KYBERNETIKA — VOLUME 7 (1971), NUMBER 2

Generalization of the Non-additive Measures of Uncertainty and Information and their Axiomatic Characterizations*

P. N. RATHIE

The object of this paper is to define generalized non-additive (i) entropy of order α and type β and (ii) information of order α and type β and to give their axiomatic characterizations. Further generalizations are indicated towards the end of the paper.

1. INTRODUCTION AND THE GENERALIZATIONS

Let $P = (p_1, \ldots, p_n)$, $n \geqq 1$ be a finite discrete probability distribution with $p_i > 0$, $W(P) = \sum_{i=1}^{n} p_i \leqq 1$. $W(P)$ is called the weight of the distribution P. Let Δ denote the set of all finite discrete generalized probability distributions. Introducing a parameter β, we call $W(P; \beta) = \sum_{i=1}^{n} p_i^\beta \leqq 1$, $\beta > 0$, as the generalized weight of the distribution P. Clearly, $W(P; 1) = W(P)$.

In what follows, $\sum$ will stand for the sum $\sum_{i=1}^{n}$ unless otherwise specified.

Now we introduce a new generalization of the non-additive entropy [2,4] as

$$(1.1) \qquad {}^{'}H_\alpha(P; \beta) = (1 - \sum p_i^{\alpha+\beta-1} / \sum p_i^\beta) / (1 - 2^{1-\alpha}) ,$$

$$\alpha \neq 1 , \quad \beta > 0 , \quad \alpha + \beta - 1 > 0 ;$$

which we shall call as the generalized non-additive entropy of order α and type β.

Let $P = (p_1, \ldots, p_n) \in \Delta$ and $Q = (q_1, \ldots, q_n) \in \Delta$ be the two generalized probability distributions, the correspondence between the elements of P and Q is that given by their subscripts. Then we define a new generalized non-additive information of

* The author is thankful to Professor A. M. Mathai of McGill University for providing financial assistance through his N.R.C. grant No. A 3057-282-08 which made this work possible.

order α and type β as

(1.2) $$I_\alpha(P; \beta \mid Q) = (1 - \sum p_i^{\alpha+\beta-1} q_i^{1-\alpha} / \sum p_i^\beta)/(1 - 2^{\alpha-1}),$$
$$\alpha \neq 1, \quad \beta > 0, \quad \alpha + \beta - 1 > 0.$$

For $\beta = 1$, (1.2) reduces to the non-additive measure of information of order α which has recently been characterized by means of a functional inequality by the author [3].

The additive entropy of order α and type β [5,6] is defined by the expression,

(1.3) $$H_\alpha^\beta(P) = (1 - \alpha)^{-1} \log_2 (\sum p_i^{\alpha+\beta-1} / \sum p_i^\beta),$$
$$\alpha \neq 1, \quad \beta > 0, \quad \alpha + \beta - 1 > 0;$$

where as the additive information of order α and type β [7] is defined as,

(1.4) $$I_\alpha^\beta(P \mid Q) = (\alpha - 1)^{-1} \log_2 (\sum p_i^{\alpha+\beta-1} q_i^{1-\alpha} / \sum p_i^\beta),$$
$$\alpha \neq 1, \quad \beta > 0, \quad \alpha + \beta - 1 > 0.$$

It is easy to find from (1.1) and (1.3) that*

(1.5) $$H_\alpha(P; \beta) = (1 - 2^{(1-\alpha)H_\alpha{}^\beta(P)})/(1 - 2^{1-\alpha});$$

and from (1.2) and (1.4), we get

(1.6) $$I_\alpha(P; \beta \mid Q) = (1 - 2^{(\alpha-1)I_\alpha{}^\beta(P|Q)})/(1 - 2^{\alpha-1}).$$

The conditions $\beta > 0$ and $\alpha + \beta - 1 > 0$ are put so that some of the p's may be allowed to take zero values.

