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Tolerance Automata*

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One thing an automata theorist must often envy a control theorist is the use of continuity. One may state the problem thus: "How can we put a topology on a discrete set which is not the discrete topology?" A person who uses automata in the form of neural nets to provide crude models of the brain might ask the question: "How can we define continuity in a form which can enter the 'life' of a finite automaton?"

In [1], we introduced the idea of tolerance automaton in response to this question. In this paper, we reformulate the basic notions and develop some preliminary aspects of the theory. The basic idea is to have a form of "continuity" at our disposal for dealing with the state set.

1. TOLERANCE AND CONTINUITY

The basic concept is that of a tolerance, introduced by Zeeman [3]: A tolerance ξ on a set X is a relation on X that is reflexive and symmetric. A tolerance space (X, ξ) is a set together with a tolerance on it.

In what follows, we only use the set-theoretic notion of a tolerance space, without making a presumptuous attempt to emulate Zeeman's intriguing development of homology theories for these spaces – yet.

Example 1. Let X be the euclidean plane, and ξ all pairs of points less than ε apart.

Example 2. Let X be the visual field. Let ξ be the visual acuity tolerance, that is to say, all pairs of points that are indistinguishable.

Lemma [3]. *A tolerance on X induces a tolerance on the lattice L_X of subsets of X as follows: Given $A, A' \subseteq X$, write $(A, A') \in \xi$, pronounced A and A' are indistinguishable, if $A \subseteq \xi A'$ and $A' \subseteq \xi A$. Then the relation ξ is a tolerance on L_X.*

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Definition. Two tolerance spaces (X, ξ) and (Y, η) have isomorphic set theories if there are order-preserving functions between the lattices

$$L_X \underset{g}{\overset{f}{\rightleftarrows}} L_Y$$

such that

(1) If $(A, A') \in \xi$ in X then $(fA, fA') \in \eta$ in Y and similarly for B, B' in Y.
(2) For all $A \subseteq X$, $(A, gfA) \in \xi$, and similarly for $B \subseteq Y$.

Definition. Two tolerance spaces (X, ξ) and (Y, η) are said to be *related* if there exists a relation $\alpha \subseteq X \times Y$ such that $\xi = \alpha \circ \alpha^{-1}$ and $\eta = \alpha^{-1} \circ \alpha$.

Lemma [3]. *Related tolerance spaces have isomorphic set theories.*

Let us now conjure with these notions. In the spirit of automata theory, let us assume that time is quantized $T = \{0, 1, 2, 3, \ldots\}$. Let (X, ξ) be a tolerance space. We will say a motion in X is a function $m : T \to X$, and we shall call $m(t)$ the position of the point (undergoing the motion m) at time t. We shall say that the motion is ξ-continuous if $[m(t), m(t + 1)] \in \xi$ for all t, i.e. "if there are no detectable jumps in the motion". We thus see that a discrete automaton M is given an intuitively acceptable idea of continuity if the input set of M is the collection L_X of subsets of a tolerance space (X, ξ). We think of M as having a "retina" with one receptor for each point of X; the input at any time being the stimulation of a number of receptors. It will help to consider T as bearing the "adjacency" tolerance: $(t, t') \in \xi_T$ if and only if $|t - t'| \leqq 1$.

Let $M = (X, Y, Q, \delta, \lambda)$ be an automaton.

With each input-sequence $u = (u_1, u_2, u_3, \ldots) \in X^\omega$ we associate the motion $m_u : T \to Q$ where

$$m_u(0) = q_0 \,, \quad \text{an initial state in } Q \,,$$

$$m_u(t + 1) = \delta(m_u(t), u_t) \,.$$

Let us fix some tolerance ξ on Q. Then we say M is a tolerance automaton if each motion m_u is continuous, for each $u \in X^\omega$.

If we take ξ to be the "smudge" tolerance $e = Q \times Q$, then every automaton is, of course, an e-tolerance automaton. Thus our theorems below apply to all automata, but are often vacuous in case the only applicable tolerance is the "smudge" tolerance.

In what follows we always use ξ to denote a tolerance, only adding distinguishing subscripts where necessary.

