KYBERNETIKA ČÍSLO 3, ROČNÍK 6/1970

On the Inversion of Moving Averages, Linear Discrete Equalizers and "Whitening" Filters, and Series Summability

LUDVÍK PROUZA

The connections of some methods of inversion of moving averages, a method of construction of linear discrete equalizers or "whitening" filters, and the Borel property of series summation methods are investigated.

1. INTRODUCTION

In 1938, interesting articles [1] and [2] have been published, containing two somewhat different points of view on the problem of inversion of a finite moving average. In a special case, a new connection with the problem of series summation has been found, but not studied in detail in [1], [2] nor in [3], where the inversion problem has been attacked by the spectral methods developed in the meantime. It seems that no further articles on the random stationary sequences have been devoted to this topic.

Recently, the problem of a linear discrete equalizer has been investigated in [4], [5]. The formulation of the problem is a slight generalization of the problem of [2], the solutions being also similar.

In this article, we will investigate the problems in some detail. The connections with more recent results of the theory of linear discrete filters and with the so-called Borel property of series summation methods [6], [7] will be shown.

2. PROBLEM FORMULATION

Let $\{x(t)\}$ be a real weakly stationary white sequence, i. e. for $t = 0, \pm 1, \pm 2, \ldots$

$$\begin{aligned} \mathsf{E}[x(t)] &= 0\,, \\ \mathsf{E}\{[x(t)]^2\} &= \sigma_x^2\,, \\ \mathsf{E}[x(t)\,x(u)] &= 0 \quad \text{for} \quad t \neq u\,, \end{aligned} \tag{1}$$

where E denotes the mean value.

Without loss of generality, we will suppose $\sigma_x^2 = 1$.

Let the sequence $\{\xi(t)\}$ be formed from $\{x(t)\}$ by the finite moving average

$$\xi(t) = b_0\, x(t) + b_1\, x(t-1) + \ldots + b_h\, x(t-h)\,, \tag{2}$$

where $h \geqq 1$, b_j are real, $b_0 \neq 0$, $b_h \neq 0$. Without loss of generality we will suppose $b_0 = 1$.

Let N be a natural number, $N \geqq h$. We will form the finite moving average

$$x_N^*(t-T) = a_{N0}\, \xi(t) + a_{N1}\, \xi(t-1) + \ldots + a_{NN}\, \xi(t-N)\,, \tag{3}$$

T fixed, $0 \leqq T \leqq N + h$.

We seek $a_{N0}, \ldots, a_{NN}$ so that

$$\mathsf{E}\{[x(t-T) - x^*(t-T)]^2\} = \Phi(a_{N0}, \ldots, a_{NN}) = \min\,. \tag{4}$$

This expression is nonincreasing in N and can be expected to decrease with N.

For $T = 0$, we have in essence the problem of Frisch [2], for $T > 0$, we have the problem of Di Toro [4].

Denote the transfer functions of the filters of (2) and (3) resp.

$$B(z) = b_0 + b_1 z^{-1} + \ldots + b_h z^{-h}\,, \tag{5}$$

$$A_N(z) = a_{N0} + a_{N1} z^{-1} + \ldots + a_{NN} z^{-N}\,. \tag{6}$$

Then

$$A_N(z)\, B(z) = C_{N+h}(z) = c_{N+h,0} + \ldots + c_{N+h,N+h} z^{-N-h} \tag{7}$$

is the transfer function from $x(t)$ to $x^*(t-T)$, and

$$\mathsf{E}\{[x(t-T) - x^*(t-T)]^2\} = \frac{1}{2\pi i} \int_{C_1} |z^{-T} - A_N(z)\, B(z)|^2 \frac{dz}{z} = \tag{8}$$

$$= c_{N+h,0}^2 + c_{N+h,1}^2 + \ldots + (1 - c_{N+h,T})^2 + \ldots + c_{N+h,N+h}^2\,.$$

C_1 is the unit circle. The sequence $\{b_j\}$ is the unit impulse response of the filter (2), $\{c_{N+h,j}\}$ the same of the cascade of (2) and (3). In what follows, we will omit the first indices and write simply $\{a_j\}$, $\{c_j\}$.

Considering the filter (2) as "distorting" and the filter (3) as equalizer, then (4) according to (8) means to make c_T near 1 and the other c_j near 0 in the sense of minimum squares. This is the interpretation of Di Toro. In what follows we will consider mainly the case of $T = 0$, solved in essence by Frisch [2].

