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The Applicability Limitation of Identification Methods

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The paper deals with the connection between the solution of identification problems and the accuracy of experimental data and the type of substitute functions describing the analysed systems. The applicability limitation of identification methods is analysed by means of the confidence intervals of transfer function coefficients describing the investigated system. Confidence intervals are derived from the identification method based on regression analysis.

1. INTRODUCTION

Most of the identification methods described in literature are based on the assumption that initial data are made available with sufficient accuracy. The question should now be posed, how to understand the "sufficiency" of accuracy of initial data. It is a known fact that the results of system identification are always adversely affected by "uncertainty" caused by noise. We shall thus endeavour to derive the "existence theorem", i.e. to determine the accuracy of initial data necessary for the existence of the solution of the identification problem for the given class of functions. It should be emphasized that this "existence theorem" is to be understood in the sense of quantitative considerations, i.e. for the given accuracy of initial data. In the sense of qualitative considerations it is natural that there always exists a solution of the identification according to the concept defined furtheron.

2. BASIC RELATIONS

Systems will be discussed which can be described by transfer functions of the type

$$F_s(p) = \frac{x(p)}{y(p)} = \frac{M(p)}{N(p)} = \frac{\sum_{i=0}^{m} b_i p^i}{\sum_{i=0}^{n} a_i p^i} \tag{1}$$

where $x(p)$ is the Laplace-Wagner transform of the output signal, and $y(p)$ the LW transform of the input signal. The poles and zeros of transfer function (1) lie in the left-hand half plane ($m < n$).

The time curves pertinent to transfer function (1) are (for non-multiple roots and for the curve shifted so that $x(\infty) = 0$)

(2) $$x(t) = \sum_{i=1}^{n^*} C_i e^{p_i t},$$

where p_i are the roots of the denominator of transfer function (1), or the roots of the denominator of the transform determining the form of the input signal.

Let us formulate relations for the identification of systems described by transfer function (1). Let us start from response curves $x(t)$ to the given input signal (e.g. unit step). Three response curves will be considered:

$x(t)$ – ideal response,
$\tilde{x}(t)$ – measured response,
$\bar{x}(t)$ – substitute response calculated by the approximation (evaluation) of response $\tilde{x}(t)$.

Let us write the relations for the distances of the above responses in space $L_2(0, T)$. Responses x, $\bar{x}$ are elements of subspace $M(0, T)$ of type (2) functions. The distance between the ideal and measured responses is given by the relation

(3) $$\delta = \left\{\int_0^T [x(t) - \tilde{x}(t)]^2 \, dt\right\}^{1/2} = \|x - \tilde{x}\| .$$

The length of the time interval will be chosen so that there is a negligibly small difference of norms in spaces $L_2(0, \infty)$ and $L_2(0, T)$.

Identification, more strictly speaking is based on the minimization of the distance between the measured and substitute response

(4) $$\varepsilon = \|\bar{x} - \tilde{x}\| .$$

The correctness of the identification result is decisively determined by the third distance, that between the ideal and substitute response

(5) $$\eta = \|\bar{x} - x\| .$$

Further on the following errors will thus be considered:

δ – error of measurement,
ε – error of identification,
η – error of approximation.

We can also write the relations for the distances of the above responses in space $l_2(0, T = t_r)$ (for the discrete time responses) [2]. For instance, the distance between two vectors $\mathbf{x}$ and $\tilde{\mathbf{x}}$ is

(6) $$\|\mathbf{x} - \tilde{\mathbf{x}}\| = E^{1/2}\{(\mathbf{x} - \tilde{\mathbf{x}})^{\mathrm{T}} (\mathbf{x} - \tilde{\mathbf{x}})\},$$

where

$$\mathbf{x}^{\mathrm{T}} = [x(t_1), x(t_2), \ldots, x(t_j), \ldots, x(t_r)],$$

and

$$\tilde{\mathbf{x}}^{\mathrm{T}} = [x(t_1), \tilde{x}(t_2), \ldots, \tilde{x}(t_j), \ldots, \tilde{x}(t_r)].$$

Measured signal $\tilde{x}(t)$ represents a random function

(7) $$\tilde{x}(t) = x(t) + n(t).$$

Let us assume that $n(t)$ is the realization of the stationary and ergodic Gaussian random process with zero mean value and dispersion σ^2. The basic mean error of measurement is determined by the relation

(8) $$\bar{\delta}^2 = \lim_{q \to \infty} \frac{1}{q} \sum_{j=0}^{q} \int_0^T n_j^2(t)\, \mathrm{d}t.$$

Before embarking on the analysis of errors according to relations (3) to (5) let us indicate the procedure of identification on the basic of smoothing calculus. This method has been selected because it permits the joining of the individual constituents of the identification process as described by relations (3) to (5).