Definition. Let $M = (X, Y, Q, \delta, \lambda)$ be an automaton for which Q is a tolerance space. We say that M is a *tolerance automaton* if for each $x \in X$, $q \in Q$, *we have* $(q, \delta(q, x)) \in \xi_Q$.

Thus a tolerance automaton has *inertia* — a sudden change of input cannot cause a sudden change of state.

Example. Consider an ordinary digital computer, M, in which the state of the machine is given by the contents of the registers. Then if two states are within tolerance only if they differ in the contents of a limited number of registers, we have that M is a tolerance automation. (Note that here, the clock pulse must serve as an input when there is no input from the "environment".)

The reader will find an interestingly different approach to the study of computers via tolerance relations in [2].

Example (Bečvář). Consider the non-finite autonomous sequential machine which is a Turing machine with its states the complete configurations on which we place the tolerance

$$(c_1, c_2) \in \xi$$

if $c_1 = c'd_1c''$, $c_2 = c'd_2c''$ and d_1 and d_2 are no more than 3 symbols long.

Example (Čulík). A machine whose states are represented by binary vectors, and we can only change so many bits per actuation, the tolerance being given by some fixed Hamming distance on the state vectors.

Definition. Let X and Y be tolerance spaces. A function $f : X \to Y$ is said to be *ξ-continuous* if $(x_1, x_2) \in \xi_X$ implies $(f(x_1), f(x_2)) \in \xi_Y$.

Recall that $(x_1, x_2) \in \xi^2$ if there is an x such that $x_1 \xi x$ and $x \xi x_2$; similarly for ξ^n. $f : X \to Y$ is said to be *n-continuous* if $(x_1, x_2) \in \xi_X$ implies $(f(x_1), f(x_2)) \in \xi^n_Y$. Thus 1-continuous is equivalent to ξ-continuous.

We now give an equivalent characterization of our notion of a tolerance automaton:

Definition. Let $M = (X, Y, Q, \delta, \lambda)$ be a tolerance automaton. Then we say that M is an *n-tolerance automaton* if $\delta(\circ, x) : Q \to Q$ is *n-continuous for each* $x \in X$.

A tolerance automaton is not, in general, a 1-tolerance automaton. However:

Theorem. *Every tolerance automaton is a* 3*-tolerance automaton.*

Proof. Let $(q, q') \in \xi_Q$. Since $(q, \delta(q, x)) \in \xi_Q$ and $(q', \delta(q', x')) \in \xi_Q$, we see that $(\delta(q, x), \delta(q', x')) \in \xi^3_Q$. Q.E.D.

Theorem. *Let M be a* 1*-tolerance automaton. Then $\delta(\circ, x) : Q \to Q$ is ξ_Q-continuous for every $x \in X^*$.*

Proof. By definition of a 1-tolerance automaton, $\delta(\circ, x) : Q \to Q$ is 1-continuous for all $x \in X$. Now $\delta(\circ, xx') = \delta(\delta(\circ, x), x')$ for $x' \in X$. But then the result is immediate by induction, on noting that if f and g are ξ-continuous, then so is their composition. Q.E.D.

1-tolerance automata thus have a *stability* property unshared by general tolerance automata; small differences in initial state cannot give rise to large differences in state at any later time. Each mapping $\delta(\circ, x) : Q \to Q$ thus is something in the nature of a contraction — *as we can see by defining a metric*

$$d(x, x') = \min \{n \mid (x, x') \in \xi^n\} .$$

We mention, for possible use in later papers, the

Definition. Let M_1 and M_2 be two m-tolerance automata. M_2 is said to be a *tolomorphic* image of M_1 if there exist maps

$$\begin{aligned} h_1 &: Q_1 \to Q_2 \quad \text{onto} \\ h_2 &: X_1 \to X_2 \quad \text{onto} \\ h_3 &: Y_1 \to Y_2 \end{aligned}$$

such that h_1 is ξ-continuous, and

$$\begin{aligned} h_1(\delta_1(q, x)) &= \delta_2(h_1(q), h_2(x)) , \\ h_3(\lambda_1(q, x)) &= \lambda_2(h_1(q), h_2(x)) . \end{aligned}$$

We say M_1 and M_2 are *isotolic* if each is a tolomorphic image of the other.