3. PROBLEM SOLUTION

From (5), (6), (7), we have the system of equations

$$(9)\qquad \begin{aligned} c_0 &= b_0 a_0\,, \\ c_1 &= b_1 a_0 + b_0 a_1\,, \\ &\vdots \\ c_T &= b_T a_0 + b_{T-1} a_1 + \ldots + b_0 a_T\,, \\ &\vdots \\ c_{N+h} &= \qquad\qquad\qquad\qquad b_h a_N\,, \end{aligned}$$

where $b_j = 0$ for $j > h$.

In the matrix notation, (9) is

$$(10)\qquad \mathbf{c} = \mathbf{B}\mathbf{a}\,,$$

where the meaning of the vectors $\mathbf{a}$, $\mathbf{c}$ and the matrix $\mathbf{B}$ is clear.

Then,

$$(11)\qquad \sum_{j=0}^{N+h} c_j^2 = \mathbf{c}'\mathbf{I}\mathbf{c} = \mathbf{a}'\mathbf{B}'\mathbf{B}\mathbf{a} = \mathbf{a}'\mathbf{M}\mathbf{a}$$

where $\mathbf{I}$ is the unit matrix and

$$(12)\qquad \mathbf{M} = \begin{pmatrix} \mu_0, & \mu_{-1}, & \mu_{-2}, \ldots, & \mu_{-h-N} \\ \mu_1, & \mu_0, & \mu_{-1}, \ldots, & \mu_{-h+1-N} \\ \ldots & \ldots & \ldots & \ldots \\ \mu_{N+h}, & \mu_{N+h-1}, & \ldots, & \mu_0 \end{pmatrix}$$

where according to the third member in (11)

$$(13)\qquad \mu_{-j} = \sum_{k=0}^{h-j} b_{k+j} b_k = \sum_{k=0}^{h-j} b_k b_{k+j} = \mu_j$$

are the "autocorrelations" or "automoments" of $\{b_j\}$. For $j > h$, there is $\mu_j = 0$.

Thus the right side of (8) can be expressed as

$$(14)\qquad \begin{aligned} \Phi(a_0, \ldots, a_N) = 1 - 2c_T &+ \\ &+ a_0(\mu_0 a_0 + \mu_{-1} a_1 + \ldots + \mu_{-h} a_h) + \\ &+ a_1(\mu_1 a_0 + \mu_0 a_1 + \ldots + \mu_{-h} a_{h+1}) + \ldots \\ &\ldots + a_N(\mu_h a_{N-h} + \ldots + \mu_0 a_N)\,. \end{aligned}$$

Necessary and sufficient conditions for the minimum of this positive definite quadratic form are the $N + 1$ equations

$$(15)\qquad \begin{aligned} \mu_0 a_0 + \mu_{-1} a_1 + \ldots + \mu_{-h} a_h &= b_T\,, \\ \mu_1 a_0 + \mu_0 a_1 + \ldots + \mu_{-h} a_{h+1} &= b_{T-1}\,, \\ &\vdots \\ \mu_T a_0 + \mu_{T-1} a_1 + \ldots + \mu_{-h} a_{h+T} &= b_0\,, \\ \mu_{T+1} a_0 + \mu_T a_1 + \ldots + \mu_{-h} a_{h+T+1} &= 0\,, \\ &\vdots \\ \mu_N a_0 + \ldots + \mu_{-h} a_{h+N} &= 0\,, \end{aligned}$$

where again $\mu_j = 0$ for $j > h$, $b_j = 0$ for $j > h$, $a_j = 0$ for $j > N$. The system has the unique solution $a_0, \ldots, a_N$, since the determinant thereof is distinct from zero, being the first principal minor of the determinant of the positive definite matrix (12). Substituting from (15) in (14) one gets

$$(16)\qquad \Phi(a_0, \ldots, a_N)_{\min} = 1 - c_T\,.$$

Especially for $T = 0$

$$(17)\qquad \Phi(a_0, \ldots, a_N)_{\min} = 1 - b_0 a_0$$

and for $b_0 = 1$

$$(18)\qquad \Phi(a_0, \ldots, a_N)_{\min} = 1 - a_0\,.$$

Since $\Phi(a_0, \ldots, a_N)$ is positive definite, there is $c_T \leqq 1$. Putting $a_0 = a_1 = \ldots$ $\ldots = a_N = 0$ in (8), one gets 1, thus from (16) $c_T \geqq 0$, and

$$(19)\qquad 0 \leqq c_T \leqq 1\,.$$

In [2], an expression slightly different from (4) has been minimised, resulting in more complicated condition than (17).

