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On Solving Some Problems of the Stochastic Theory of Configurations by Modelling

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The paper points out the possibilities offered by an experimental solution in the case of problems solved in references [1, 2, 3, 4]. It describes the realization of the model, discusses the outlooks of this conception and outlines the method of experimental solution. For the model itself it considers the random process generator developed at the Institute of Information Theory and Automation, and a digital computer. In the conclusion it presents the algorithm of the solution to the general problem cleared in [1], and a concrete example of the realization.

INTRODUCTION

The solution starts from studies [1 to 4]. The nature of the problem dealt with in them is briefly this: As applied, the term configuration denotes a system of interconnected components, so-called centres. Each centre contains a certain number of elements. This number varies at random because the studies consider the possibility of both "birth" and "death" of the elements inside the centre, and random travel – migrations – of the elements between centres in a network connecting them. The studies deal with the mathematical description of the networks thus formed, and the description of the changes in the number of particles in the centres. Their task is to find suitable criteria for estimating further development in a particular configuration, given the initial state in the centres and a given network. The j-th ($j = 1, 2, \ldots, n$) centre is illustrated schematically in Fig. 1. In our further considerations two random sources of particles representing "birth" and "death", are characterized by the Poisson distribution with intensities μ and δ.

Assume that in the centre j containing N_j particles, the probability of birth of an additional particle in the time interval $(t, t + \Delta t)$ is given by the expression

$$N_j \mu_{jj} \Delta t + O(\Delta t) \tag{1}$$

and analogously, the probability of death by

$$N_j \delta_{jj} \Delta t + O(\Delta t)\,. \tag{2}$$

To the centre there is also attached an appropriate number of interconnecting channels realizing random emigration – immigration of particles between centres. Assume analogously to the above mentioned, that the probability of transition from the j-th to the i-th centre in that interval $(t, t + \Delta t)$ is

$$N_j D_{ji} \Delta t + O(\Delta t) . \tag{3}$$

In the above the values of D_{ji} give the intensities for emigration – immigration. The problem we intend to solve by means of our model conforms to the above description and references [1 to 4]. It is evident that the model will enable us to

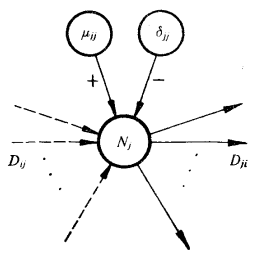

Fig. 1.

solve also problems with other than the quoted types of probability distributions. Thus for example the intensities may be constant, functions of the states of one or two interconnected centres, or finally functions of the state of particles or the whole network.

It is quite clear that by choosing a more complicated network, and considering the intensities as functions of the states of the individual centres or, alternatively, of the whole network, and the functions themselves of a not too simple character (nonlinearities, etc.) we can turn the model into a fairly universal device adaptable not only to studies of different methods of control but also to examinations of the changes in strategy and structure in dependence on time, various criteria, etc.

Since a mathematical solution of complicated networks is a very exacting requirement practically nearly to impossible to attain in some cases, the question of experimental solution by modelling is one of main interest. In what follows we shall describe a so-called hybrid model using a random process generator developed at the Institute of Information Theory and Automation, and a digital computer.

DESCRIPTION OF THE MODEL

As just mentioned, the model consists of the following two sections:

1. a complete source of information about the motion of particles (birth, death, emigration, immigration) in all centres; this first section thus generates "input data" for the second section;

2. a model of the network of all the centres, including the evaluating and output units.

1. Using the random process generator GENAP and its accessories for the source of random variables with exponential distribution

In the realization of random numbers with exponential distribution we shall also use a probability transformer in addition to the GENAP II generator and a pulse counter.

The GENAP II Random Process Generator serves to form a sequence of pulses of two types ("0" and "1") at chosen and regular intervals, the occurence of both types of pulses being random with the probability $P(0) = P(1) = 0{\cdot}5$ and independent of the preceding pulses. Thus the generator serves as a basic supply source with very precisely defined statistical parameters [cf. 5]. The probability transformer enables us to realize the random sequence of "0's" and "1's" where $P(0) = p$ and $P(1) = 1 - p$, and $p \ll 0{\cdot}5$. In such a sequence we shall ascertain the distribution of groups composed solely of pulses of type "1".