A tolerance ξ is simply a subset of $X \times X$ such that

(i) $\quad$ for all $x \in X$, $(x, x) \in \xi$,

(ii) $\quad (x, y) \in \xi \Rightarrow (y, x) \in \xi$.

Hence if ξ and η are tolerances, then so are $\xi \cap \eta$ and $\xi \cup \eta$. The tolerances thus form a sub-lattice $\mathscr{T}(X)$ of the lattice of subsets of $X \times X$. This sub-lattice has $0 = \text{Diag } X = \{(x, x) \mid x \in X\}$ and $1 = X \times X$. For any subset $R \subseteq X \times X$ we define

$$\text{Tol } R = \text{g.l.b. } \{\xi \mid \xi \in \mathscr{T}(X) \text{ and } R \subseteq \xi\} ,$$
$$R^{-1} = \{(x, y) \mid (y, x) \in R\} .$$

Then Tol $R = R \cup R^{-1} \cup \text{Diag } X$ is the weakest tolerance containing R.

Given a machine M, we now wish to define ξ_M, the smallest tolerance on Q with respect to which M is a 1-tolerance automaton. It is immediate from the definition of 1-tolerance automaton that for all $x \in X$ and all $q, q' \in Q$ we must have $(q, \delta(q, x)) \in \xi_M$; and $(q, q') \in \xi_M$ implies $(\delta(q, x), \delta(q', x)) \in \xi_M$.

Define

$$R_1 = \{(q, \delta(q, x)) \mid q \in Q, x \in X\} ,$$

$$R_{n+1} = \{(\delta(q, x), \delta(q', x)) \mid (q, q') \in R_n, x \in X\} - \bigcup_{j=1}^{n} R_j \quad \text{for} \quad n > 1 .$$

Clearly $R_n = \emptyset$ for $n > m = [\#(Q)]^2$.*

* $\#\, Q$ denotes the number of elements of the set Q.

Set

$$R = \bigcup_{n=1}^{m} R_n .$$

Then

(x) $$\xi_M = \text{Tol}\, R .$$

Adopting (x) as the *definition* of ξ_M, we immediately have:

Lemma. *An automaton $M = (X, Y, Q, \delta, \lambda)$ for which Q is a space with tolerance ξ_Q, is a 1-tolerance automaton if and only if*

$$\xi_M \subseteq \xi_Q .$$

Let $M = (X, Y, Q, \delta, \lambda)$. For each $q \in Q$ we define $M_q : X^* \to Y$ by $M_q(x) = \lambda(q, x)$, and we define for each x' in X^* $L_{x'} : X^* \to X^*$ by $L_{x'}x = x'x$. $M_{\delta(q,x)} = M_q L_x$.

There exists a reduced state-output machine equivalent to M, and any two such machines are renamed copies of one another. One such machine is the state-output reduction of M,

$$M^\circ = (X, Y, F^\circ, \delta^\circ, \lambda^\circ)$$

where $F^\circ = \{M_q \mid q \in Q\}$,

$\delta^\circ(f, a) = f L_a$ for $f \in F^\circ$, $a \in X$,

$\lambda^\circ(f, a) = f L_a(\Lambda) = f(a)$.

Let M be a 1-tolerance automaton – in particular, assume ξ_M is the tolerance on Q. What can we say about M° as a 1-tolerance automaton?

Definition. Let (X, ξ) be a tolerance space and let $\{S_\alpha\}$ be a partition of X. We define a tolerance (ξ, S) on $\{S_\alpha\}$ by

$$(S_\alpha, S_\beta) \in (\xi, S) \Leftrightarrow (x_\alpha, x_\beta) \in \xi \quad \text{for at least one pair} \quad (x_\alpha, x_\beta) \in S_\alpha \times S_\beta .$$

Regarding F° as a partition on Q we obtain a tolerance (ξ_M, F°) on F°.

Theorem. *Let M be an automation, with the tolerance ξ_M with which it is a 1-tolerance automaton.*

(i) *M° is a 1-tolerance automaton w. r. t. the tolerance (ξ_M, F°) on F°*
(ii) *Moreover*, $(\xi_M, F^\circ) = \xi_{M^\circ}$.