Let us consider the substitute (regression) function of $\bar{x}(t)$ in the following form (for the known numerator $M(p)$ in transfer function (1), e.g. $M(p) = 1$):

(9) $$\bar{x}(t) = \sum_{i=0}^{n} \bar{a}_i\, v^{(i)}(t),$$

where

$$L\{v^{(i)}(t)\} = \frac{p^i}{N^2(p)}.$$

The suitability of the form of substitute function $\bar{x}(t)$ follows from the relation

(10) $$L\{\bar{x}(t)\}\Big|_{\bar{a}_i = a_i} = \frac{\sum_{i=0}^{n} a_i p^i}{N^2(p)} = \frac{N(p)}{N^2(p)} = \frac{1}{N(p)} = L\{x(t)\}.$$

Functions $v^{(i)}(t)$ were derived in reference paper [6] in connection with deriving the linearized functionals of the sensitivity of coefficients $\bar{a}_i$.

(11) $$\frac{\partial E}{\partial \bar{a}_i} \doteq 2(\bar{a}_i - a_i)\, \|v^{(k)}\|^2,$$

where $E = \eta^2$.

Superscript k is determined by the form of the input signal [6]. For the Laplace-Wagner transform and for unit-step input signal $k = i$.

Let us assume that functions $v^{(i)}(t)$ are known and that these functions do not depend on coefficients $\bar{a}_i$. The best unbiased estimate of $\bar{x}(t)$ is then determined from the condition of

$$\sum_{j=1}^{r} [\tilde{x}(t_j) - \sum_{i=0}^{n} \bar{a}_i \, v^{(i)}(t_j)]^2 = \min . \tag{12}$$

The calculation of the estimates of coefficients $\bar{a}_i$ from condition (12) can be described by the matrix equation

$$\boldsymbol{A}\bar{\boldsymbol{a}} = \boldsymbol{\alpha} , \tag{13}$$

where $\boldsymbol{A} = \boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}$ and $\boldsymbol{\alpha} = \boldsymbol{G}^{\mathrm{T}}\tilde{\boldsymbol{x}}$.

Matrix $\boldsymbol{G}^{\mathrm{T}}$ can be written in the form

$$\boldsymbol{G}^{\mathrm{T}} = \begin{vmatrix} v^{(0)}(t_1) & v^{(0)}(t_2) & \dots & v^{(0)}(t_j) & \dots & v^{(0)}(t_r) \\ v^{(1)}(t_1) & v^{(1)}(t_2) & \dots & v^{(1)}(t_j) & \dots & v^{(1)}(t_r) \\ \vdots & \vdots & & \vdots & & \vdots \\ v^{(n)}(t_1) & v^{(n)}(t_2) & \dots & v^{(n)}(t_j) & \dots & v^{(n)}(t_r) \end{vmatrix}$$

and matrix

$$\tilde{\boldsymbol{x}}^{\mathrm{T}} = |\tilde{x}(t_1), \tilde{x}(t_2), \dots, \tilde{x}(t_j), \dots, \tilde{x}(t_r)| .$$

The estimates of coefficients $\bar{a}_i$ are now determined by the relation

$$\bar{\boldsymbol{a}} = \boldsymbol{A}^{-1}\boldsymbol{\alpha} = (\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G})^{-1} \boldsymbol{G}^{\mathrm{T}}\tilde{\boldsymbol{x}} \tag{14}$$

where $\boldsymbol{A}^{-1}$ is an inverse matrix.

Let us calculate the deviation of the estimates of coefficients $\bar{a}_i$. The covariance matrix is given by the following relation [2]

$$\begin{aligned} \boldsymbol{\Psi} &= E\{(\bar{\boldsymbol{a}} - \boldsymbol{a})(\bar{\boldsymbol{a}} - \boldsymbol{a})^{\mathrm{T}}\} = \\ &= E\{[(\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G})^{-1} \boldsymbol{G}^{\mathrm{T}}\tilde{\boldsymbol{x}} - \boldsymbol{a}] [(\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G})^{-1} \boldsymbol{G}^{\mathrm{T}}\tilde{\boldsymbol{x}} - \boldsymbol{a}]^{\mathrm{T}}\} = \\ &= (\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G})^{-1} \boldsymbol{G}^{\mathrm{T}} E\{\boldsymbol{nn}\} \boldsymbol{G}(\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G})^{-1} . \end{aligned} \tag{15}$$

In practical problems of the smoothing calculus the matrix of noise $E\{\boldsymbol{nn}\}$ is approximated by the residual sum of squares

$$\boldsymbol{\Psi} \doteq (\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G})^{-1} \frac{S_0}{r - n - 1} . \tag{16}$$

The procedure of identification just described here has, however, one shortcoming. The assumption of functions $v^{(i)}(t)$ being known is fulfilled only for a sufficiently small $|\bar{a}_i - a_i|$. Consequently no linear regression function is represented by relation (9). Function $v^{(i)}(t)$ depends on the sought for coefficients $\bar{a}_i$.