The probability that the length of the group N will be $N \leqq k - 1$ (where k is the number of all pulses "1" + "0") is

$$P(N \leqq k - 1) = \sum_{j=0}^{k-1} (1 - p)^j \, p = p \frac{1 - (1 - p)^k}{1 - (1 - p)} = 1 - (1 - p)^k = 1 - \mathrm{e}^{-\lambda k}$$

where

$$\lambda = -\lg (1 - p) .$$

If a pulse counter recording the lengths of groups N_i is connected behind the probability transformer, the individual numbers N_i form the values of random variables $\{\eta_i;\ i = 1, 2, 3, \ldots\}$ having the appropriate exponential distribution. As an example, Fig. 2 shows the histogram of distribution of 2 000 random numbers thus formed. The theoretical distribution curve is given there for the purposes of comparison.

The agreement with the theoretical distribution was tested using the Kolmogorov-Smirnov test (for 1000 random numbers N_i is $\lambda = 0{\cdot}60$; $K(\lambda) = 0{\cdot}1357$; $1 - K(\lambda) = 0{\cdot}8643$).

We shall now show how this method is used in the solution of our problem. Since the generator GENAP II is a physical source of *independent* random variables, random numbers N_i, too, are mutually independent. Consequently we can in this case replace the necessary number k of mutually independent random generators by a single generator and distribute its output successively into k channels.

This will, of course, result in sequences of random numbers with identical statistical

properties (the same intensity λ) in all k channels. We shall outline a simple method of changing intensity λ as desired. It is necessary for the solution of the problem on hand that the individual random generators (of outputs from the channels) should

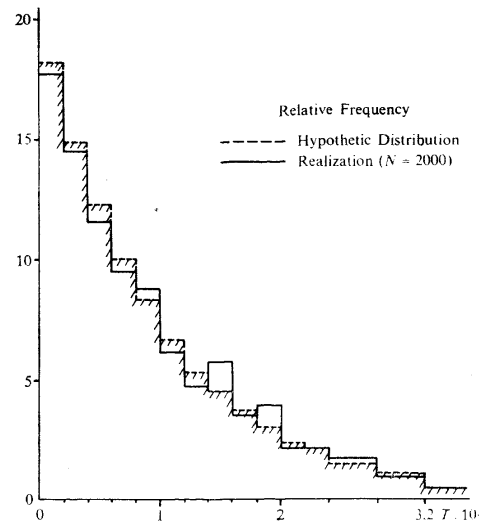

Fig. 2.

produce random variables on the time scale. If we plan to use in the next step a digital computer, we can effect this transformation as follows:

A) Pass from a continuous to a discrete model. This is not an essential restriction because the transition from the continuous time into discrete one with the increments Δt can be refined as required.

B) The proper transformation of numbers N_i to the random variable on the time scale consists in subtracting the chosen constant C from the number N_i at each step of the solution (Δt). The value of the new random variable η_i is the time $\tau_i = (N_i \Delta t)/C$ in which the value of number N_i becomes zero. This instant corresponds to the arrival of pulses from the originally considered generators; on its basis the centre makes the appropriate change, the computer finds the next random number N_{i+1} in the respective channel, and the process is repeated.

This conception has several advantages:

1. Any number of channels is adequately taken care of by a single generator and a single random independent sequence of numbers $\{N_i\}$.

2. The change of the intensity λ is effected by a simple method, i.e. by a change of the value of the substracted constant C, with the result that in different channels the random variable η has different intensity (and this holds equally for the time variables) even when the numbers N_i have identical statistical properties.

3. With no essential complications such cases can be considered in which the random variables η will have not only the exponential distributions but also even other distributions. In this case only the primary part containing the GENAP generator will be modified.

We have already taken up and solved this problem for the following types of distribution: normal, uniform and Rayleigh.

4. In the considered case, the use of pseudo-random sequences generated by the computer itself has not been found advantageous, as it involves first the transformation of the sequence of random variables with the uniform distribution to the exponential distribution, and then the distribution into the appropriate number of channels. The latter operation is rather difficult, however, mainly because of problems relating to the independence of pseudo-random numbers, and it is not clear enough – especially when dealing with a large number of channels – under what circumstances the input random variables will interact one with another as they do in our case. Testing such a multi-dimensional problem would be a very complicated affair indeed.

