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On Directable Automata

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The paper is concerned with the shortest directing word estimates for non-initial Medvedev automata.

Let $\mathscr{A} = (A, X, \delta)$ be a Medvedev automaton with the set of states A, the set of input signals X and the transition function δ.

δ maps the set $A \times X^*$ into the set A (where X^* is the set of all words over X). The definition of δ can be extended to the set $A' = 2^A$ of all subsets of A. This extended mapping we designate δ', that means

$$\delta'(B, p) = \{\delta(b, p) : b \in B\} .$$

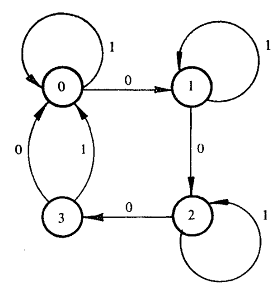

Fig. 1.

If $C = \delta'(B, p)$, we shall sometimes use the brief designation $B \underset{p}{\rightarrow} C$. For every $B \in 2^A$, $|B|$ will designate the number of elements in B. Putting $a'_0 = A$ and using δ' and A', we can define the total initial automaton $\mathscr{A}' = (A', X, \delta', a'_0)$ corresponding to $\mathscr{A}$.

Example 1. Let $\mathscr{U}_4 = (\{0; 1; 2; 3\}; \{0; 1\}, \delta)$ where δ is defined on the fig. 1. The multigraph of the corresponding $\mathscr{U}_4'$ is on the fig. 2.

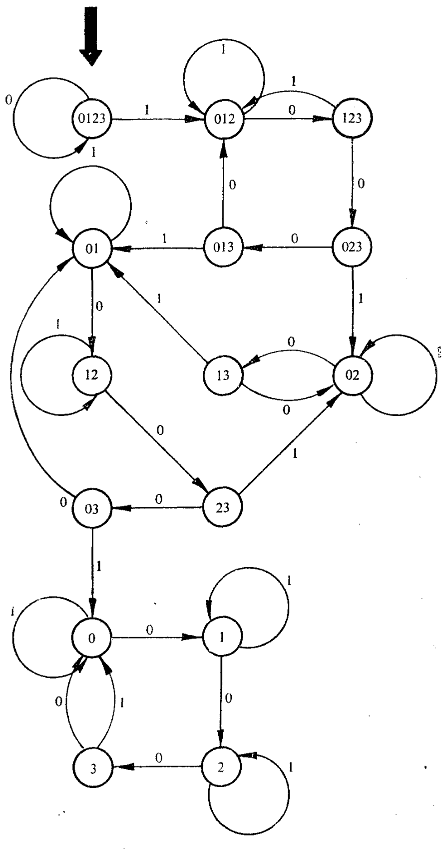

Fig. 2.

If there exists a word $p \in X^*$ and a state $a \in A$ such that $A \xrightarrow[p]{} \{a\}$, we shall call $\mathscr{A}$ directable and p the directing word of $\mathscr{A}$.

If $\mathscr{A}$ is directable then there exists a path from A to $\{a\}$ on the multigraph of $\mathscr{A}'$. Let us designate $l(p)$ the length of the word p and put

$$n(\mathscr{A}) = \min l(p)$$

where the minimum is taken over the set of all directing words of $\mathscr{A}$.

In example 1 $n(\mathscr{U}_4) = 9$ and the shortest directing word is 100010001.

Let $\boldsymbol{\Pi}_k$ be the set of all directable automata with k states. Let

$$n(k) = \sup_{\mathscr{A} \in \boldsymbol{\Pi}_k} n(\mathscr{A}) .$$

In [1] it was proved that

$$(1) \qquad (k - 1)^2 \leqq n(k) \leqq 2^k - k - 1 \; ; \quad k = 1, 2, \ldots$$

In [2] the following inequality was found:

$$(2) \qquad n(k) \leqq 1 + \tfrac{1}{2}k(k - 1)(k - 2)$$

which is better than the abovementioned one for $k \geqq 7$.