Proof. (i) is immediate from the definition of (ξ_M, F°).

(ii) follows on checking the inductive construction of ξ_M and ξ_{M°. Q.E.D.

In the "black-box" approach to automata, in which our attention focuses on input-output behavior rather than that of the states, it makes sense to focus on new notions of tolerance which relate "small changes" in input to "small changes" in ouput. We thus make the new definitions:

Definition. Let $M = (X, Y, Q, \delta, \lambda)$ be an automaton for which the set of input strings is a tolerance space (X^*, ξ_{X^*}), and the set of output string is a tolerance space (Y^*, ξ_{Y^*}). We say M is an *i/o tolerance* automaton if for all states $q \in Q$ and all $x, x' \in X^*$

$$(x, x') \in \xi_{X^*} \Rightarrow (\lambda(q, x), \lambda(q, x')) \in \xi_{Y^*} .$$

We are often interested in the case where ξ_{X^*} is induced directly by a tolerance ξ_X on X, where:

Definition. Let (A, ξ) be a tolerance space. Then (A^*, ξ^*) is A^* made into a tolerance space by defining $a, a' \in A^*$ to be within the tolerance ξ^* if and only if

$$a = (a_1, \ldots, a_n) , \quad a' = (a'_1, \ldots, a'_n)$$

satisfy $n = n'$, $(a_1, a'_1) \in \xi, \ldots, (a_n, a'_n) \in \xi$.

Example (Bečvář). Let M be a machine with continuous time, but finite input and output sets. For $a(t)$, $a'(t)$ two input (or two output) functions defined on the same interval let $d(a, a') =$ length of $\{t \mid a(t) \neq a'(t)\}$. We might consider M to act continuously if small differences of input pulses give small differences of output pulses. This would correspond to M being an i/o tolerance automaton with

$$(x, x') \in \xi_{X^*} \Leftrightarrow d(x, x') < \varepsilon ,$$

$$(y, y') \in \xi_{Y^*} \Leftrightarrow d(y, y') < \delta$$

for suitable ε and δ. The point emphasized here is that we may want to put a tolerance on X^* different from ξ_X^* — we thus used the notation ξ_{X^*} above.

Given an output tolerance ξ_{Y^*} for an automaton M, we can define a tolerance ξ^x, for each $x \in X^*$, on Q by

$$(q, q') \in \xi^x \Leftrightarrow (\lambda(q, x), \lambda(q', x)) \in \xi_{Y^*} .$$

Definition. The tolerance ξ_K on Q, the *state-tolerance induced* by ξ_Y *with "attention-span"* K is

$$\bigcap_{l(x) \leq K_1} \xi^x$$

where $l(x)$ is the length of x. i. e.

$$(q, q') \in \xi_K \Leftrightarrow (\forall x)\,(l(x) \leqq K \Rightarrow (q, q') \in \xi^x)\,,$$
$$\Leftrightarrow (\forall x)\,(l(x) \leqq K \Rightarrow (\lambda(q, x), \lambda(q', x)) \in \xi_{Y^*})\,.$$

If $K \geqq 2$, and $x \in X$, then

$$(q, q') \in \xi_K \Rightarrow (\delta(q, x), \delta(q', x)) \in \xi_{K-1}\,.$$

If $\xi_\infty = \bigcap_K \xi_K$, we have for all $x \in X$

$$(q, q') \in \xi_\infty \Rightarrow (\delta(q, x), \delta(q', x)) \in \xi_\infty\,.$$

Assertion. If Q has n states, $\xi_\infty = \xi_n$.

Let $\hat{\xi} = \xi_M \cap \xi_\infty$. Then M is both a 1-tolerance automaton and an i/o tolerance automaton w.r.t. $\hat{\xi}$, for the given tolerances ξ_{X^*} and ξ_{Y^*}.

Let us use LP(X, Y) to denote the length-preserving functions from X^* to Y^*. Then ξ_X and ξ_Y induce a natural subset of LP(X, Y); the set of functions continuous w.r.t. ξ_X^* and ξ_Y^*:

$$f \in \mathrm{LP}_\xi(X, Y) \Leftrightarrow : (x, x') \in \xi_X^* \Rightarrow (f(x), f(x')) \in \varrho_Y^*\,.$$

Thus we see that M is an i/o tolerance automaton iff each $M_q \in \mathrm{LP}_\xi(X, Y)$.

