{\eta_m^2}{1+\eta_m^2} \tag{26}$$

and the lemma follows from (21), (26) and theorem 1.

Introducing now the substitution (12), one sees from (19) that (17) is identical with

$$\lim_{m\to\infty} p\{|\sum_{n=0}^{n_m} t_{mn}x_n - \tfrac{1}{2}\sum_{n=0}^{n_m} t_{mn}| < \varepsilon\} = 1 \tag{27}$$

for arbitrary $\varepsilon > 0$.

Lemma 5. (10) c) *is a necessary condition for the transform $\mathscr{T}$ to possess the weak Borel property.*

Proof. The lemma follows from (11) and (27).

Lemma 6. (15) *and* (10) c) *are sufficient conditions for $\mathscr{T}$ to possess the weak Borel property.*

Proof. The lemma follows from (27) and lemma 3.

Lemma 7. *For a transform $\mathscr{T}$ the weak Borel property and* (10) a) *are sufficient conditions for* (15) *and* (10) c).

Proof. The lemma follows from (11), (27), (17) and lemma 4.

Theorem 2. *Let $\mathscr{T}$ possess the property* (10) a). *Then* (15) *and* (10) c) (*or* (14) *and* (10) c)) *are necessary and sufficient $\mathscr{T}$ to possess the weak Borel property.*

Proof. The theorem follows from lemma 6, lemma 7 and corollary 1.

It is now clear that since (14) and (15) are equivalent in this case and (14) is stronger than (10) b), a transform $\mathscr{T}$ satisfying the property (10) a) and the weak Borel property is necessarily regular.

Further, it is clear that supposing in advance $\mathscr{T}$ regular is not a too severe restriction with respect to theorem 2. For transforms with nonnegative t_{mn}, (10) a) follows from (10) c).

Theorem 3. *Let $\mathscr{T}$ be regular. Then* (14) *is necessary and sufficient for $\mathscr{T}$ to possess the weak Borel property.*

Proof. Follows immediately from theorem 2.

Theorem 3 has been found by Lorentz.

5. MORE GENERAL CONDITIONS FOR THE WEAK BOREL PROPERTY

With respect to (7) and (8), we will be always interested in the properties (15) and (10) c) in connection with the weak Borel property. By lemma 2, (14) and (10) b) follow from (15), by lemma 6, the weak Borel property follows from (15) and (10) c). Thus the weak Borel property and (14) follow from (15) and (10) c).

Thus, generalizing theorem 2, the property (10) a) must be replaced by a weaker one.

Let us consider summing transforms $\mathscr{T}$ with row-finite matrices and satisfying

$$\sum_{n=0}^{n_m} t_{mn}^2 < C < \infty\,, \tag{28}$$

where C is independent on m.

This condition is not too restrictive, since it is fulfilled if (15) holds. But now, (15) follows no more from (14) and (28) in contrast to the situation in lemma 1.

Theorem 4. *Let $\mathscr{T}$ be a transform satisfying* (28). *Then* (15) *and* (10) c) *are necessary and sufficient for the weak Borel property and* (14).

Proof. The sufficiency follows from lemma 2 and lemma 6. The necessity of (10) c) follows from lemma 5. There remains to prove the necessity of (15).

To this end, one may use theorems on limiting distributions of sequences of sums of independent random variables (see [3], pp. 232–236).

Denoting in (19)

$$t_{mn}\xi_n = \zeta_{mn}\,, \quad (n = 0, \ldots, n_m) \tag{29}$$

one sees that (28) and (14) are the conditions of "elementariness" of the system $\{\zeta_{mn}\}$. For the sums (19), theorem 9 of [3], p. 236 is applicable. Denoting $F_{mn}(x)$ the distribution function of ζ_{mn} from (29), one sees from (20), (21) that

$$\sum_{n=0}^{n_m} \int x \, \mathrm{d}F_{mn}(x) = 0 \quad \text{for every } m \,, \tag{30}$$

$$\sum_{n=0}^{n_m} \int x^2 \, \mathrm{d}F_{mn}(x) = \sum_{n=0}^{n_m} t_{mn}^2 \,. \tag{31}$$

Let us now suppose for a moment that (14) and the weak Borel property hold and (15) is not satisfied. From (28) follows that in this case

$$\overline{\lim_{m\to\infty}} \sum_{n=0}^{n_m} t_{mn}^2 = t^2 > 0 \tag{32}$$

exists and one may choose a subsequence $\{m_j\}$ such that

$$\lim_{m_j\to\infty} \sum_{n=0}^{n_{m_j}} t_{m_j n}^2 = t^2 \,. \tag{33}$$

For this subsequence, (31) has thus the limit t^2.

Further, since the distribution function of ζ_{mn} is constant for every x with exception of $x = \pm t_{mn}$, where it has two jumps of the heights $\frac{1}{2}$, it follows from (14) that

$$\lim_{m_j\to\infty} \sum_{n=0}^{n_{m_j}} \int_{-\infty}^{u} x^2 \, \mathrm{d}F_{m_j n}(x) \begin{array}{ll} = 0 & \text{for} \quad u < 0 \,, \\ = t^2 & \text{for} \quad u > 0 \,. \end{array} \tag{34}$$

Now, (30) is identical with 3) of the cited theorem, (31) with 2) and (34) with 1). There follows that the distributions of η_{m_j} tend to the normal distribution with mean 0 and variance $t^2 > 0$. Thus (17), (27) and the weak Borel property cannot be satisfied. This is a contradiction and theorem 4 is proved.

6. CONCLUDING REMARKS

From the proof of theorem 4, somewhat more concrete statement is obvious, that is, the distributions of the expressions (19), standardized by the factors

$$k_m = 1/\sqrt{\left(\sum_{n=0}^{n_m} t_{mn}^2\right)} \tag{35}$$

tend to the normal distribution with mean 0 and variance 1.

Theorem 2 can be proved easily also for $\mathcal{T}$ with row-infinite matrices. With (10) a), η_m in (19) is defined as a random variable with probability 1 also for $n_m = \infty$. In proving the necessity of (15) in theorem 4, the cited theorem from [3] should be slightly modified and this is out of the scope of the present article.

Further, since the (strong) Borel property is identical with the strong law of large numbers, it would be interesting to compare the sufficient conditions of Hill and Lorentz with the known conditions of Kolmogorov.

From the practical viewpoint one is interested in the transform properties for a given m (possibly not too large). The efficiency of a transform can be measured by comparing the left side of (3) for given m with the minimum attained for $\mathscr{C}_1$. The property (6) is the known property of unbiasedness.

Transforms with row-infinite matrices are not out of interest, since they may be eventually easily realized with the aid of feedback, as is the case of the Abel transform (see [1]).

In all cases, it seems to be clear from the preceding sections that transforms regular or only slightly more general are practically useful in constructing digital filters.

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VÝTAH

O slabé Borelově vlastnosti sumačních metod

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V článku se vyšetřuje souvislost Toeplitzových podmínek regularity a slabé Borelovy vlastnosti (speciálního případu slabého zákona velkých čísel) sumačních transformací.

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