The subject matter of this paper is the problem of reliable estimates. It is not concerned with designing a method of identification. Thus we can start from the modified condition (12)

$$\|\tilde{x}(t) - \sum_{i=1}^{n} a_i \, v^{(i)}(t)\|^2 = \min . \tag{17}$$

The formerly described identification process will be reversed. It will be assumed that the transfer function of the system and sufficiently accurate estimate of basic mean error of measurement $\hat{\delta}$ are known and investigation will be directed towards the influence of the measurement error, and of the type of function describing the system, on the dispersion of coefficients. In this way we shall bypass both difficulties encountered in the previous procedure, i.e. the evaluation of the noise matrix $E\{\mathbf{nn}\}$, and the non-linearity of the regression function. Moreover, for a sufficiently fine division of time it will be possible to pass over to continuous functions $v^{(i)}(t)$. Under these assumptions the covariant matrix will be determined by relation

$$\boldsymbol{\Psi} = \hat{\delta}^2 \, \mathbf{A}^{-1} . \tag{18}$$

where $\hat{\delta}^2 = T\hat{\sigma}^2$ and $\mathbf{A}^{-1} = [\alpha^{hi}]$.

The elements of inverse matrix α^{hi} can be calculated from

$$\alpha^{hi} = \frac{|A_{hi}|}{|A|} , \tag{19}$$

where $|A_{hi}|$ is the complement of element α_{hi} in matrix $\mathbf{A}$ and $|A|$ is the determinant of matrix $\mathbf{A}$. Elements α_{hi} of matrix $\mathbf{A}$ are determined by relation

$$\alpha_{hi} = (v^{(h)}, v^{(i)}) \quad (h, i = 0, 1, 2, \ldots, n) .$$

The confidence measure of the estimates of coefficients $\bar{a}_i$ will be decided upon according to the relation

$$D(\bar{a}_i) = \hat{\delta}^2 \alpha^{ii} . \tag{20}$$

It is thus possible to discuss the confidence measure (or the confidence intervals) of the estimates of coefficients separately from the influence of measurement accuracy (characterized by the estimate of the basic mean measurement error $\hat{\delta}$) and from the influence of the type of function describing the system (determined by elements α^{hi} or the norms of functions $v^{(i)}(t)$). Into error $\hat{\delta}$ we concentrate all data concerning the accuracy of initial data including the number of measured points, or the number of the realization of responses $\tilde{x}(t)$. In practice the measurement error cannot be reduced without limit. We shall consider this error, or the corresponding deviation $\hat{\sigma}$, only to decimal places where it can be assumed that no systematic error is involved. Identification results are thus always burdened by some error.

Before discussing the analysis of the applicability limitation of identification methods of some selected classes of functions $x(t)$, let us derive the interrelations of errors in measurement, approximation and identification.

3. THE ORDER OF THE SUBSTITUTE FUNCTION

The method of least squares is used for finding the shortest distance between vectors $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$. Let us start from the theorem of vector projection [2] and consider the vector $\bar{\boldsymbol{x}} \in m$, where m is the subspace of vectors pertinent to functions of type (2).

The distance $\|\tilde{\boldsymbol{x}} - \bar{\boldsymbol{x}}\|$ will be minimum, if $\bar{\boldsymbol{x}}$ is the projection of $\tilde{\boldsymbol{x}}$ onto m.

This follows from decomposition theorem [2]. Random vector $\tilde{\boldsymbol{x}}$ can be decomposed into the sum of vectors

$$\tilde{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{n}\,, \tag{21}$$

where $\boldsymbol{x} \in m$ and $\boldsymbol{n} \perp m$.