There is a natural tolerance on LP(X, Y), namely

$$(f, f') \in \xi_{\mathrm{LP}} \Leftrightarrow (\forall x \in X^*)\,((f(x), f'(x)) \in \xi_Y^*)\,.$$

We say M is a *natural tolerance automaton* (w.r.t. input tolerance ξ_X and output tolerance ξ_Y) if for each $q \in Q$, and each $x \in X$

$$(M_q, M_q L_x) \in \xi_{\mathrm{LP}}\,.$$

We could also define a tolerance $\tilde{\xi}$ on X in terms of a tolerance ξ_Y on Y by

$$(x, x') \in \tilde{\xi} \Leftrightarrow (\forall q \in Q)\,((\lambda(q, x), \lambda(q, x')) \in \xi_Y)\,.$$

Note that we would normally require of a tolerance on X^* (or Y^*) that it be right-invariant, i.e.

$$(x, x') \in \xi_{X^*} \quad \text{and} \quad x'' \in X^* \Rightarrow (xx'', x'x'') \in \xi_{X^*}\,.$$

All the tolerances introduced explicitly above share this tolerance property. In fact, we have always had the even stronger property that

$$(x, x') \in \xi_{X^*} \quad \text{and} \quad (x'', x''') \in \xi_{X^*} \Rightarrow (xx'', x'x''') \in \xi_{X^*}\,.$$

Definition. C is called a *cost space* if

(a) C is a tolerance space w.r.t. ξ;

(b) C is an abelian group under $+$, partially ordered w.r.t. $\leqq$;

(c) For each $c \in C$, there exist $a, b \in C$ such that $a < c < b$; and $a < c' < b$ implies $(c, c') \in \xi$.

In fact we shall usually think of a cost space as the reals under a tolerance of the form $(c, c') \in \xi \Leftrightarrow |c - c'| < \varepsilon$, for some fixed $\varepsilon > 0$.

Given an automaton $M = (X, Y, Q, \delta, \lambda)$, and a cost space C, a *cost function* for M is a function $p : Q \times X \to C$. We extend p to $Q \times X^*$ by

$$p(q, xx') = p(q, x) + p(\delta(q, x), x') .$$

The *optimal control problem for automata* may be stated as follows:

Let q_0 and q_1 be two states of Q, called the initial state and terminal state respectively. We shall say that $u = (u_1, \ldots, u_n) \in X^*$ transfers M from q_0 to q_1 if

$$\delta(q_0, u) = q_1 \quad \text{whereas} \quad \delta(q_0, u_1, \ldots, u_k) \neq q_1 \quad \text{for all} \quad k < n .$$

Among all sequences u in X^* which transfers M from q_0 to q_1 find that for which $p(q_0, u)$ is minimal.*

Let $M = (X, Y, Q, \delta, \lambda)$ be an automaton with cost function p. We define the (usually infinite) automaton

$$(M, p) = (X, Y, Q \times C, \delta_p, \lambda_p)$$

by

$$\delta_p(q, c, x) = (\delta(q, x), c + p(q, x)) ,$$
$$\lambda_p(q, c, x) = \lambda(q, x) .$$

We define the attainable set R_{q_0} in $Q \times C \times T$ to be

$$\{(\delta_p(q_0, 0, u), l(u)) \mid u \in X^*\}$$

(where $\delta_p(q_0, 0, \Lambda) = (q_0, 0)$ and $l(u)$ is the length of u).

Definition. Let X and Y be tolerance spaces. Then the product tolerance space is the cartesian product $X \times Y$ together with the tolerance ξ defined by

$$((x, y), (x' y')) \in \xi \Leftrightarrow (x, x') \in \xi_X \quad \text{and} \quad (y, y') \in \xi_Y .$$

Now let M be a 1-tolerance automaton, and let us consider $Q \times C$ as a tolerance space with the product tolerance. For each n we consider the cross-section of R_{q_0} at time n,

$$R_{q_0}^n = \{\delta_p(q_0, 0, u) \mid u \in X^n\} \subseteq Q \times C .$$

* A better formulation would be: from q_0 to within tolerance of q_1. The present formulation is completely tentative.