The object of this paper is to prove some characterization theorems for the generalized non-additive measures of uncertainty (1.1) and information (1.2) respectively by assuming certain sets of postulates. On specializing the parameter β (i.e. $\beta = 1$), one can easily obtain similar results for the ordinary non-additive measures of uncertainty and information.

2. CHARACTERIZATION OF THE GENERALIZED UNCERTAINTY

This section deals with the characterizations of the generalized non-additive measures of uncertainty, $H_\alpha(P; \beta)$ by two sets of postulates. The axiomatic characterizations are given below in the form of two theorems which generalize the recent results of [4].

Postulate 1. $\lim_{p \to 0+} H_\alpha(1 - p; \beta)/p = A, \; p \in \Delta$.

* The author thanks I. Vajda, the reviewer of this paper, for suggesting the relationship between $H_\alpha(P; \beta)$ and $H_\alpha^\beta(P)$.

Postulate 2. $H_\alpha(\frac{1}{2}; \beta) = 1$.

Postulate 3. If $p, q \in \Delta$, then

$$H_\alpha(pq; \beta) = H_\alpha(p; \beta) + H_\alpha(q; \beta) + (2^{1-\alpha} - 1)\, H_\alpha(p; \beta)\, H_\alpha(q; \beta)\,.$$

Postulate 4. If $P = (p_1, \ldots, p_n) \in \Delta$, $Q = (q_1, \ldots, q_m) \in \Delta$ and $W(P; \beta) + W(Q; \beta) \leqq \leqq 1$, then

$$H_\alpha(P \cup Q; \beta) = \frac{W(P; \beta)\, H_\alpha(P; \beta) + W(Q; \beta)\, H_\alpha(Q; \beta)}{W(P; \beta) + W(Q; \beta)}\,,$$

where $P \cup Q = (p_1, \ldots, p_n, q_1, \ldots, q_m)$.

It is sufficient to assume postulate 4 for $n = m = 1$, the result for the general case follows by induction.

Theorem 1. *A function $H_\alpha(P; \beta)$ satisfying the postulates* 1, 2, 3 *and* 4 *is given by* (1.1) *for* $n \geqq 2$.

Proof. For $p = 1$ the postulate 3 takes the following form,

$$H_\alpha(1; \beta)\,[1 + (2^{\alpha-1} - 1)\, H_\alpha(q; \beta)] = 0\,. \tag{2.1}$$

Taking $q = \frac{1}{2}$ and using the postulate 2, we find that

$$H_\alpha(1; \beta) = 0\,. \tag{2.2}$$

Now with $q = 1 - \delta p/p$, the postulate 3 takes the form,

$$H_\alpha(p; \beta) - H_\alpha(p - \delta p; \beta) = H_\alpha(1 - \delta p/p; \beta)\,[(1 - 2^{1-\alpha})\, H_\alpha(p; \beta) - 1]\,. \tag{2.3}$$

Dividing (2.3) by δp and taking limits as $\delta p \to 0$, we get

$$\mathrm{d}H_\alpha(p; \beta)/\mathrm{d}p = (A/p)\,[(1 - 2^{1-\alpha})\, H_\alpha(p; \beta) - 1]\,, \tag{2.4}$$

by using the postulate 1.

Solving the differential equation (2.4) under the boundary conditions given in the postulate 2 and (2.2), we arrive at

$$H_\alpha(p; \beta) = (p^{\alpha-1} - 1)/(2^{1-\alpha} - 1)\,. \tag{2.5}$$

Hence using (2.5) in postulate 4 proves theorem 1.

Postulate 1 implies that $H_\alpha(p; \beta)$ is differentiable. We can weaken this postulate by assuming the following postulate of continuity:

Postulate 1′. $H_\alpha(p; \beta)$ is a continuous function of $p \in (0,1]$.

 Now we prove the following theorem:

Theorem 2. *A function $H_\alpha(P; \beta)$ satisfying the postulates* 1′, 2, 3 *and* 4 *is given by* (1.1) *for* $n \geqq 2$.