Instead of solving the system (15), it were apparently possible according to (8) to seek $c_0, \ldots, c_{N+h}$ fulfilling

$$(20)\qquad c_0^2 + c_1^2 + \ldots + (1 - c_T)^2 + \ldots + c_{N+h}^2 = \min$$

with h supplementary conditions expressing the fact that the equation

$$(21)\qquad c_0 z^{N+h} + c_1 z^{N+h-1} + \ldots + c_{N+h} = 0$$

(and possibly their derivations) are satisfied by the roots (possibly multiple) of

$$(22)\qquad b_0 z^h + b_1 z^{h-1} + \ldots + b_h = 0\,.$$

The solution may be found by the method of Lagrange multipliers and then $a_0, \ldots, a_N$ be computed from (9).

4. THE CASE $T = 0$

For $T = 0$ one gets from (15) the system

$$
\begin{aligned}
(23) \qquad & \mu_0 a_0 + \mu_{-1} a_1 + \ldots + \mu_{-h} a_h && = b_0 , \\
& \mu_1 a_0 + \mu_0 a_1 + \ldots + \mu_{-h+1} a_h + \mu_{-h} a_{h+1} && = 0 , \\
& \vdots \\
& \mu_h a_0 + \mu_{h-1} a_1 + \ldots + \mu_0 a_h + \mu_{-1} a_{h+1} + \ldots + \mu_{h-N} a_N && = 0 , \\
& \mu_h a_{N-h} + \ldots + \mu_0 a_N && = 0 .
\end{aligned}
$$

Let us consider for a moment the system (23) with exception of the first equation. It is obvious that the solution of this reduced system is precisely each multiple of the (unique) solution of (23). Putting in the reduced system $a_0 = 1$, one gets the system of Frisch [2]. The unique solution thereof is thus a multiple of the solution of (23). Only if $a_0 = 1$ in (23), both solutions are the same. Generally, the error (8) is greater for the Frisch case.

To solve (23), we use the same method as in [2]. We replace (23) by the homogeneous linear difference equation of the order $2h$ for $\{a_j\}$

$$
(24) \qquad \mu_{-h} a_n + \mu_{-h+1} a_{n-1} + \ldots + \mu_0 a_{n-h} + \mu_1 a_{n-h-1} + \ldots + \\
\ldots + \mu_h a_{n-2h} = 0
$$

with $2h$ boundary conditions

$$
(25) \qquad a_{-1} = 0 , \quad a_{-2} = 0, \ldots, a_{-h+1} = 0 , \quad a_{-h} = -\frac{b_0}{\mu_h} \neq 0 ,
$$

$$
(26) \qquad a_{N+1} = 0 , \quad a_{N+2} = 0 , \ldots, a_{N+h} = 0 .
$$

From the solution of (24), one will use $a_0, \ldots, a_N$.

Clearly (25), (26) are satisfied by the solution of (23). Thus if (24) with the boundary conditions (25), (26) has unique solution, this solution includes that of (23).

The uniqueness of the solution of (24) with (25), (26) is, generally, an unsolved nontrivial problem. However, in special simple cases, it may be solved easily. The great advantage of solving (24), (25), (26) instead of (23) lies in the fact that $2h$ may be considerably smaller than N.

Consider the characteristic equation to (24):

$$
(27) \qquad \mu_h z^{2h} + \mu_{h-1} z^{2h-1} + \ldots + \mu_0 z^h + \mu_1 z^{h-1} + \ldots + \mu_h = \\
= B(z)\, B(z^{-1})\, z^h = 0 .
$$

It is reciprocal and its roots $\zeta_1, \ldots, \zeta_{2h}$ are the roots $z_1, \ldots, z_h$ of (22) and the reciprocal values thereof. Suppose the roots arranged according to

$$|\zeta_1| \leqq |\zeta_2| \leqq \dots \leqq |\zeta_{2h}| \,. \tag{28}$$

From (13) one gets $\mu_h = b_0 b_h$, so that the last condition of (25) is $a_{-h} = -1/b_h$. Since from (22) $b_h = (-1)^h z_1 z_2 \dots z_h$ for $b_0 = 1$, one gets

$$a_{-h} = \frac{(-1)^{h-1}}{z_1 z_2 \dots z_h} \,. \tag{29}$$

Consider the case where the roots of (22) are simple and none lies precisely on the unit circle C_1.