2. The digital computer model of the system

1. *The initial conditions of the problem, and the importance of their proper choice*

a) The number of centres is n(n is a number of the integer type, $n > 0$). The number of elements in the i-th centre in the time t is denoted by $N_i(t)$, ($N_i(t)$ is of the integer type, $N_i(t) \geqq 0$). $N_i(0)$, $i = 1, 2, 3, \ldots, n$ forms part of the initial conditions.

b) The realization of the control random variable η_i ($i = 1, 2, \ldots$) is placed in the input means No. 2 of computer. The variable η_i is characterized by the parameter λ. The generators g_i, ($i = 1, 2, \ldots, n^2 + n$) of time instants of the changes of state in the centres (i.e. of birth, death and migration of elements) are derivated from the control random variable. The generator g_i generates the random variable τ_i equal to the time interval between two realizations of the phenomenon. The intensity λ determines the basic time step $\Delta t = \lambda$ for the evaluation of the dependence of the state in the centres on time. Theoretically, the distribution function of the random variable τ_i has an exponential course (for example of the type $1 - e^{-\mu_{ii}N_i(t)t}$). It is approximated in our model by a step function with $1/\lambda$ segments per time unit. This step approximation is the closer, the smaller is the λ.

c) The probabilistic properties of the generators g_i ($i = 1, 2, \ldots, n^2 + n$) are described by equations (1), (2), (3). The quantities μ_{ii}, δ_{ii}, D_{ij} ($i, j = 1, 2, \ldots, n$,

$j \neq i$) are constant, non-negative numbers of the real type, forming part of the initial conditions. If we wish to solve a problem involving dependence of these constants on, say, time or on the number of elements in the centre, the dependence must be approximated to adequate accuracy by a step function, and to use in each part the presently described algorithmus. If a generator is missing in the network being modelled, the constant belonging to it is specified as zero.

A definite time sequence of "birth", "death" and "migration" of elements is always realized during the basic time step $\Delta t = \lambda$. Within the interval, however, the probabilistic properties of the system (see eqs. (1), (2), (3)) do not occure because of the fact that the characteristic coefficients $\mu_{ii} N_i(t)$, $\delta_{ii} N_i(t)$, $D_{ij} N_i(t)$ $(i, j = 1, 2, \ldots, n, i \neq j)$ are reduced all at once, after the "lapse" of the interval Δt. The exact instants of changes are recorded, however, and hence even several realizations of one phenomenon can take place during Δt. If the characteristic products are small enough, the error is but slight. It is possible simply to estimate the number of phenomena realized in Δt, and in turn the ensuing error, and on the basis of this estimate to choose a sufficiently small λ and the scale of coefficients μ_{ii}, δ_{ii}, D_{ij} $(i, j = 1, 2, \ldots, n, i \neq j)$.

The physical dimension (scale) of the constants μ_{ii}, δ_{ii}, D_{ij}, determines the dimension of the time unit of the result. Thus for example for constants in ms^{-1}, the unit of the result is in ms.

d) If all the coefficients are already specified, the size of the step Δt (but not the approximation of the distribution function of the variables τ_i) can be changed with the aid of the coefficient q (q is of the real type, $q > 0$) to $\Delta \tilde{t} = q \Delta t$. A shorter step ($q < 1$) is chosen whenever higher accuracy is called for; a longer step ($q > 1$) means faster calculation. The coefficient q plays an essential role in cases in which the step Δt needs to be changed automatically during the calculation (e.g. in the control of the frequency of occurrence of the number of phenomena during Δt), or in the case of multiple use of the algorithm to which we have referred earlier. The error arising owing to the occurrence of several phenomena during a time interval can be wholly eliminated by the extension of the algorithm. As the gain in accuracy resulting from the action is, however, completely overshadowed by the loss of speed, this alternative has not been considered by us so far.

e) The constant s is a number of the integer type, $s > 0$, stating the number of steps $\Delta t = \lambda$ the computer has to make in the given phase. After this the calculation is terminated.

f) The initial conditions are introduced in appropriate form into the input means of No. 1 of the computer in the following order:

n, q, s

$N_1(0), N_2(0), \ldots, N_n(0)$,

$\mu_{11}, \delta_{11}, 0, D_{12}, D_{13}, \ldots, D_{1,n-2}, D_{1,n-1}, D_{1,n}$,

$\mu_{22}, \delta_{22}, D_{21}, 0, \ldots, D_{2n}$,

$\vdots$

$\mu_{nn}, \delta_{nn}, D_{n1}, D_{n2}, \ldots, D_{n,n-1}, 0$.