Proof. Let

$$g_\alpha(p; \beta) = 1 + (2^{1-\alpha} - 1) H_\alpha(p; \beta), \tag{2.6}$$

then from postulate 3, we have

$$g_\alpha(pq; \beta) = g_\alpha(p; \beta)\, g_\alpha(q; \beta). \tag{2.7}$$

Since $H_\alpha(p; \beta)$, by postulate 1′, is continuous in (0,1] and therefore $g_\alpha(p; \beta)$ is also continuous. Hence the only non-zero continuous solutions [1, p. 41] of (2.7) are given by

$$g_\alpha(p; \beta) = p^a, \tag{2.8}$$

where a is a real arbitrary constant which may depend on α and β.

Now the use of postulate 2 yields $a = \alpha - 1$ giving (2.5). Hence as before, the postulate 4 proves the theorem.

3. CHARACTERIZATION OF THE GENERALIZED INFORMATION

In this section we characterize the generalized non-additive measure of information of order α and type β. We start by assuming the following postulates.

Postulate 1. $\lim_{q \to 0^+} I_\alpha(1; \beta \mid 1 - q)/q = A, \ q \in \Delta.$

Postulate 2. $I_\alpha(p; \beta \mid 1)$ is a continuous function of $p \in (0,1]$.

Postulate 3. $I_\alpha(1; \beta \mid \frac{1}{2}) = 1.$

Postulate 4. $I_\alpha(\frac{1}{2}; \beta \mid \frac{1}{2}) = 0.$

Postulate 5. If $p_1, p_2, q_1, q_2 \in \Delta$, then

$$I_\alpha(p_1 p_2; \beta \mid q_1 q_2) = I_\alpha(p_1; \beta \mid q_1) + I_\alpha(p_2; \beta \mid q_2) + \\ + (2^{\alpha-1} - 1) I_\alpha(p_1; \beta \mid q_1) I_\alpha(p_2; \beta \mid q_2).$$

Postulate 6. If $P, Q \in \Delta$, then

$$I_\alpha(P; \beta \mid Q) = \frac{W(P_1; \beta) I_\alpha(P_1; \beta \mid Q_1) + W(P_2; \beta) I_\alpha(P_2; \beta \mid Q_2)}{W(P_1; \beta) + W(P_2; \beta)}$$

where $P = P_1 \cup P_2$ and $Q = Q_1 \cup Q_2$.

Theorem 3. *A function $I_\alpha(P; \beta \mid Q)$ satisfying the postulates* 1, 2, 3, 4, 5 *and* 6 *is given by* (1.2) *for* $n \geqq 2$.

Proof. Taking $p_1 = p$, $p_2 = q_1 = 1$ and $q_2 = q$ in postulate 5, we have

$$(3.1)\qquad I_\alpha(p; \beta \mid q) = I_\alpha(p; \beta \mid 1) + I_\alpha(1; \beta \mid q) + (2^{\alpha-1} - 1)\, I_\alpha(p; \beta \mid 1)\, I_\alpha(1; \beta \mid q)$$

Postulate 5 for $p_1 = p_2 = 1$ gives

$$(3.2)\qquad I_\alpha(1; \beta \mid q_1 q_2) = I_\alpha(1; \beta \mid q_1) + I_\alpha(1; \beta \mid q_2) + \\ + (2^{\alpha-1} - 1)\, I_\alpha(1; \beta \mid q_1)\, I_\alpha(1; \beta \mid q_2)\,.$$

Now for $q_2 = 1$, (3.2) yields

$$(3.3)\qquad I_\alpha(1; \beta \mid 1)\, [1 + (2^{\alpha-1} - 1)\, I_\alpha(1; \beta \mid q_1)] = 0\,.$$

Taking $q_1 = \frac{1}{2}$ and using the postulate 3, we have

$$(3.4)\qquad I_\alpha(1; \beta \mid 1) = 0\,.$$

Again taking $q_1 = q$, $q_2 = 1 - \delta q/q$ in (3.2), we get

$$I_\alpha(1; \beta \mid q) - I_\alpha(1; \beta \mid q - \delta q) = I_\alpha(1; \beta \mid 1 - \delta q/q)\, [(1 - 2^{\alpha-1})\, I_\alpha(1; \beta \mid q) - 1]\,;$$

which on dividing by δq, taking limits as $\delta q \to 0$ and using the postulate 1 gives the following differential equation

$$(3.5)\qquad \mathrm{d}I_\alpha(1; \beta \mid q)/\mathrm{d}q = (A/q)\, [(1 - 2^{\alpha-1})\, I_\alpha(1; \beta \mid q) - 1]\,.$$