If u is optimal, then it has minimal C-coordinate of any point in $R^n_{q_0}$, and thus a *necessary* condition that u be optimal is that $\delta_p(q_0, 0, u)$ be a point of the *boundary* of R_{q_0}.

At this stage, a digression is necessary to say what we mean by "boundary" when talking of tolerance, rather than topological, spaces. Analogous to the usual definition for topological spaces we have:

Definition. Let S be a subset of a tolerance space X. Then we define:

The *ξ-closure* of S,

$$\bar{S} = \{x \mid (x, y) \in \xi \quad \text{for some} \quad y \in S\}.$$

The *ξ-interior* of S,

$$\operatorname{int}(S) = \{x \mid (x, y) \in \xi \quad \text{implies} \quad y \in S\}.$$

The *ξ-boundary* of S,

$$\begin{aligned}\beta S &= \bar{S} - \operatorname{int}(S) \\ &= \{x \mid (x, y) \in \xi \quad \text{for some} \quad y \in S, \quad \text{but} \\ &\quad (x, z) \in \xi \quad \text{for same} \quad z \notin S\}.\end{aligned}$$

Definition. The *component* of an element s of a tolerance space X is

$$C(s) = \{t \mid (s, t) \in \xi^m \quad \text{for some} \quad m \geqq 1\}.$$

Definition. A tolerance space X is *connected* if $X = C(x)$ for some element x (and hence all elements) of X.

Assertion. (i) If S is a subset of a tolerance space X, it is *not* necessary that $\bar{\bar{S}} = \bar{S}$. (ii) $\bar{S} = \bar{\bar{S}}$ implies that $\bar{S} = C(s)$ for some element (and hence all elements) of S. (iii) $C(s) = \overline{C(s)}$ for all $s \in X$.

Definition. A tolerance space X is said to be *pathwise connected* if given any x, $y \in X$ we can pass from x to y in a finite continuous motion: i.e. there exists $m > 0$ and a ξ-continuous map $p : \{1, 2, \ldots, m\} \to X$ such that $p(1) = x$ and $p(m) = y$.

Assertion. A tolerance space X is connected if and only if X is pathwise connected.

Theorem. *For every state q of a 1-tolerance automaton, M, and every $n \in T$, R^n_q is connected with respect to ξ^2_Q.**

Proof. Let $q_1, q_2 \in R^n_q$. Then there exist $x = x_1, \ldots, x_n$ and $x' = x'_1, \ldots, x'_n$ in X^n such that $\delta(q, x) = q_1$ and $\delta(q, x') = q_2$. Let $p(m) = \delta(q, x_1, \ldots, x_m, x'_{m+1}, \ldots, x'_n)$. Then $p(0) = q_2$, $p(n) = q_1$ and $(p(m), p(m+1)) \in \xi^2_Q$. Q.E.D.

* The improvement from the ξ^4_Q of [1] is due to L. A. M. Verbeek.

Definition. The "*mesh*" of a connected tolerance space X is defined to the minimum m such that $(x, y) \in \xi^m$ for all $x, y \in X$; and ∞ if no such m exists.

For a general tolerance space X, we define mesh $(X) = \sum[\text{mesh}\,(C(x))]$ where the sum extends over disjoint components.

Note. If mesh $(X) = \infty$, then X has infinitely many elements.

If Y is a connected tolerance space with mesh N then every map $f: X \to Y$ is N-continuous. Thus N-continuity is uninteresting. By continuity, we shall always mean m-continuity for some $m \ll N$, but our choice of m/N will depend on the fineness of discrimination we demand.

Lemma. $X - \bar{S} = \text{int}\,(X - S)$.

Proof. $x \in X - \bar{S} \Leftrightarrow (x, y) \in \xi$ implies $y \notin S$

$\Leftrightarrow x \in \text{int}\,(X - S)$. Q.E.D.