We see that the upper and lower estimates of $n(k)$ in (1) are equal for $k = 1, 2, 3$, but their difference is an increasing function of k for $k \geqq 4$ (see table 1).

Table 1.

k	1	2	3	4	5	6
$(k - 1)^2$	0	1	4	9	16	25
$2^k - k - 1$	0	1	4	11	26	57
$3 . 2^{k-2} - 2$		1	4	10	22	46
$(k/3) - (3k/2) + (25k/6) - 4$		1	4	10	21	39
$n(k)$ in Theorem 2				9		
$n(k)$ in Theorem 3					16	

If $\mathscr{A} = (A, X, \delta) \in \boldsymbol{\Pi}_k$ we can choose a directing word $x_1, \ldots, x_n$ such that in the sequence $\{\delta'(A, x_1 \ldots x_j)\}$ we have 1. no two identical terms, 2. no couple of terms with the equal numbers of elements containing a state pair b, c such that $\delta(b, x) = \delta(c, x)$ for some x. Thus

$$n(k) \leqq 1 + \sum_{j=2}^{k-1} \left[\binom{k}{j} - \binom{k-2}{j-2} + 1 \right] = 3 . 2^{k-2} - 2 .$$

Improving the Starke's method from [2] we can obtain the following theorem.

Theorem 1. *For every integer $K \geqq 2$ the following inequality holds:*

$$n(k) \leqq \tfrac{1}{3}k^3 - \tfrac{3}{2}k^2 + \tfrac{25}{6}k - 4 . \tag{3}$$

Proof. Let $\mathscr{A} = (A, X, \delta) \in \boldsymbol{\Pi}_k$. We are going to look for a directing word p of $\mathscr{A}$ such that

$$l(p) \leqq \tfrac{1}{3}k^3 - \tfrac{3}{2}k^2 + \tfrac{25}{6}k - 4 .$$

Let us suppose that $p = x_1 \dots x_{l(p)} = x_{h_{k-1}} x_2 \dots x_{h_{k-2}} \dots x_{h_1}$ where h_j has the property that

$$|\delta'(A, x_1 \dots x_{h_{j-1}})| \geqq j + 1 \,; \quad |\delta'(A, x_1 \dots x_{h_j})| \leqq j \quad (j = 1, \dots, k - 1) .$$

Note that the case of $h_j = h_{j-1}$ is possible. Then

$$l(p) = 1 + \sum_{j=2}^{k-1} (h_{j-1} - h_j) .$$

First we are going to prove that p can be chosen such that

$$h_{j-1} - h_j \leqq \binom{k}{2} - \binom{j}{2} - j + 3 \quad (j = 2, \dots, k - 1) \tag{4}$$

(the right hand side is obviously not less than 3).

Because $\mathscr{A}$ is directable, there exists such $x_1 \in X$ that $|\delta'(A, x_1)| = \bar{k} < k$. Then we put $x_1 = x_{h_{k-1}} = x_{h\bar{k}}$, $(h_{k-1} = \dots = h_{\bar{k}})$.

Let us assume that $x_1 \dots . x_{h_j}$ has been already chosen.

a) If $|\delta'(A, x_1 \dots x_{h_j})| < j$ then we put $h_{j-1} = h_j$ and

$$h_{j-1} - h_j = 0 < \binom{k}{2} - \binom{j}{2} - j + 3 .$$

b) If $|\delta'(A, x_1 \dots x_{h_j})| = j$ and if there exists such $x \in X$ that $|\delta'(A, x_1 \dots x_{h_j} x)| \leqq \leqq j - 1$ then we put $h_{j-1} = h_j + 1$ and $x_{h_{j-1}} = x$. Obviously

$$h_{j-1} - h_j = 1 < \binom{k}{2} - \binom{j}{2} - j + 3 .$$

c) If $|\delta'(A, x_1 \dots x_{h_j})| = j$ and no such $x \in X$ exists that $|\delta'(A, x_1 \dots x_{h_j} x)| \leqq j - 1$ then because of the directability of $\mathscr{A}$ there exists such $x \in X$ that

$$B_j = \delta'(A, x_1 \dots x_{h_j}) \neq C_j = \delta'(A, x_1 \dots x_{h_j} \bar{x}) .$$