Then the general solution of (24) is

$$a_n = A_1 \zeta_1^n + A_2 \zeta_2^n + \dots + A_{2h} \zeta_{2h}^n \tag{30}$$

and from the boundary conditions (25), (26) one gets a system of $2h$ equations for $A_1, \dots, A_{2h}$

$$\begin{aligned} \zeta_1^{-h} A_1 + \zeta_2^{-h} A_2 + \dots + \zeta_{2h}^{-h} A_{2h} &= a_{-h} \,, \\ \zeta_1^{-h+1} A_1 + \zeta_2^{-h+1} A_2 + \dots + \zeta_{2h}^{-h+1} A_{2h} &= 0 \,, \\ &\vdots \\ \zeta_1^{-1} A_1 + \zeta_2^{-1} A_2 + \dots + \zeta_{2h}^{-1} A_{2h} &= 0 \,, \\ \zeta_1^{N+1} A_1 + \zeta_2^{N+1} A_2 + \dots + \zeta_{2h}^{N+1} A_{2h} &= 0 \,, \\ &\vdots \\ \zeta_1^{N+h} A_1 + \zeta_2^{N+h} A_2 + \dots + \zeta_2^{N+h} A_{2h} &= 0 \,. \end{aligned} \tag{31}$$

The determinant of this system is

$$\Delta = (\zeta_1 \dots \zeta_{2h})^{-h} \begin{vmatrix} 1 & ,\dots, & 1 \\ \zeta_1, & \dots, & \zeta_{2h} \\ & \vdots & \\ \zeta_1^{h-1}, & \dots, & \zeta_{2h}^{h-1} \\ \zeta_1^{h+N+1}, & \dots, & \zeta_{2h}^{h+N+1} \\ & \vdots & \\ \zeta_1^{2h+N}, & \dots, & \zeta_{2h}^{2h+N} \end{vmatrix} = \tag{32}$$

$$= (\zeta_1 \dots \zeta_{2h})^{-h} \cdot \left[V_{\zeta_1 \dots \zeta_h} \cdot V_{\zeta_{h+1} \dots \zeta_{2h}} \cdot (\zeta_{h+1} \dots \zeta_{2h})^{h+N+1} + \dots \right] .$$

The determinant on the left side is a generalization of the known Vandermonde determinant. On the right side, this determinant is developed according to Laplace theorem. $V_{\zeta_1 \dots \zeta_h}$ and $V_{\zeta_{h+1} \dots \zeta_{2h}}$ are Vandermonde determinants to the respective roots and only the first term of the Laplace development is explicitly shown in the bracket.

According to (28) and since the roots of (22) are simple, this term is clearly "dominant" for $N \to \infty$. Thus, at least for sufficiently great N the determinant (32) is distinct from zero and the solution of (24) is unique.

Now, with respect to (18), we are interested in the behaviour of a_0 for $N \to \infty$. From (30) and (31) it is seen that $a_0 = a_{-h} \,.\, \Delta_0/\Delta$, where Δ_0 is a determinant analogous to (32). Developing again this determinant one gets

(33) $$\Delta_0 = (-1)^{h-1} (\zeta_1 \ldots \zeta_{2h})^{-h+1} \,.\, [V_{\zeta_1 \ldots \zeta_h} \,.\, V_{\zeta_{h+1} \ldots \zeta_{2h}} (\zeta_{h+1} \ldots \zeta_{2h})^{h+N} + \ldots] \,,$$

where again only the "dominant" term is shown in the bracket.

From (29), (32), (33) there follows

(34) $$\lim_{N \to \infty} a_0 = \frac{(-1)^{2(h-1)}}{z_1 \ldots z_h} \frac{\zeta_1 \ldots \zeta_{2h}}{\zeta_{h+1} \ldots \zeta_{2h}} = \frac{\zeta_1 \ldots \zeta_h}{z_1 \ldots z_h} \,.$$

Theorem 1. *If all roots of* (22) *are simple and none lies on* C_1, *then*

$$\lim_{N \to \infty} a_0 = 1$$

if and only if all roots lie inside of C_1. *If at least one root lies outside of* C_1, *then*

$$\lim_{N \to \infty} a_0 < 1 \,,$$

and vice versa. In no case there is

$$\lim_{N \to \infty} a_0 = 0 \,.$$

Proof follows immediately from (34).

Simple examples and the reasoning that (34) must very continuously with continuous vatiation of the roots of (22) give some evidence that the formula (34) and Theorem 1 are of general validity without the premise about the roots of (22), but the precise proof thereof seems to be difficult.