Thus we do indeed have, using condition (c) of the Definition of a cost space, that if u is optimal, then $\delta_p(q_0, 0, u) \in \beta R^n_{q_0}$.

Theorem. *Let M be a 1-tolerance automaton, and let $u = u_1, \ldots, u_n$ be in X^n. If $\delta_p(q_0, u) \in \beta R^n_{q_0}$, then*

$$(q_k, c_k) = \delta_p(q_0, 0, u_1, \ldots, u_k) \in \beta R^k_{q_0}\,, \quad 1 \leqq k \leqq n\,.$$

Proof. Suppose there were some $k < n$ such that $(q_k, c_k) \in \text{int}\,(R^k_q)$ whereas $(q_{k+1}, c_{k+1}) \in \beta R^{k+1}_q$. Then $(q_k, c') \in R^k_q$ for some $c' < c_k$. Let $(q_k, c') = \delta_p(q_k, 0, b)$; $b \in X^k$. Then $\delta_p(q_0, 0, bx_{k+1}) = (\delta(q_k, x_{k+1}), c' + c(q_k, x_{k+1}))$ contradicting the optimality of u. Q.E.D.

The control theorist will recognize this as the analogue of the theorem on which rests the Pontryagin Maximum Principle of Optimal Control.

We have only explored here a few rather superficial properties of tolerance automata. We conclude by setting-up a tool which should prove valuable in later work:

A subset S of a linear space is convex if, for each $x, y \in S$, the line $\overline{xy}$, lying between them, is contained in S.

Let us say S is *ε-convex* if, for each $x, y \in S$, and each point $z \in \overline{xy}$, z is within distance ε of some point of S.

Let us say S is *almost convex* if it is ε-convex for every $\varepsilon > 0$, i.e. if S is dense in its convex hull.

ε-convexity for a subset of a linear space suggests the following:

Definition. A subset S of a tolerance space (X, ξ) is said to be ξ-convex if for each $x, y \in S$ we have:

If $n = \min \{m \mid (x, y) \in \xi^m\}$ then there is a sequence $x = x_0, x_1, x_2, \ldots, x_n = y$ satisfying $(x_k, x_{k+1}) \in \xi$ for $k = 0, \ldots, n - 1$, and, for any such sequence, $(x_k, z) \in \xi$ for at least one $z \in S$ i.e. each $x_k \in \bar{S}$.

The reader may verify that if S is a subset of a linear space X, and if ξ_η is the tolerance

$$(x, x') \in \xi_\eta \Leftrightarrow \|x - x'\| < \eta$$

on X, then ξ_η-convexity implies ε-convexity if $\eta < 2\varepsilon/3$.

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REFERENCES

[1] M. A. Arbib: Automata Theory and Control Theory — A Rapprochement. Automatica *3* (1966), 161–189.

[2] W. DeBacker, L. Verbeek: Study of Analog, Digital and Hybrid Computers Using Automata Theory. ICC Bulletin *5* (1966), 215–244.

[3] E. C. Zeeman: The Topology of the Brain and Visual Perception. In: The Topology of 3-Manifolds. Ed. by M. K. Fort, pp. 240–256.

VÝTAH

Toleranční automaty

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V článku je zaveden pojem tolerance (reflexivní a symetrické relace), tolerančního prostoru a na jejich základě i pojem tolerančního automatu. Lze doufat, že tento pojem tolerance bude možno rozšířit tak, aby vystihoval řadu jevů, jež souvisí se situacemi, kdy automat pracuje ne zcela deterministicky, nebo kdy je příliš veliký pro bezprostřední popis, nebo kdy nás zajímá jeho chování pouze s jistou přesností. Pro automaty se pak dá formulovat řada problémů, které jsou blízké otázkám řešeným v teorii automatické regulace (stability, princip maxima apod.).

Obsahem první části článku je vyšetřování základních vlastností tolerančních automatů. Ve druhé části se studují i/o toleranční automaty, tj. zhruba řečeno takové automaty, kde malé změny vstupních řetězů působí malé změny výstupních. Ve třetí části se zavedených pojmů používá k řešení jistého optimalizačního problému pro automaty. Na konci práce autor formuluje a naznačuje další směry bádání o podobných problémech.

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