In B_j there are $\binom{j}{2} = j(j - 1)/2$ different pairs of its elements and in C_j there are at least further $j-1$ pairs. Let $\{b, c\}$ be such a pair form these $\binom{j}{2}+j-1$ pairs which possesses the shortest word q with the property that $\delta'(\{b, c\}, q)$ is one point set.

Since the number of all pairs from A is $\binom{k}{2}$,

$$l(q) \leqq \binom{k}{2} - \binom{j}{2} - j + 2 .$$

If $b, c \in B_j$ then we put $x_{h_j+1} \dots x_{h_{j-1}} = q$. If $b, c \in C_j$ then $x_{h_j+1} \dots x_{h_{j-1}} = xq$. In both this cases

$$h_{j-1} - h_j \leqq \binom{k}{2} - \binom{j}{2} - j + 3 .$$

Thus we have found the word p by induction. Obviously p fulfils the condition (4). Then

$$l(p) = 1 + \sum_{j=2}^{k-1} \left[\binom{k}{2} - \binom{j}{2} - j + 3\right] = \tfrac{1}{3}k^3 - \tfrac{3}{2}k^2 + \tfrac{25}{6}k - 4$$

which concludes the proof.

In table 1 there are calculated the first values of

$$\tfrac{1}{3}k^3 - \tfrac{3}{2}k^2 + \tfrac{25}{6}k - 4 .$$

Corollary 1. *If an automaton $\mathscr{A} \in \Pi_k$ possesses two different pairs of states $\{a, b\}$ and $\{c, d\}$ such that*

$$|\delta'(\{a, b\}, x)| = |\delta'(\{c, d\}, y)| = 1$$

for some $x, y \in X$ then

$$n(\mathscr{A}) \leqq \tfrac{1}{3}k^3 - \tfrac{3}{2}k^2 + \tfrac{25}{6}k - 4 - (k-2) .$$

Corollary 2. *If an automaton $\mathscr{A} \in \Pi_k$, $k \geqq 4$ possesses two disjoint pairs $\{a, b\}$ and $\{c, d\}$ such that $\{c, d\} \to \{a, b\} \to \{f\}$ for some $x, y \in X$ and $f \in A$, then*

$$n(\mathscr{A}) \leqq \tfrac{1}{3}k^3 - \tfrac{3}{2}k^2 + \tfrac{25}{6}k - 4 - (k-3) .$$

The proof of this assertion is based on the inequalities

$$h_{k-2} - h_{k-1} \leqq \binom{k}{2} - \binom{k-1}{2} - (k-1) + 3 - 1 = 2 ,$$

$$h_{k-3} - h_{k-2} \leqq \binom{k}{2} - \binom{k-2}{2} - (k-2) + 3 - (k-4) = 6$$

which immediately follow from the fact that every $k-1$ tuple of states must contain $\{a, b\}$ or $\{c, d\}$ and $k-2$ tuple, excluding at most 4 must also contain $\{a, b\}$ or $\{c, d\}$.

Theorem 2. $n(4) = 9$.

Proof. We shall prove that for every automaton $\mathscr{A} \in \Pi_4$, $\mathscr{A} = (A = \{1, 2, 3, 4\}, X, \delta)$ the inequality $n(\mathscr{A}) \leqq 9$ is valid. Therefore considering (1) $n(4) = 9$.

