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A Note on One-Sided Context-Sensitive Grammars

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In this note it is proved that so called one-sided context-sensitive grammars can generate languages which cannot be generated by any context-free grammar.

This fact is not quite new. It has been proved in [3], [4] and [5] (as far as the author knows). In [3] it is proved that a special one-sided context-sensitive grammar suggested by Dr. Friš ([1]) generates a language

$$\{a^m b^m c^n;\ 1 \leqq n \leqq m\}$$

which is not context-free.

In [5] an example of a one-sided context-sensitive grammar is given and in [4] there is proved, concerning this grammar, that it generates a well-known language

$$\{a^n b^n c^n;\ n \geqq 1\}\ .$$

The proofs given in [3] and [4] are rather complicated though the grammars in question contain about 20 rules only.

The aim of the present note is to give a simple proof of the above-mentioned statement.

Let us define a one-sided context-sensitive grammar $G = \langle V_T, V_N, R, S\rangle$ as follows:

$$V_T = \{a, b, c\}\ ,$$
$$V_N = \{A, B, C, D, E\}\ ,$$

R:

1. $S \to a\,a\,A\,B\,B\,c\,c\ ,$
2. $A \to a\,A\,B\ ,$
3. $A \to a\,b\ ,$
4. $b\ B \to b\ C\ ,$
5. $C\,B \to C\,C\ ,$
6. $b\ C \to b\ D\ ,$

$$\begin{aligned}
&7.\quad b\ D \to b\ b\,,\\
&8.\quad D\ C \to D\ B\,,\\
&9.\quad B\ C \to B\ B\,,\\
&10.\quad B\ C \to B\ B\ c\,.
\end{aligned}$$

We shall prove

$$L(G) = \{a^m b^m c^n;\ 1 < n < m\}\,, \tag{1}$$

$L(G)$ not being context-free. (It may be easily proved directly or derived from general theorems in [2].) In what follows $\overset{*}{\Rightarrow}$ (resp. $\Rightarrow$) denotes derivability (resp. immediate derivability) in G.

Assertion 1. *For any m, n, $1 < n < m$*

$$S \overset{*}{\Rightarrow} a^m b^m c^n\,.$$

Proof. For $m > 3$ and $2 \leqq i \leqq m - 2$ we have

$$a^m b^{i-1} B^{m-i+1} c^i \overset{*}{\Rightarrow} a^m b^i B^{m-i} c^{i+1}\,, \tag{2}$$

for

$$a^m b^{i-1} B^{m-i+1} c^i \Rightarrow a^m b^{i-1} C B^{m-i} c^i \overset{*}{\Rightarrow} a^m b^{i-1} C^{m-i+1} c^i \Rightarrow$$
$$\Rightarrow a^m b^{i-1} D C^{m-i} c^i \Rightarrow a^m b^{i-1} D B C^{m-i-1} c^i \overset{*}{\Rightarrow}$$
$$\overset{*}{\Rightarrow} a^m b^{i-1} D B^{m-i-1} C c^i \Rightarrow a^m b^i B^{m-i-1} C c^i \Rightarrow a^m b^i B^{m-i} c^{i+1}\,.$$

Suppose $1 < n < m$. Using (2) several times we obtain

$$S \Rightarrow a^2 A B^2 c^2 \overset{*}{\Rightarrow} a^m b B^{m-1} c^2 \overset{*}{\Rightarrow} a^m b^2 B^{m-2} c^3 \overset{*}{\Rightarrow} a^m b^{n-1} B^{m-n+1} c^n \Rightarrow$$
$$\Rightarrow a^m b^{n-1} C B^{m-n} c^n \Rightarrow a^m b^{n-1} D B^{m-n} c^n \Rightarrow a^m b^n B^{m-n} c^n \Rightarrow a^m b^n C B^{m-n-1} c^n \Rightarrow$$
$$\Rightarrow a^m b^n D B^{m-n-1} c^n \Rightarrow a^m b^{n+1} B^{m-n-1} c^n \overset{*}{\Rightarrow} a^m b^m c^n\,.$$

In order to prove (1), we need the following

Assertion 2. *If*

$$S = x_0 \Rightarrow x_1 \Rightarrow \ldots \Rightarrow x_p \in V_T^*\,, \tag{3}$$

then there are m, n $(1 < n < m)$ such that $x_p = a^m b^m c^n$.