Thus it is possible to write

$$\|\tilde{\boldsymbol{x}} - \bar{\boldsymbol{x}}\|^2 = E\{(\tilde{\boldsymbol{x}} - \bar{\boldsymbol{x}})^{\mathrm{T}} (\tilde{\boldsymbol{x}} - \bar{\boldsymbol{x}})\} = \tag{22}$$
$$= E\{[(\tilde{\boldsymbol{x}} - \boldsymbol{x}) + (\boldsymbol{x} - \bar{\boldsymbol{x}})]^{\mathrm{T}} [(\tilde{\boldsymbol{x}} - \boldsymbol{x}) + (\boldsymbol{x} - \bar{\boldsymbol{x}})]\}\,.$$

Since $\tilde{\boldsymbol{x}} - \boldsymbol{x} = \boldsymbol{n}$, and $\boldsymbol{n} \perp m$, relation (22) can be written in the form

$$\|\tilde{\boldsymbol{x}} - \bar{\boldsymbol{x}}\|^2 = \|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|^2 + \|\boldsymbol{x} - \bar{\boldsymbol{x}}\|^2 \tag{23}$$

or in the form of inequity

$$\|\tilde{\boldsymbol{x}} - \bar{\boldsymbol{x}}\| \geqq \|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|\,. \tag{24}$$

Equation (24) indicates that the confidence measure calculated according (20) will be the maximum attainable one.

Let us further examine the relationship between the measurement error and the error of approximation. The error of approximation closely depends on the choice of the type or the order of the substitute function. The method of least squares offers the minimum solution for the given order of the substitute function. However, no criterion exists for the comparison of two minimum solutions for different orders of substitute functions. From the viewpoint of the approximation of functions it can be stated that the order of the substitute function must be adequate to the measurement error. This means that the estimates of coefficients must characterize the investigated system and not the properties of noise.

Let the problem be analyzed by means of the zero hypothesis [1]. Consider a substitute function $\bar{x}(t)$ of the n-th order. According to the assumption made in section 2, the estimate of coefficient $\bar{a}_n$ has the normal distribution

$$\bar{a}_n = a_n + e_n \,, \tag{25}$$

where e_n is a random parameter. The question is now posed, whether coefficient a_n differs from zero. Let us state that

$$a_n = 0 \,. \tag{26}$$

If the zero hypothesis holds good, the estimate of coefficient $\bar{a}_n$ is determined by measurement errors only

$$\bar{a}_n = e_n \,. \tag{27}$$

Let us limit the confidence interval for a standard variable $\tau = (a_n - \bar{a}_n)/\delta_n$

$$P\{a_n - \tau\delta_n < \bar{a}_n < a_n + \tau\delta_n\} = \Gamma(\tau) \tag{28}$$

or

$$P\{\bar{a}_n - \tau\delta_n < a_n < \bar{a}_n + \tau\delta_n\} \geqq \Gamma(\tau) \,,$$

where $\delta_n = [D(\bar{a}_n)]^{1/2}$ and $\Gamma(\tau)$ is the confidence coefficient. It is the probability that the estimate of parameter $\bar{a}_n$ will be within the given limits. For instance, the estimate of parameter $\bar{a}_n$ will be within the limits of $\pm\delta_n$ (for $\tau = \pm 1$) round the correct value of a_n with the probability of $P = 68\%$.

Let us revert to the zero hypothesis. For the assessment of the zero hypothesis we shall use the following rule [1]:

If it holds that (for $\tau = \pm 1$)

$$|\bar{a}_n| < \delta_n \tag{29}$$

the zero hypothesis will be accepted. Coefficient a_n either equals zero, or the measurement was not sufficiently accurate to establish its value.

If the zero hypothesis is assessed according to the selective mean error, it is necessary to apply Student's t-distribution.

The assessment of the applicability limitation of identification methods will be based on relations (20) and (29). The confidence measure of the last coefficient a_n (or several last coefficients) will be assessed on the assumption that all the other coefficients are correct. In fact, in this way the confidence measure of the order of the substitute function will be assessed. According to relation (20) we obtain

$$D(\bar{a}_n) = \hat{\delta}^2 \hat{\alpha}^{nn} = \delta_n^2 = \frac{\hat{\delta}^2}{\|v^{(n)}\|^2} \,. \tag{30}$$

In this case the inverse matrix $\mathbf{A}^{-1}$ has only one element $\hat{\alpha}^{nn} = 1/\|v^{(n)}\|^2$. The dependence of the mean error of the estimate of coefficient $\bar{a}_n$ on the errors of other coefficients will thus be disregarded. It will be shown that this is just the case of the "minimum" mean error of coefficient $\bar{a}_n$.