Because $\mathscr{A} \in \Pi_4$ is directable, there are 2 possibilities:

1. There exist two such pairs $\{a, b\}$ that $|\delta'(\{a, b\}, x)| = 1$ for some $x \in X$. Then according to the corollary 1, $n(\mathscr{A}) \leqq 8$.

2. There exists exactly one pair with that quality, say $\{1, 2\}$. Then we must solve 2 cases:

2.1. There exists $y \in X$ that $\{3, 4\} \xrightarrow[y]{} \{1, 2\}$. Then by the corollary 2 $n(\mathscr{A}) \leqq 9$.

2.2. For every $y \in X$ $\delta'(\{3, 4\}, y) \neq \{1, 2\}$. Then because of directability of $\mathscr{A}$ there exists such a pair $\{i, j\}$ that $i \in \{1, 2\}$, $j \in \{3, 4\}$ and $\delta'(\{i, j\}, y) = \{1,2\}$ for some $y \in X$. If there are 2 such pairs, then there exists a word q that

$$h_1 - h_2 \leqq 5, \quad h_2 - h_3 \leqq 3 \quad \text{and} \quad l(q) \leqq 9 .$$

Thus let us assume, that the pair $\{i, j\}$ is only one.

In the further cosiderations every $x \in X$ which fulfils $|\delta'(\{1, 2\}, x)| = 1$ we shall denote $\bar{x}$. It is impossible that $\{i, j\} \xrightarrow[\bar{x}]{} \{1, 2\}$ because then $\delta'(\{1, j\}, \bar{x}) = \delta'(\{2, j\}, \bar{x}) = \{1, 2\}$ what is in contradiction with our assumption.

Thus there are only 2 possibilities for $\bar{x}$:

a) There exists a state b that $\delta(b, \bar{x}) = 1$ and $\delta(c, \bar{x}) \neq 2$ for every state c.

b) There exists a state b that $\delta(b, \bar{x}) = 2$ and $\delta(c, \bar{x}) \neq 1$ for every $c \in A$.

Therefore 2 cases are to be solved:

2.2.1. There exists $\bar{x}$ that a, is valid.

2.2.1.1. $\{i, j\} = \{1, 3\}$ or $\{1, 4\}$. Then $\{1, 2, 3, 4\} \xrightarrow[\bar{x}]{} \{1, 3, 4\} \xrightarrow[y]{} \{1, 2, a\}$; $a = 3$ or 4 and there exists a directing word p that $l(p) \leqq 9$.

2.2.1.2. $\{i, j\} = \{2, 3\}$. (The multigraphs of $y \in X$, $\delta(\{2, 3\}, y) = \{1, 2\}$ are on the figures 3, 4, 5, 6.)

If there is $\bar{x}'$ that b, holds, then $A \xrightarrow[\bar{x}'y]{} \{1, 2, a\}$, $a \in \{3, 4\}$ and $n(\mathscr{A}) \leqq 9$.

Let us assume that for every $\bar{x}$ only a) is valid. Then there exists $z \in X$ that $\{1, 3, 4\} \xrightarrow[z]{} \{2, 3, 4\}$ and obviously $|\delta'(\{1, 2\}, z)| = 2$.

2.2.1.2.1. There exists z with abovementioned quality such that $\{2, 3\} \xrightarrow[z]{} \{1, 2\}$. The corresponding multigraph of z is on the figure 3 (it is z_1) or fig. 4 (it is z_2).

We shall construct the directing word $p = x_1x_2 \ldots x_n$, $n \leqq 9$.

$$z_1 : x_1x_2x_3x_4 = \bar{x}z_1z_1\bar{x} \quad \text{or} \quad x_1x_2x_3x_4x_5 = \bar{x}z_1z_1z_1\bar{x}$$

so that $A \xrightarrow[x_1x_2.x]{} \{1, 4\}$ or $\{3, 4\}$ what is allways possible. Then the others $x_i = z_1$ for $i \leqq n - 1$, $x_n = \bar{x}$ and $h_2 - h_3 \leqq 4$, $h_1 - h_2 \leqq 4$.

$$z_2 : x_1x_2x_3x_4 = xz_2z_2x \, .$$

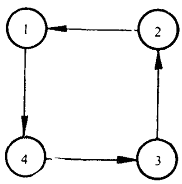

Fig. 3.