5. EXAMPLES

Example 1. Let

(35) $$\xi(t) = x(t) - \tfrac{1}{2}x(t-1) \,.$$

By the methods of the section 4, one gets

(36) $$a_n = \frac{2^{N+3}(\frac{1}{2})^n - 2^{n+2}(\frac{1}{2})^{N+1}}{2^{N+3} - (\frac{1}{2})^{N+1}} \,,$$

so that

(37) $$\lim_{N\to\infty} a_n = (\tfrac{1}{2})^n$$

especially

(38) $$\lim_{N\to\infty} a_0 = 1$$

in accordance with (34).

Example 2. Let

(39) $$\xi(t) = x(t) - 2x(t-1).$$

By the methods of section 4, one gets

(40) $$a_n = \frac{2^{2(N+1)-n} - 2^n}{2^{2(N+2)} - 1},$$

so that

(41) $$\lim_{N\to\infty} a_n = (\tfrac{1}{2})^{n+2},$$

especially

(42) $$\lim_{N\to\infty} a_0 = \tfrac{1}{4}$$

in accordance with (34).

For $N = 1$, we have

(43) $$x^*(t) = \tfrac{5}{21}\xi(t) + \tfrac{2}{21}\xi(t-1).$$

This case has been solved in [3] with the unnecessary restriction $a_0 = 1$ (see the note after (23)) and the nonoptimal solution

$$x^*(t) = \xi(t) + \tfrac{2}{5}\xi(t-1)$$

has been found.

Example 3. Let

(44) $$\xi(t) = x(t) - x(t-1).$$

Now, the characteristic equation (24) has double root $z_{1,2} = 1$. Thus

(45) $$a_n = A_1 + A_2 \,.\, n$$

and with (25), (26) one gets

(46) $$a_n = \frac{N+1-n}{N+2},$$

so that

$$\lim_{N\to\infty} a_n = 1\,, \tag{47}$$

especially

$$\lim_{N\to\infty} a_0 = 1 \tag{48}$$

in accordance with (34). According to (9)

$$1 - c_0 = 1 - a_0 = \frac{1}{N+2} \tag{49}$$

and

$$c_n = a_n - a_{n-1} = -\frac{1}{N+2} \tag{50}$$

for $n = 1, 2, \ldots, N + 1$.

Thus

$$x(t) - x^*(t) = \frac{1}{N+2}\left[x(t) + x(t-1) + \ldots + x(t-N-1)\right]. \tag{51}$$

From (46) and (51) we see that in this case we have in essence the known Cesàro summation method $\mathscr{C}_1$. According to (48)

$$\lim_{N\to\infty} \left[(1 - c_0)^2 + \sum_{j=1}^{N+1} c_j^2\right] = 0\,. \tag{52}$$

Example 4. Let

$$\xi(t) = x(t) - 2x(t-1) + x(t-2)\,. \tag{53}$$

Now, the characteristic equation (24) has quadruple root $z_{1,2,3,4} = 1$ and with boundary conditions (25), (26)

$$a_n = (1 + n)\left(1 - \frac{n+2}{N+3}\right)\left(1 - \frac{n+2}{N+4}\right), \tag{54}$$

so that

$$\lim_{N\to\infty} a_n = 1 + n\,, \tag{55}$$

especially

$$\lim_{N\to\infty} a_0 = 1 \tag{56}$$

in accordance with (34).

According to (9)

$$1 - c_0 = 1 - a_0 = \frac{4N + 10}{(N + 3)(N + 4)} \tag{57}$$

and

$$c_n = -\frac{4N + 10 - 6n}{(N + 3)(N + 4)} \tag{58}$$

for $n = 1, \ldots, N + 2$. Thus

$$x(t) - x^*(t) = \frac{1}{(N + 3)(N + 4)} [(4N + 10)\,x(t) + \tag{59}$$

$$+ (4N + 4)\,x(t - 1) + \ldots + (-2N - 2)\,x(t - N - 2)] .$$

From (54) and (55) is clear that we have to do with the "Cesàro" analogy for double summation.

The matrix of coefficients of (59) (for increasing N) is row-finite. Denoting these coefficients $c_0^*, c_1^*, \ldots, c_{N+2}^*$ we may prove easily

$$\sum_{j=0}^{N+2} c_j^* = 1 \quad \text{for every } N , \tag{60}$$

$$\lim_{N \to \infty} c_n^* = 0 \quad \text{for every } n ,$$

$$\sum_{j=0}^{N+2} |c_j^*| < 4 \quad \text{for every } N ,$$

so that all Toeplitz conditions are fulfilled and the transform is regular.