Determinant $|A|$ is Gram's determinant [3]. For these determinants Hadamard's inequity holds, i.e.

$$|A| \leqq \prod_{k=0}^{z} |A_{kk}| , \tag{31}$$

where $|A_{kk}|$ are diagonal square blocks obtained by the division of matrix $\mathbf{A}$. By using relation (31) equation (19) can be rewritten as follows:

$$\alpha^{nn} = \frac{|A_{nn}|}{|A|} \geqq \frac{|A_{nn}|}{|A_{nn}| \, \|v^{(n)}\|^2} = \frac{1}{\|v^{(n)}\|^2} = \hat{\alpha}^{nn} . \tag{32}$$

4. SYSTEMS OF THE SECOND ORDER

The application of the relations derived will be demonstrated in the analysis of systems of the second order with real poles. These systems can be described by the transfer function (for $a_0 = b_0 = 1$)

$$F_s(p) = \frac{1}{(T_1 p + 1)(T_2 p + 1)} = \frac{1}{a_2 p^2 + a_1 p + 1} = \frac{1}{a_1^2 \varkappa p^2 + a_1 p + 1} , \tag{33}$$

where $\varkappa = a_2 / a_1^2$.

Let us first derive the relative mean error of the estimate of coefficients $\bar{a}_n$

$$s_n = \frac{\delta_n}{a_n} = \frac{\hat{\delta}(\alpha^{nn})^{1/2}}{a_n} . \tag{34}$$

According to rule (29) for the assessment of the zero hypothesis, for $s_n > 1$ no reliable estimate exists for the system of the n-th order. Relation (34) indicates whether there exists a solution of identification for the given function $x(t)$, i.e. whether it is possible to derive a sufficiently reliable estimate of the function of n-th order from the measured function $\tilde{x}(t)$ with the measurement error given.

The measurement error will be chosen as

$$\hat{\delta}^2 = T\hat{\sigma}^2 = T \,.\, 10^{-4} = 2a_1 \,.\, 10^{-4} . \tag{35}$$

In connection with relation (35) let it be emphasized that the estimate of accuracy, i.e. the estimate of deviation $\hat{\sigma}$ (or of the measurement error $\hat{\delta}$) has never been, and apparently never will be, solved unequivocally. It can be generally stated that small sets of measured data normally do not comply with the Gaussian distribution. On the other hand, large sets do not comply with the requirement of stationarity. In identification problems an important role will be played by the accuracy and presicion of instruments used (transmitters, recorders), the accuracy of the evaluation of data (large set of values will be normally handled), and finally the internal noise of the investigated system. Instruments (transmitters) and parasite disturbances (caused, e.g. by the unsuited characteristics of control elements) will be the primary sources of systematic errors. From the quoted adverse effects it is possible only to estimate within a narrow band the accuracy of transmitters ($\sigma^* = 0{\cdot}005$ to $0{\cdot}01$). With regard to these values it can be stated that results given in relation (35) can be regarded as the lower limit of the measurement error in the identification of industrial plants.

Norms $\|v^{(i)}\|$ are calculated for an infinite integration interval according to the algorithm in reference [5]. For a sufficiently large integration interval, e.g. $T = 8a_1$, we may write (with an error of the order of 10^{-3})

$$\|v^{(i)}\|_{T\to\infty} \doteq \|v^{(i)}\|_{T=8a_1}\,. \tag{36}$$

The integration interval of the noise is smaller as the integration interval of the useful signal. We consider so the favourable alternative.

The relative error of the estimate of coefficient $\bar{a}_2$ (calculated for a response to unit step according to relation (34)) depends only on the dimensionless parameter $\varkappa$, i.e.

$$^0s_2 = \frac{0{\cdot}02\sqrt{2}}{\varkappa}\,. \tag{37}$$

For the relative error calculated from responses to unit impulses the following relation is obtained:

$$^1s_2 = 0{\cdot}02\, a_1 \sqrt{\frac{2}{\varkappa}}\,. \tag{38}$$

Further on only two forms of input signal are considered and denoted by a superscript on the left. The unit step input signal carries the superscript 0, and unit impulse input signal is denoted by superscript 1.

Norms $\|v^i\|$ or the functionals of sensitivity were linearized at the point $\bar{a}_n = a_n$. But we can linearize at the point $\bar{a}_n = 0$. This point is decisive by the assessment of the confidence measure of the order of the substitute function. In this case the error of approximation is determined by the relation

$$\|x - \bar{x}\|^2 = a_n^2\|\bar{v}^{(n)}\|^2 = \int_0^T \left[L^{-1}\left\{\frac{M(p)\,a_n}{N(p)\,\bar{N}(p)}\right\}\right]^2 \mathrm{d}t\,, \tag{39}$$

where $\bar{N}(p)$ is the denominator of the transfer function pertinent to the substitute response $\bar{x}(t)$. For coefficients $\bar{a}_0$ to $\bar{a}_{n-1}$ it holds that $\bar{a}_i = a_i$ and for coefficients $\bar{b}_0$ to $\bar{b}_m$ it holds that $\bar{b}_i = b_i$.