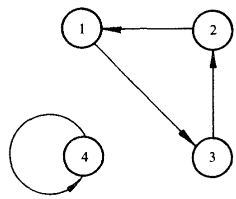

Fig. 4.

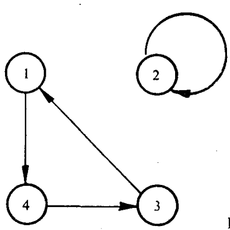

Fig. 5.

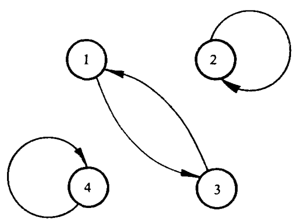

Fig. 6.

If $A \xrightarrow[\bar{x}z_2z_2\bar{x}]{} \{1, 3\}$, then $x_i = z_2$ for $i \geqq 5$.

If $A \xrightarrow[\bar{x}z_2z_2\bar{x}]{} \{1, 4\} \xrightarrow[z_2]{} \{3, 4\}$, then there exists $u \in x$ that $\delta(4, u) = 1$ or 3 and we put $x_5 = u$ or $x_5x_6 = z_2u$ so that we should get $\{1, 2\}$, $\{1, 3\}$ or $\{2, 3\}$. Then the others $x_i = z_2$, for $i \leqq n - 1$ and $x_n = \bar{x}$.

If $A \xrightarrow[\bar{x}z_2z_2\bar{x}]{} \{3, 4\}$ it is possible to use the preceding method.

In every case with z_2, $h_2 - h_3 = 3$ and $h_1 - h_2 \leqq 5$.

2.2.1.2.2. For every z, $\{1, 2\} \xrightarrow[z]{} \{1, 2\}$. Then obviously $3 \xrightarrow[y]{} 1$ and $2 \xrightarrow[y]{} 2$ (only the figure 5 and 6). The corresponding multigraph of z is on the figure 7 (z_3) or 8 (z_4)

$$z_3 : x_1x_2x_3 = \bar{x}z_3y \, .$$

If $A \xrightarrow[X_1x_2x_3]{} \{1, 2, 3\} \xrightarrow[z_3]{} \{1, 2, 4\}$, then $x_4 = \bar{x}$ or $x_4x_5 = z_3\bar{x}$ so that we should get $\{1, 3\}$ or $\{1, 4\}$ and then the others x_i equal z_3 or y for $i \leqq n - 1$, $x_n = \bar{x}$.

If $A \xrightarrow[x_1x_2x_3]{} \{1, 2, 4\} \xrightarrow[z_3]{} \{1, 2, 3\}$, the preceding method can be used.

In the both cases $h_2 - h_3 \leqq 4$ and $h_1 - h_2 \leqq 4$.

z_4: It can be solved by means of similar considerations as the preceding ones so that $h_2 - h_3 = 3$ and $h_1 - h_2 \leqq 5$.

2.2.1.3. $\{i, j\} = \{2, 4\}$. The same procedure as in 2.2.1.2, we only change the states 3 and 4.

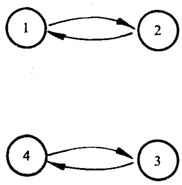

Fig. 7.

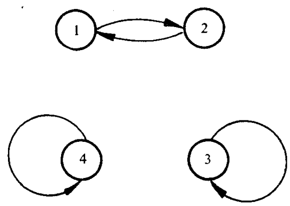

Fig. 8.