6. THE CONNECTIONS WITH SERIES SUMMATION METHODS

The fact that the inversion of (44) leads to a summation method is not surprising. Since (44) is a "first difference" filter the formal inversion of which is the unstable summing filter, there is plausible that from the postulate (4) with increasing N a sequence of stable filters representing a regular summation method results. What is interesting is that this method is precisely Cesàro $\mathscr{C}_1$.

Now, it is natural to abandon the postulate (4) and to require only

$$\lim_{N \to \infty} \sum_{j=0}^{N+1} c_j^{*2} = 0 \tag{61}$$

(identical with (52)) and

$$\sum_{j=0}^{N+1} c_j^* = 1 \quad \text{for every } N \tag{62}$$

(since (22) has the root $z = 1$) and to seek regular transforms fulfilling (61) and (62).
We obtain in fact a wider class, since e. g. the known Euler $\mathscr{E}_1$ transform fulfills (61) and (62), as we will show.

Put according to $\mathscr{E}_1$

(63) $$x^*(t) = \frac{1}{2^N}\left\{\left[\binom{N}{1} + \binom{N}{2} + \ldots + \binom{N}{N}\right]\xi(t) + \right.$$
$$\left. + \left[\binom{N}{2} + \ldots + \binom{N}{N}\right]\xi(t-1) + \ldots + \binom{N}{N}\xi(t-N+1)\right\}.$$

Then

(64) $$x(t) - x^*(t) = \frac{1}{2^N}\left[\binom{N}{0}x(t) + \binom{N}{1}x(t-1) + \ldots + \binom{N}{N}x(t-N)\right].$$

From (64), (62) follows at once. Moreover,

(65) $$\frac{\binom{N}{0}^2 + \binom{N}{1}^2 + \ldots + \binom{N}{N}^2}{2^{2N}} = \frac{\binom{2N}{N}}{2^{2N}} \approx \frac{1}{\sqrt{(\pi N)}}$$

asymptotically for great N, so that (61) is also fulfilled. Moreover, all $c_j^* > 0$ in $\mathscr{E}_1$, similarly as in $\mathscr{C}_1$.

Now, let us consider another interesting transform due to Wold [1] and Frisch [2]. Let

(66) $$x^*(t) = \xi(t) + \varrho\xi(t-1) + \ldots + \varrho^N\xi(t-N), \quad (0 < \varrho < 1).$$

Then

(67) $$x(t) - x^*(t) = (1-\varrho)\left[x(t-1) + \varrho x(t-2) + \ldots + \varrho^{N-1}x(t-N)\right] +$$
$$+ \varrho^N x(t-N-1).$$

It is clear that for $\varrho \to 1$ with $N \to \infty$ this is a "truncated" Abel summation with the transform matrix

(68) $$\mathbf{F} = \begin{pmatrix} 1-\varrho_1, & \varrho_1, & & 0, \ldots \\ 1-\varrho_2, & (1-\varrho_2)\varrho_2, & \varrho_2^2, & 0, \ldots \\ 1-\varrho_3, & (1-\varrho_3)\varrho_3, & (1-\varrho_3)\varrho_3^2, \varrho_3^3, & 0, \ldots \end{pmatrix}.$$

This matrix is row-finite, fulfills (62) and for $\varrho_N \to 1$ is regular. Moreover

(69) $$\mathsf{E}\{[x(t) - x^*(t)]^2\} = \frac{1-\varrho}{1+\varrho} + \frac{2}{1+\varrho}\varrho^{2N+1},$$

 so that (61) is fulfilled only if

(70) $$\varrho_N^N \to 0 \quad \text{for } N \to \infty\,.$$

According to Cramér (see [1] and [2]) $\varrho_N \to 1$ and $\varrho_N^N \to 0$ simultaneously for $N \to \infty$ e. g. for

(71) $$\varrho_N = 1 - \frac{1}{\sqrt{N}}\,.$$

The first term on the right side of (69) is valid for the Abel transform (see [78]), where the rapidity of the convergence $\varrho_N \to 1$ to fulfill the analogon of (61) (Abel matrix is not row-finite) may be arbitrary.

Consider now the postulate (61) for regular transforms generally, without the restriction to row-finite ones.

Remember that according to (8) ($T = 0$) this means that $x^*(t)$ converges to $x(t)$ in the quadratic mean for $N \to \infty$.