By using relation (39) and the algorithm in reference [5] the following relations are obtained

$$(40) \qquad {}^{0}\bar{s}_2 = \frac{0{\cdot}02\sqrt{(2 + \varkappa)}}{\varkappa}\,,$$

$$ {}^{1}\bar{s}_2 = 0{\cdot}01\, a_1 \sqrt{\frac{2(2 + \varkappa)}{\varkappa}}\,.$$

It follows from (37) that coefficient $\bar{a}_2$ cannot be, at the given accuracy $\hat{\delta}$, determined for $\varkappa \leqq 0{\cdot}0282$ (i.e. for the ratio of time constants $\lambda = (T_1/T_2) \geqq 33{\cdot}5$). From the practical viewpoint it is obvious that systems with $\varkappa \leqq 0{\cdot}0282$ can be well approximated by systems of the first order.

Now, the question is naturally posed how to assess the quality, or more exactly the sufficiency of approximation in more complex systems. Without the criterion of the assessment of the quality of approximation it is impossible to assess the results of system identification. The development of such a criterion is the subject matter dealt with in the next section.

5. THE APPROXIMATION INDEX

In reference [6] identification is formulated as the problem of the approximation of functions. On the basis of the functionals of the sensitivity of transfer function coefficients it was shown that identification is closely connected with problems of synthesis. On the basis of reference papers [6, 7] we shall derive the criterion for the assessment of the quality of approximation. Let us first look at the assessment of the order of the substitute function. Let us start from the transfer function of a control loop composed of systems with real poles and from a proportional controller (for disturbances at the input to the system)

$$(41) \qquad F_z(p) = \frac{b_0}{a_n p^n + a_{n-1} p^{n-1} + \ldots + a_1 p + a_0 + K}\,.$$

References [6, 7] showed the importance of critical gain coefficient in judging the approximation of transfer functions of systems of the type considered. The significance of coefficients a_i will therefore be assessed by the approximation index

$$(42) \qquad I_j = \frac{|\Delta K_n + \Delta K_{n-1} + \ldots + \Delta K_j|}{K_k}\,,$$

where K_k is the critical gain coefficient and $\Delta K_n, \ldots, \Delta K_j$ can be calculated as the differences of the critical gain coefficients of transfer functions formed by the gradual omission of the highest coefficients up to the j-th one.

In the approximation of transfer functions of type (41) the importance of coefficients a_i is given by their approximation index. This follows from the development of transfer function (41) into exponential series (for $p = 0$) (or from frequency transfer functions or directly from frequency characteristics). Let us write the exponential development for the inverse value $1/F_z(p)$:

$$(43) \qquad \frac{1}{F_z(p)} = \sum_{i=0}^{n} \varphi_i p^i = \frac{1}{b_0} (a_0 + K + a_1 p + \ldots + a_{n-1} p^{n-1} + a_n p^n) .$$

The approximation error obtained by means of development (43) is the smaller, the lower is the approximation index of the omitted coefficients. Naturally, this applies to coefficients of the highest exponentials of the complex variable p according to relation (42). For instance, the approximation index of coefficient a_4 of a system of the fourth order equals

$$(44) \qquad I_4 = I_{a_4} = \frac{a_4 \omega_k^2}{K_k} ,$$

where ω_k is the critical frequency.

Coefficients a_3 of systems of the third order (and naturally of all higher orders) have the largest approximation index

$$(45) \qquad I_3 = \frac{\infty}{K_k} = \infty .$$

From the viewpoint of the approximation index according to relation (42), systems of the third order can be regarded as the touchstone of identification methods. They are the simplest systems with transfer functions of type (1) with a constant in the numerator, which cannot be approximated by simpler systems, and at the same time, they are not trivial from the viewpoint of synthesis problems.

6. SYSTEMS OF THE THIRD ORDER

In this section conditions will be investigated under which there exists a reliable solution of the identification of systems of the third order according to relation (29) for the practically important interval of critical gain coefficient $K_k \in \langle 8, 60 \rangle$. Let us start from the transfer function (for $b_0 = a_0 = 1$)

$$(46) \qquad F_s(p) = \frac{1}{\prod_{k=1}^{3} (T_k p + 1)} = \frac{1}{a_3 p^3 + a_2 p^2 + a_1 p + 1} =$$

$$= \frac{1}{\varrho \varkappa a_1^3 p^3 + \varkappa a_1^2 p^2 + a_1 p + 1} ,$$

where

$$\varrho = \frac{1}{1 + K_k} = \frac{a_3}{a_1 a_2}.$$

For the measurement error according to relation (35), and for the approximation error according to relation (39), let us calculate the relative least square error of the estimate of coefficient $\bar{a}_3$ (for $\bar{a}_1 = a_1$, $\bar{a}_2 = a_2$)