Solving the differential equation (3.5) under the boundary conditions given in (3.4) and the postulate 3, we have

$$(3.6)\qquad I_\alpha(1; \beta \mid q) = (q^{1-\alpha} - 1)/(2^{\alpha-1} - 1)\,.$$

Taking $q_1 = q_2 = 1$ in postulate 5, we get

$$(3.7)\qquad I_\alpha(p_1 p_2; \beta \mid 1) = I_\alpha(p_1; \beta \mid 1) + I_\alpha(p_2; \beta \mid 1) + \\ + (2^{\alpha-1} - 1)\, I_\alpha(p_1; \beta \mid 1)\, I_\alpha(p_2; \beta \mid 1)\,.$$

Let

$$(3.8)\qquad g_\alpha(p; \beta \mid 1) = 1 + (2^{\alpha-1} - 1)\, I_\alpha(p; \beta \mid 1)\,,$$

then from (3.7) we have

$$(3.9)\qquad g_\alpha(p_1 p_2; \beta \mid 1) = g_\alpha(p_1; \beta \mid 1)\, g_\alpha(p_2; \beta \mid 1)\,.$$

By postulate 2 the continuity of $I_\alpha(p; \beta \mid 1)$ implies the continuity of $g_\alpha(p; \beta \mid 1)$ and hence the non-zero continuous solutions of (3.9) are given by [1, p. 41],

$$(3.10)\qquad g_\alpha(p; \beta \mid 1) = p^a\,,$$

where a is a real arbitrary constant. Hence

$$I_\alpha(p;\ \beta \mid 1) = (p^a - 1)/(2^{\alpha-1} - 1)\,. \tag{3.11}$$

Thus (3.1) on using (3.6) and (3.11) gives

$$I_\alpha(p;\ \beta \mid q) = (p^a q^{1-\alpha} - 1)/(2^{\alpha-1} - 1)\,. \tag{3.12}$$

The use of postulate 4 yields $a = \alpha - 1$ giving

$$I_\alpha(p;\ \beta \mid q) = (p^{\alpha-1} q^{1-\alpha} - 1)/(2^{\alpha-1} - 1)\,. \tag{3.13}$$

Theorem 3 can now be obtained on using (3.13) and the postulate 6.

Now we replace the postulate 1 by a weaker postulate assuming the continuity of $I_\alpha(1;\ \beta \mid q)$.

Postulate 1′. $I_\alpha(1;\ \beta \mid q)$ is a continuous function of $q \in (0,1]$.

Theorem 4. *A function* $I_\alpha(P;\ \beta \mid Q)$ *satisfying the postulates* 1′, 2, 3, 4, 5 *and* 6 *is given by* (1.2) *for* $n \geqq 2$.

Proof. As done in the later part of the proof of theorem 3, it is easy to prove in this case that

$$I_\alpha(p;\ \beta \mid 1) = (p^a - 1)/(2^{\alpha-1} - 1) \tag{3.14}$$

and

$$I_\alpha(1;\ \beta \mid q) = (q^b - 1)/(2^{\alpha-1} - 1) \tag{3.15}$$

giving

$$I_\alpha(p;\ \beta \mid q) = (p^a q^b - 1)/(2^{\alpha-1} - 1)\,. \tag{3.16}$$

The use of postulate 3 and 4 yields $a = \alpha - 1$ and $b = 1 - \alpha$ giving (3.13) from which theorem 4 follows by postulate 6.