Proof. Let a derivation (3) of grammar G be given. There are i and j $(0 < i < p$, $2 < j)$ such that x_i in (3) is of the form $a^j b B^{j-1} c^2$. Actually, the only rule which can be applied to x_0 is the rule 1 whose application results in $x_1 = a^2 A B^2 c^2$. To x_1 only the rules 2 or 3 can be applied. The application of the rule 3 yields x_2 of the desirable form ($i = 2$), the rule 2 results in $x_2 = a^3AB^3c^2$ to which only rules 2 or 3 may be applied again. The repeated application of the rule 2 yields strings of the form $a^nAB^nc^2$ ($n > 3$) and cannot result in the terminal string $x_p \in V_T^*$, hence the rule 3 has to be applied at least once. The first (and only possible) application of the rule 3 results in x_i of the desirable form.

Lemma. *If $j > 2$ and $a^jbB^{j-1}c^2 \overset{*}{\Rightarrow} \eta$, then either there is an occurrence of the string cB resp. cC in η, or*

$$\eta = a^jb^k\tilde{D}\varphi c^l, \tag{4}$$

where $k > 0$, $l \geqq 2$, $\tilde{D}$ is either empty or $\tilde{D} = D$, φ is a string (maybe empty) built of B and C, $|b^k\tilde{D}\varphi| = j$ and if we denote by $\gamma(\tilde{D}\varphi)$ the number of distinct occurrences of strings BC and DC in $\tilde{D}\varphi$ (with the only exception: we put $\gamma(DC) = 0$), then

$$\operatorname{Max}(|\varphi| - 2, 0) + \gamma(\tilde{D}\varphi) + l < j\,. \tag{5}$$

Note. Assertion 2 can be easily derived from the lemma: in (3) we have

$$S = x_0 \Rightarrow \ldots \Rightarrow x_i = a^jbB^{j-1}c^2 \Rightarrow \ldots \Rightarrow x_p\,.$$

There are no occurrences of cB (resp. cC) in x_p, $\tilde{D}$ and φ are empty, therefore $x_p = a^jb^jc^l$; (5) yields $l < j$.

Proof. We shall prove the lemma by induction on the length of the derivation of η.

I. A string $a^jbB^{j-1}c^2$ is obviously of the needed form.

II. Suppose $a^jbB^{j-1}c^2 \overset{*}{\Rightarrow} \eta \Rightarrow \eta'$; we shall prove the statement of the lemma for η' assuming it valid for η.

If there are occurrences of cB or cC in η, then such occurrences are in η', too (this may be easily seen from the set of rules). Suppose that η is of the form (4); let us investigate all possible cases generating η' from η:

a) the rule 4 is applied to η; in this case $\tilde{D} = \Lambda$, $\varphi = B\varphi_1$, hence $\eta' = a^jb^kC\varphi_1c^l$, $|b^kC\varphi_1| = j$ and (5) holds, since γ did not increase;

b) the rule 6 is applied to η ($\tilde{D} = \Lambda$, $\varphi = C\varphi_1$); $\eta' = a^jb^kD\varphi_1c^l$, the length of φ decreased (-1), γ, if changed, increased ($+1$), hence the inequality (5) remains valid. The case $\varphi = CC$ requests a special consideration: $\gamma(DC) = 0$ and (5) holds;

c) the rule 7 is applied to η, then $\tilde{D} = D$, (5) holds;

d) the rule 5 is applied to η; it does not affect $|\varphi|$, γ does not increase, (5) holds, too.

It may be easily seen that η' is of the desirable form also when the rule 8 or 9 is applied to η.

e) The rule 10 is applied to η; there are two possibilities to be considered. Either it is applied to an occurrence of BC which is immediately followed by B or C, then η' contains an occurrence of cB resp. cC; or the rule 10 is applied to the last two symbols of the string φ, γ decreases (-1), l (i.e. the number of c's) increases $(+1)$.

No rule of $1-3$ may be applied to η. The lemma, Assertion 2 and also (1) are proved.

The main problem concerning one-sided context-sensitive grammars is that of comparison of generative power of such grammars and context-sensitive grammars in a usual sense.

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VÝTAH

Poznámka o jednostranně kontextových gramatikách

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V práci se dokazuje, že tzv. jednostranně kontextové gramatiky, které jsou v Chomského klasifikaci mezi typy 2 a 1 (tj. mezi gramatikami bezkontextovými a gramatikami kontextovými), mohou generovat více než jen bezkontextové jazyky. Všechna pravidla jednostranně kontextové gramatiky jsou tvaru $\varphi A \to \varphi\omega$, kde $\varphi \in V^*$, $A \in V_N$, $\omega \in V^* - \{\Lambda\}$. Sestrojuje se jednostranně kontextová gramatika o 10 pravidlech a dokazuje se o ní, že generuje jazyk $\{a^m b^m c^n; 1 < n < m\}$, který není bezkontextový.

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