$$(47) \qquad {}^0\bar{s}_3 = \frac{\|x - \tilde{x}\|}{\|x - \bar{x}\|} = \frac{\hat{\delta}}{a_3\|\bar{v}^{(3)}\|} = \frac{10^{-2}(2a_1)^{1/2}}{a_3\|\bar{v}^{(3)}\|},$$

$$(48) \qquad {}^1\bar{s}_3 = \frac{\hat{\delta}}{a_3\|\bar{v}^{(4)}\|} = \frac{10^{-2}(2a_1)^{1/2}}{a_3\|\bar{v}^{(4)}\|}.$$

Norms $\|\bar{v}^{(i)}\|$ are calculated for an infinite integration interval according to the algorithm in reference [5].

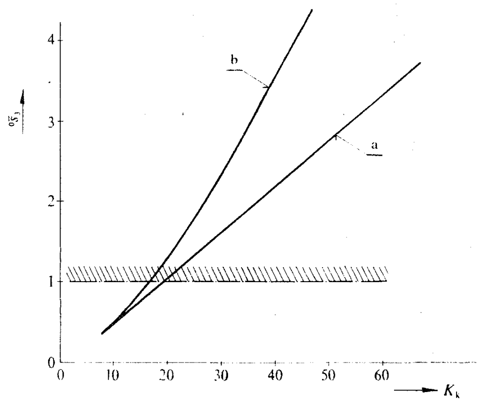

Fig. 1. Relative mean error of the estimates of coefficients $\bar{a}_3$ in systems of the third order (measurement error $\hat{\delta} = (2a_1)^{\frac{1}{2}} . 10^{-2}$, calculated from responses to unit step).

The boundary of reliable solutions of identification are represented in Figs. 1 and 2 (${}^0\bar{s}_3 = {}^1\bar{s}_3 = 1$). The drawing shows only curves pertinent to the majorants and minorants of the third order of type (46). In reference [7, 8] it was shown that majorants and minorants of step response curves have the transfer function

$$(49) \qquad F_s(p) = \frac{1}{(T_1 p + 1)(T_2 p + 1)^2}.$$

The majorants are determined by the condition $T_2 > T_1$ (curves a in Figs. 1 and 2), while the minorants fulfil the condition $T_1 > T_2$ (curves b in Figs. 1 and 2).

For the coefficients of the denominator of transfer function (46) the following inequities hold good [4]:

$$4(a_2^2 - 3a_3a_1)(a_1^2 - 3a_2a_0) \geqq (a_1a_2 - 9a_0a_3)^2 , \tag{50}$$

$$a_2^2 \geqq 3a_1a_3 , \qquad a_1^2 \geqq 3a_0a_2 . \tag{51}$$

Relations (51) are known under the designation of Euler's inequities. For minorants and majorants the sign of equity holds in relation (50). After substituting the dimensionless parameters of $\varkappa$ and ϱ, relation (50) can be written in the form

$$4(\varkappa - 3\varrho)(1 - 3\varkappa) - \varkappa(1 - 9\varrho^2)^2 = 0 . \tag{52}$$

The curves in Figs. 1 and 2 were calculated by means of the quoted relations. Relative mean errors of the estimates of coefficients $\bar{a}_3$ of other transfer functions of type (46) lie in the zone between both curves in Figs. 1 and 2.

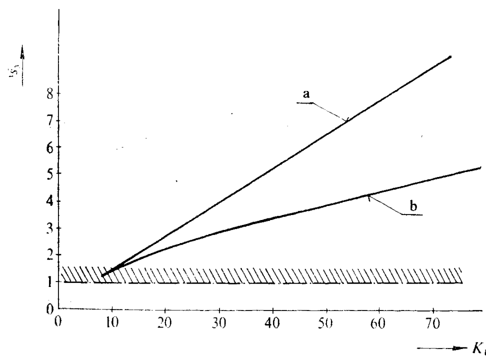

Fig. 2. Relative mean error of the estimates of coefficients $\bar{a}_3$ in systems of the third order ($\hat{\delta} = (2a_1)^{\frac{1}{2}} . 10^{-2}$; $a_1 = 10$, calculated from responses to unit impulse).

It can be seen from both figures that systems of the third order lie largely beyond the boundary of reliable solutions of identification problem (beyond the boundary of identifiability). It can thus be stated: For the majority of third order systems there does not exist a solution of identification in the sense of relation (29) for an accuracy of measurement close to the value according to relation (35). Figs. 1 and 2 also represent the explanation why literature contains so few examples of the practical application of identification methods. At first sight these conclusions may sound pessimistic. However, it can be stated that a number of assumptions was idealized (a favourable choice was made of the measurement error, noise interval, the integration interval of norms $\|v^{(i)}\|$, the continuous responses were considered). Therefore, practical results may be expected to be far less favourable.