2. *The resultant quantities and their evaluation*

In the course of computations, output device *1* prints the time as the product of the basic step of the time scale Δt, whose value equals $q\lambda$, as well as the states in all n centres in successive rows whenever a change is effected in one of the centres.

3. *The algorithm of the model in Algol*

The following algorithm was verified on the computer Elliot 4 100.

```
begin comment Variables according to description : n, q, s ... see points ad 2.1 — a), d), e), coeff ...
        probability coefficients (see c)), gener ... generators g_i (see b)), number ... N_i(t)
        (see a)). Time is measured with the variable "time". Working variables are write,
        help, i, k, and array nber [i];
    integer n;
    INTEGER (1, n);
    begin integer time, i, k, s; real q, help; Boolean write;
        array coeff, gener[1 : n, —1 : n];
        integer array number, nber[1 : n];
        INTEGER (1, q); INTEGER (1, s);
        inarray (1, number); inarray (1, coeff); inarray (2, gener);
        for i := 1 step 1 until n do nber[i] := number[i];
        write := true; time := 0;
    L1: if write then
        begin OUTINTEGER (1, time); outsymbol (1, '≡<', —1);
            for i := 1 step 1 until n do
            OUTINTEGER (1, number[i]); outsymbol (1, '≡<', —1);
            write := false
        end;
        time := time + 1; if time > s then go to L5;
        for i := 1 step 1 until n do
        begin if number[i] ≦ 0 then go to L2;
            for k := —1 step 1 until n do
            begin if i = k ∨ coeff[i, k] ≦ 0 then go to L3;
                help := gener[i, k] — coeff[i, k] × number[i] × q;
            L4: gener[i, k] := help; if help > 0 then go to L3;
                if k = —1 then nber[i] := nber[i] + 1
                else begin nber[i] := nber[i] — 1;
                if k > 0 then nber[k] := nber[k] + 1
                    end;
                inreal (2, help); help := help + gener[i, k];
                go to L4;
            L3: end;
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        L2: end;
            for i := 1 step 1 until n do
            begin if number[i] ≠ nber[i] then
                  begin if nber[i] ≦ 0 ∧ number[i] > 0 then
                        for k := —1 step 1 until n do inreal (2, gener[i, k]);
                        write := true; number[i] := nber[i]
                  end
            end; go to L1;
    L5: end
end of programe.
```

INTEGER and OUTINTEGER are procedures respectively for input and output of an integer number.

4. *Verification of the algorithm — an example*

Initial conditions: $n = 3$, $\lambda = 1/512$, $q = 1$, $s = 150$, $N_1(0) = N_2(0) = N_3(0) = 10$;

$$\mu_{11} = 10\,,\quad \delta_{11} = 5\,,\quad 0\,,\quad D_{12} = 2\,,\quad D_{13} = 2\,,$$
$$\mu_{22} = 5\,,\quad \delta_{22} = 10\,,\quad D_{21} = 2\,,\quad 0\,,\quad D_{23} = 2\,,$$
$$\mu_{33} = 10\,,\quad \delta_{33} = 10\,,\quad D_{31} = 2\,,\quad D_{32} = 2\,,\quad 0\,.$$

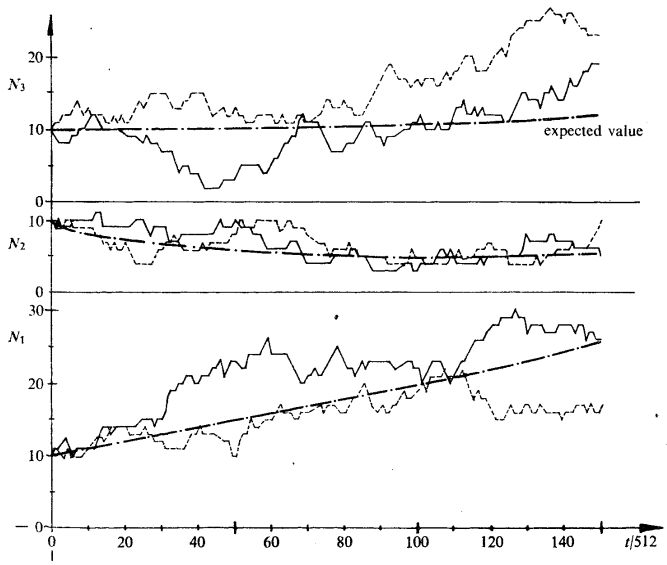

Fig. 3.