2.2.2. For every $\bar{x}$ b), is valid. It is symmetric with 2.2.1.

Remark 1. Except of the automaton $\mathscr{U}_4$ (example 1) we have found another automaton $\mathscr{P}_4$ (fig. 9) not isomorphic with $\mathscr{U}_4$, such that $n(\mathscr{P}_4) = 9$. For $\mathscr{U}_4$ it is valid that for the shortest directing word $h_2 - h_3 = h_1 - h_2 = 4$ while for $\mathscr{P}_4$ it is

$$h_2 - h_3 = 3\,, \quad h_1 - h_2 = 5\,.$$

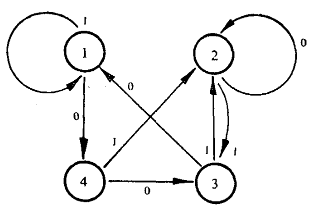

Fig. 9.

Remark 2. Adding the new state 4 to $\mathscr{U}_4$ we can easily obtain the automaton $\mathscr{U}_5$ for which $n(\mathscr{U}_5) = 16$ (see [1]). On the other hand no automaton $\mathscr{P}_5$ with $n(\mathscr{P}_5) \geqq$ $\geqq 16$ we could find by adding a new state to $\mathscr{P}_4$. The following theorem is valid for the automata with 5 states.

Theorem 3. $n(5) = 16$.

Proof. Because the proof is analogous as in theorem 2, we are not going to explain it in details. It is divided into the following main points:

1. Every mapping $\delta(0, x)$ is compressive, i.e. for every input symbol $x|\delta'(A, x)| < |A|$.

1.1. There exist four states i, j, k, l, such that for some

$$x, y \in X|\delta'(\{i, j\}, x)| = |\delta'(\{k, l\}, y| = 1 .$$

1.2. There exist three states i, j, k such that for some

$$x \in X|\delta'(\{i, j, k)| = 1 .$$

1.3. There exist three states i, j, k such that for some

$$x, y \in X|\delta'(\{i, j\}, x| = |\delta'(\{j, k\}, y)| = 1 .$$

1.4. There exists the pair i, j such that for every $x \in X$ $|\delta'(\{i, j\}, x)| = 1$ and for every $\{k, l\} \neq \{i, j\}$ and every $x \in X|\delta'(\{k, l\}, x)| = 2$.

2. There exist such a symbol $x \in X$ that the mapping $\delta(0, x)$ is a permutation of the elements of A.

2.1. The permutation is of the type 1; 2; 3; 4; 5 (every state is mapped into itself).

2.2. Of the type 12; 3; 4; 5 ($1 \rightarrow 2$; $2 \rightarrow 1$ and others into themselves).

2.3. Of the type 12; 34; 5.

2.4. Of the type 123; 4; 5 (a cycle from 123).

2.5. Of the type 123; 45.

2.6. Of the type 1234; 5.

2.7. Of the type 12345 (the total cycle).

Remark 3. The hypothesis $n(k) = (k - 1)^2$ may be found not valid only for $k \geqq 6$.

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O usmerniteľných automatoch

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V článku sa študujú odhady pre čísla $n(k)$, definované v [1] ako $\sup_{\mathscr{A} \in \Pi_k} \min_{p \in P(\mathscr{A})} l(p)$, kde $\mathscr{A} = (A, X, \delta)$ je neiniciálny Medvedevov automat, $P(\mathscr{A})$ je množina jeho usmerňujúcich slov, definovaných v [1], a Π_k je množina všetkých usmerniteľných automatov s k stavmi. Zlepšujú sa odhady pre $n(k)$ získané v [1] a [2], a to tým, že sa dokážu tieto vety:

Veta 1. *Pre všetky* $k \geqq 2$ *je* $n(k) \leqq k^3/3 - 3k^2/2 + 25k/6 - 4$.

Veta 2. $n(4) = 9$.

Veta 3. $n(5) = 16$.

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