The identification of systems of an order higher than the third can be analyzed similarly. For assessing the confidence measure of the estimates of coefficients of these systems it would be necessary to introduce an approximation index based on a proportional derivative controller. This line of discussion is already outside the scope of this paper.

7. CONCLUSION

In conclusion let us summarize the results of this paper.

a) The definition of identification must be complemented as follows. To identify a system described by transfer function (1) means to obtain the sufficiently reliable estimates of coefficients $\bar{a}_i$ (and $\bar{b}_i$).

b) On functions of the third order and on functions of higher order which can be approximated by those of third order [8] it was demonstrated that the practical possibilities of identification depend on the measurement error (Figs. 1 and 2). For measurement accuracy normally encountered in practice no reliable solution of identification problems exist for most of these systems. Relations derived in this paper explain the failure of most identification methods in the solution of practical problems.

c) It was shown that system identification must be assessed from the viewpoint of the approximation of functions. It is possible to identify only such coefficients of transfer functions whose share in the sense of approximation error exceeds the measurement error (relations (39) and (47)).

d) From the foregoing discussion it follows that in practical problems it is always necessary to expect the uncertainty of the transfer function of the system. The aspects of the synthesis of control loops must, therefore, be revalued.

e) Now, what are the perspectives of indetification in the light of the results of this paper? According to relations (30) and (34) there are two possibilities of improving identification results.

First is the reduction of the measurement error. This way of improving identification results will be troublesome from the technical – as well as primarily from the economic point of view. It seems there is no perspective for the practical exploitation of methods based on the improvement of measurement errors.

The second possibility is offered by changing the error of approximation or norms $\|v^{(i)}\|$. According to relation (11) the change of norms $\|v^{(i)}\|$ means the change of the corresponding functionals of sensitivity. It was shown in reference [6] that functionals of sensitivity vary under the influence of feedback. In agreement with reference [6] more reliable estimates can be obtained only by identification based on the evaluation of responses of closed control loops.

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VÝTAH

Meze použitelnosti identifikačních metod

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V práci je analyzován vliv přesnosti výchozích údajů a typu funkce (popisující soustavu) na řešení úloh identifikace. Jde vlastně o „existenční větu" tzn. o zjištění přesnosti výchozích údajů, při které existuje řešení úlohy identifikace pro zadanou třídu funkcí.

Práce je omezena na soustavy, které lze popsat přenosem (1). Uvažovány jsou tři odezvy: $x(t)$ ideální odezva, $\tilde{x}(t)$ změřená odezva a $\bar{x}(t)$ náhradní odezva. Vzdálenosti mezi jednotlivými odezvami jsou definovány vztahy (3) až (5). Rozptyl odhadů koeficientů $\bar{a}_i$ přenosu (1) podle vztahů (20) byl určen pomocí identifikační metody založené na regresní analýze.

V odst. 3 je zdůrazněna důležitost řádu náhradní funkce. Na základě nulové hypotézy byl odvozen vztah (29) pro posouzení spolehlivosti odhadu koeficientu $\bar{a}_n$ a tím vlastně řádu náhradní funkce.

Pro chybu měření podle vztahu (35) je analyzována spolehlivost odhadu koeficientu $\bar{a}_2$ přenosu příslušného soustavám druhého řádu. Pro posouzení výsledků identifikace je rozhodující chyba aproximace podle vztahu (5) popř. (39). Pro hodnocení chyby aproximace byl v odst. 5 odvozen aproximační index (vztah (42)). Na základě aproximačního indexu je ukázáno, že pro posuzování identifikačních metod jsou rozhodující soustavy třetího řádu.

V odst. 6 je ukázáno na soustavách třetího řádu a soustavách, které lze aproximovat třetím řádem, že praktické možnosti identifikace závisí na chybě měření. U převážné většiny těchto soustav neexistuje řešení identifikace pro přesnosti běžné v technické praxi. Vztahy odvozené v této práci vysvětlují selhání většiny identifikačních metod v praxi.

V závěru jsou diskutovány perspektivy identifikace se zřetelem k odvozeným výsledkům. Na základě vztahu (34) a práce [6] lze ukázat, že nejslibnější cestou pro zvětšení spolehlivosti výsledků identifikace budou metody založené na vyhodnocení regulačních pochodů tzn. odezev uzavřených regulačních obvodů.

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