4. FURTHER GENERALIZATIONS

In this section we give some further generalizations of the non-additive measures of uncertainty and information. They are:

(i) The generalized non-additive entropy of order α and type $\{\beta_i\}$,

$$H_\alpha(P;\ \beta_i \mid Q) = \left(1 - \sum p_i^{\alpha+\beta_i-1} / \sum p_i^{\beta_i}\right)/(1 - 2^{1-\alpha})\,, \tag{5.1}$$
$$\alpha \neq 1\,, \quad \beta_i > 0\,, \quad \alpha + \beta_i - 1 > 0\,.$$

(ii) The generalized non-additive information of order α and type $\{\beta_i\}$,

$$(5.2)\qquad I_\alpha(P; \beta_i \mid Q) = \left(1 - \sum p_i^{\alpha+\beta_i-1} q_i^{1-\alpha} / \sum p_i^{\beta_i}\right) / (1 - 2^{\alpha-1}),$$

$$\alpha \neq 1, \quad \beta_i > 0, \quad \alpha + \beta_i - 1 > 0.$$

Clearly (5.1) and (5.2) yield (1.1) and (1.2) respectively for $\beta_i = \beta$ for all $i = 1, \ldots, n$. It is proposed to study (5.1) and (5.2) in subsequent papers.

(Received August 12, 1970.)

REFERENCES

[1] J. Aczél: Lectures on Functional equations and their applications. Academic Press, New York 1966.

[2] J. Havrda, F. Charvát: Quantification method of classification processes. The concept of structural a-entropy. Kybernetika, *3* (1967), 1, 30–35.

[3] P. N. Rathie: Some characterization theorems for generalized measures of uncertainty and information. Proceedings of the Meeting on Information Measures, University of Waterloo, Waterloo, Ontario, Canada, April 10–14, 1970.

[4] I. Vajda: Axiomy a-entropie zobecněného pravděpodobnostního schématu. Kybernetika *4* (1968), 2, 105–112.

[5] J. Aczél, Z. Daróczy: Über verallgemeinerte quasilineare Mittelwerte, die mit Gewichtsfunktionen gebildet sind. Publ. Math., Debrecen, *10* (1963), 171–190.

[6] J. N. Kapur: Generalized entropy of order α and type β. Math. Seminar *4* (1967), 78–94.

[7] J. N. Kapur: Information of order α and type β. Proc. Indian Acad. Sci. *A 68* (1968), 65–75.

VÝTAH

Zobecnění neaditivních měr nejistoty a informace a jejich axiomatické charakteristiky

P. N. Rathie

Budiž $P = (p_1, \ldots, p_n)$ konečné diskrétní rozložení pravděpodobnosti pro $p_i > 0$, $\sum p_i \leqq 1$. Nechť Δ znamená množinu všech konečných diskrétních rozložení pravděpodobnosti. Pak zobecněná neaditivní entropie řádu α a typu β je definována vztahem

$$(1.1)\qquad H_\alpha(P; \beta) = \left(1 - \sum p_i^{\alpha+\beta-1} / \sum p_i^\beta\right) / (1 - 2^{1-\alpha}),$$

$$\alpha \neq 1, \quad \beta > 0, \quad \alpha + \beta - 1 > 0.$$

Rovněž pro $P = (p_1, \ldots, p_n) \in \Delta$ a $Q = (q_1, \ldots, q_n) \in \Delta$ je definována zobecněná neaditivní informace řádu α a typu β vztahem

$$I_\alpha(P; \beta \mid Q) = (1 - \sum p_i^{\alpha+\beta-1} q_i^{1-\alpha} / \sum p_i^\beta)/(1 - 2^{\alpha-1}), \quad \alpha \neq 1, \quad \beta > 0, \quad \alpha + \beta - 1 > 0. \tag{1.2}$$

Pro (1.1) a (1.2) jsou dokázány čtyři charakterizační věty při uvážení určitých souborů postulátů. Je naznačeno další zobecnění (1.1) a (1.2). První dvě věty zobecňují výsledky získané I. Vajdou.

Dr. P. N. Rathie, Visiting Scientist, Department of Mathematics, McGill University, Montreal 110, Quebec. Canada.