A graphical record of the resultant dependences for two different realizations of the control variable η_i with the same value of λ is shown in Fig. 3.

5. *Expected values*

For a given network and known intensities μ, δ, D, one can – according to [4] – calculate the so-called *expected values*, actually the mean values of the number of particles $(\bar{N})$ in the various centres in dependence on time.

The system of differential equations to be solved is of the following type:

$$\frac{\mathrm{d}\bar{N}_j}{\mathrm{d}t} = \left(\mu_{jj} - \delta_{jj} - \sum_{i \neq j} D_{ji}\right) \bar{N}_j + \sum_{k \neq j} D_{kj} \bar{N}_k \,.$$

When substituting in the above the appropriate values of μ, δ, D we get for our example the following system of equations:

$$\frac{\mathrm{d}\bar{N}_1}{\mathrm{d}t} = \bar{N}_1 + 2\bar{N}_2 + 2\bar{N}_3 \,,$$

$$\frac{\mathrm{d}\bar{N}_2}{\mathrm{d}t} = 2\bar{N}_1 - 9\bar{N}_2 + 2\bar{N}_3 \,,$$

$$\frac{\mathrm{d}\bar{N}_3}{\mathrm{d}t} = 2\bar{N}_1 + 2\bar{N}_2 - 4\bar{N}_3 \,.$$

The approximate expressions of $\bar{N}_1$, $\bar{N}_2$, $\bar{N}_3$ obtained with the aid of the Laplace-Wagner transformation are:

$$\bar{N}_1 = 10[1{\cdot}349\mathrm{e}^{2.28t} - 0{\cdot}07\mathrm{e}^{-9.83t} - 0{\cdot}28\mathrm{e}^{-4.45t}] \,,$$

$$\bar{N}_2 = 10[0{\cdot}333\mathrm{e}^{2.28t} + 0{\cdot}519\mathrm{e}^{-9.83t} + 0{\cdot}148\mathrm{e}^{-4.45t}] \,,$$

$$\bar{N}_3 = 10[0{\cdot}534\mathrm{e}^{2.28t} - 0{\cdot}158\mathrm{e}^{-9.83t} + 0{\cdot}624\mathrm{e}^{-4.45t}] \,.$$

To illustrate the degree of correspondence between the realizations measured on the model and the expected values, the latter are also drawn in Fig. 3. It is clear that the two sets of values are in good correspondence.

CONCLUSION

The problem to be dealt with in the modelling of networks composed of n stations is how to obtain $n^2 + n$ independent generators of random variables that determine the instants of the change of state. The use of pseudo-random generators is seriously restricted by unwarranted independence and labouriousness of the process of arriving at the sought results. Neither it is realistic in case of large n to plan on acquiring a sizeable number of stable physical generators and setting up an analogue model. The main advantage of the procedure just described lies in that satisfactory inde-

pendent generators are formed from a single, adequately stable physical generator by means of reduction effected on the digital computer that models the whole network. Confining the pertinent considerations to processes with the Poisson distribution as done in the paper, detracts nothing from the versatility of the solution.

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VÝTAH

Řešení některých problémů ze stochastické teorie konfigurací modelováním

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Článek popisuje některé možnosti modelování náhodných procesů množení, vymírání, emigrace a imigrace mezi několika zdroji – středisky, které vzájemně tvoří svým propojením určitou síť. Článek navazuje na teoretické práce akademika J. Kožešníka [1–4]. Pro vlastní model je uvažováno použití zařízení pro generování náhodných procesů (zejména generátor náhodného procesu GENAP II), vyvinutá v ÚTIA a číslicový počítač. Model poskytuje možnosti řešení i v některých těch případech, kdy nelze výpočet získat analytickou cestou. Vedle Poissonových náhodných procesů, které jsou uvažovány, lze v modelu užít i jiné typy náhodných procesů, lze uvažovat i složitější závislosti jednotlivých intenzit procesů zrodu, vymírání atd. na okamžitém stavu příslušného střediska. Závěrem článku je uveden algoritmus řešení obecné úlohy z [1], je přiložen program pro číslicový počítač a úloha je ilustrována konkrétním příkladem realizace.

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