Thus, suppose we are given a regular transform $\mathscr{T}$ with the matrix

(72) $$\boldsymbol{T} = \begin{pmatrix} t_{11}, t_{12}, \ldots, \\ t_{21}, t_{22}, \ldots \\ \ldots\ldots\ldots\ldots \end{pmatrix}.$$

Then, Hill [6] has been shown that

(73) $$\lim_{j \to \infty} \sum_{k=1}^{\infty} t_{jk}^2 = 0$$

is a necessary condition that the transform $\mathscr{T}$ has the so-called Borel property, which may be defined as follows.

Let $\{x_n\}$ be a sequence of 0's and 1's with infinite number of 1's. We connect a binary number $0, x_1, x_2 \ldots$ in the interval $(0, 1\rangle$ with this sequence. Introducing the usual Lebesgue measure on $(0, 1\rangle$, one says that $\mathscr{T}$ possess the Borel property, if in the sense of this measure it transforms almost every sequence of 0's and 1's to the value $\frac{1}{2}$.

Various sufficient conditions for $\mathscr{T}$ to possess the Borel property have been found by Hill [6] and Lorentz [7]. Each of such transforms which is row-finite and fulfills (62) may be used to define a sequence of nonrecursive stable filters to invert (44). But according to (4) Cesàro $\mathscr{C}_1$, which has been shown in 1909 by Borel himself to possess the Borel property, possess the greatest rapidity of convergence of (61).

The condition (73) may be expressed in a simpler form.

Theorem 2. *Let* $\mathscr{T}$ *be a regular transform with the matrix* (72). *Then* (73) *holds if and only if*

(74) $$\lim_{j \to \infty} \tau_j = \lim_{j \to \infty} \max_k |t_{jk}| = 0\,.$$

Proof. Let (74) hold. In each row

$$t_{j1}^2 + t_{j2}^2 + \ldots \leqq \tau_j \,.\, \sum_{k=1}^{\infty} |t_{jk}| \leqq \tau_j \,.\, K\,, \tag{75}$$

where $K < \infty$ is independent on j, $\mathcal{T}$ being regular. From (74) and (75), there follows (73). Let further (74) is not valid. Then there exists a $\delta > 0$ so that for some subsequence $\{j_l\}$ there holds $\tau_{j_l} > \delta$. In this case the left side of (75) is greater than $\delta^2 > 0$ for each j_l and (73) is not valid.

It is seen that the property (74) is a generalization of (70).

7. INVERSION BY RECURSIVE FILTERS

Let us consider example 1. The transfer function of the filter with "limit" coefficients (37) is

$$1 + \tfrac{1}{2}z^{-1} + \tfrac{1}{4}z^{-2} + \ldots = \frac{1}{1 - \frac{1}{2}z^{-1}} \tag{76}$$

and this is the transfer function of a stable recursive filter obtained by formal inversion of (35).

In example 2, we obtain by formal inversion a useless unstable filter. But the transfer function of the filter with "limit" coefficients (41) is

$$\tfrac{1}{4}[1 + \tfrac{1}{2}z^{-1} + \tfrac{1}{4}z^{-2} + \ldots] = \frac{1}{4}\frac{1}{1 - \frac{1}{2}z^{-1}} \tag{77}$$

and this is again the transfer function of a stable recursive filter.

In example 3, the filter with "limit" coefficients (47) is the same unstable summing filter as obtained by formal inversion of (44).

To make an approximation by stable filter, we shift the pole (1, 0i) to the point (ϱ, 0i) inside C_1 ($0 < \varrho < 1$). One may expect that for $\varrho \to 1$ the error analogous to (4) will tend to 0. In fact, this follows from

$$\frac{1}{2\pi i}\int_{C_1} \left|1 - \frac{z - 1}{z - \varrho}\right|^2 \frac{dz}{z} = \frac{1 - \varrho}{1 + \varrho}\,. \tag{78}$$

Developing

$$A_\varrho(z) = \frac{1}{1 - \varrho z^{-1}} \tag{79}$$

in a series, one sees that this procedure is identical with the Abel summation method, as has been pointed out in [1]. But, replacing the Abel matrix by the matrix (68), one may not choose ϱ_N independently from the "truncation" (see (71)).

The filter (79) is used in radar in the form of a "delay-line video-integrator".

In cases where (22) has roots outside C_1, as in example 2, one may find the transfer function of the recursive inverse filter with advantage directly with the aid of a theorem of Walsh ([8], p. 183)), as has been shown in similar situations in [9], [10].

Substitute $z^{-1} = v$ and rearrange the integral in (8) as follows ($T = 0$):

$$\frac{1}{2\pi i}\int_{C_1}\left|\frac{1}{B^*(v)} - A^*(v)\right|^2 |B^*(v)|^2 \frac{dv}{v} = \min\,, \tag{80}$$

where $A^*(v) = A(v^{-1})$, $B^*(v) = B(v^{-1})$.

According to Theorem 17 of Walsh,

$$A^*(v) = \frac{1}{2\pi i N(v)}\int_{C_1}\frac{N(t)\,.\,dt}{B^*(t)\,(t - v)}\,, \tag{81}$$

where $N(v)$ is an analytic function without zeros inside C_1 and for which

$$|N(v)|^2 = |B^*(v)|^2 \tag{82}$$

on C_1. We find $N(v)$ so that each linear factor $v - v_j$ of $B^*(v)$, where v_j is the zero inside C_1, is multiplied by the Blaschke factor

$$\Pi(v) = \frac{1 - \bar{v}_j v}{v - v_j}\,. \tag{83}$$

For example 2, $N(v) = 2 - v$ and

$$A^*(v) = \frac{1}{4 - 2v} = \frac{1}{4}\,.\,\frac{1}{1 - \frac{1}{2}z^{-1}} \tag{84}$$

as in (77).

8. CONCLUDING REMARKS

Thus trivially, the formal inversion by recursive filter is "precise" and with zero error, if the roots of (22) lie inside C_1. If they lie outside C_1, the nonformal inversion by recursive filter resulting from the Walsh theorem is "optimal" but gives nonzero quadratic error resulting from (34).

If the roots of (22) lie inside and outside C_1, but not on C_1, both methods can be combined to obtain the optimum recursive filter.

If the roots of (22) lie on C_1, only sequences of stable recursive or nonrecursive filters can be constructed approximating the operation of inversion. For the root $z = 1$, these sequences are e. g. Abel and Cesàro summation methods. For $|z| = 1$, $z \neq 1$, the generalization seems obvious, but a general theory of such filters is not ready.

It seems that some important questions arising in connection with the present article are unsolved.

An Appendix concerning one such question will appear in the next number of "Kybernetika".

The existence and uniqueness of the solution of a linear difference equation, given boundary conditions of various complexity, seems to be an open problem and its solution is of great importance for $T > 0$. For the optimum nonrecursive filter one can argue that solving the system (15) gives always result, but since the structure of the inversion filter is influenced by the roots of (22), the solution through the difference equation as in the section 4 is more natural and important.

Considering the existence of additive noise will make the solution of inversion filters more realistic and more complicated. Some results are known from [4], but a general theory remains to be created.

(Received January 22, 1970.)

REFERENCES

[1] Wold, H.: On the Inversion of Moving Averages. Skand. Aktuarietidskr. (1938), 208—217.

[2] Frisch, R.: On the Inversion of a Moving Average. Skand. Aktuarietidskr. (1938), 218—225.

[3] Nagabhushanam, K.: The primary process of a smoothing relation. Ark. för Mat. (1951), 421—488.

[4] Di Toro, M. J.: Communication in Time-Frequency Spread Media Using Adaptive Equalization. Proc. IEEE (1968), 1653—1679.

[5] Di Toro, M. J.: Note on the Optimal Transversal Equalizer for a Dual Multipath Noiseless Channel. Proc. IEEE (1969), 809—810.

[6] Hill, J. D.: Summability of sequences of 0' s and 1's. Ann. of Math. (1945), 556—562.

[7] Lorentz, G. G.: Borel and Banach properties of methods of summation. Duke Math. J. (1955), 129—141.

[8] Walsh, J. L.: Interpolation and approximation by rational functions in the complex domain (Russian translation). Inoizdat, Moscow 1961.

[9] Prouza, L.: "Closed"-form formulas for the Kolmogorov-Wiener optimal prediction and filtration of stationary random sequences (in Czech). Technika el. přístrojů (1965), 2, 45—46.

[10] Prouza, L.: On the Smoothing of a Discrete Random Autoregressive Process. Kybernetika (1966), 5, 423—434.

O inverzi klouzavých průměrů, lineárních diskrétních vyrovnávacích a „bělících" filtrech a sumabilitě řad

LUDVÍK PROUZA

V článku se vyšetřují souvislosti několika metod inverze klouzavých průměrů, metody konstrukce lineárních diskrétních vyrovnávacích a „bělících" filtrů a tzv. Borelovy vlastnosti metod sumace řad.

Dr. Ludvík Prouza, CSc., Ústav pro výzkum radiotechniky, Opočínek, p. Lány na Důlku.