Information Transfer in Sensory Channels with an Application in Auditory Sensorial Communication

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Based on the assumption that the link between *stimulus* and *percept* is of statistical nature, the subject of sensory communication is tackled from the standpoint of the Shannon's theory of information. After introduction of the general mathematical model identifying the given source of stimuli to an abstract information source and the given channel of sensorial perception to an abstract communication channel on the input of which is directly applied the source, the concept of *information* on the stimulus, contained in the percept, is defined as well as its *transmission rate*. This model is specified for single-parameter stimulus and two-parameter stimulus (size and duration) in the region of Weber-Fechner law, and it is applied in the field of auditory sensorial communication, namely, in discerning a finite count of tonic signals.

CONTENTS

O. INTRODUCTION

Sensorial communication is a branch of science which examines sense organs as receivers of outside information. There has been lately remarkable development of this branch of science due, in particular, to the fact that the control of modern machinery still requires the participation of human operator. Man is in this process both the receiver of information from machines and a source of information for machines and it is therefore desirable from the point of view of information transfer that information-theoretic characteristics of men are in agreement with those of

158   machines (similarly as e.g. we match impedances in electronics in order to optimize the transfer of signal energy). From this point of view, the information-theoretic characteristics of sense organs are investigated mainly by the so called engineering psychology.

In the quantitative analysis of sensorial channels very little is up to now done about the mathematical description of the actual mechanism of the individual sensory organs. From the point of view of sensorial communication, sensorial physics (psychophysics) should answer mainly two questions:

(a) How does human consciousness react to outside stimuli reaching sense organs?

(b) How are the individual physical and chemical processes in the sensorial organs registered and interpreted in the nervous system of these organs?

The present paper is concerned with some aspects related to the first question which is, obviously, of much more phenomenological character than the second question. The great number of investigations in different countries on sensorial communication and intelligibility from the point of view of information theory which try to answer question (a) is explained by the needs arising in different applications similar to those mentioned above. Observing this fact ten years ago one of the authors of the present paper has formulated the opinion* that these investigations could be much more fruitfull if their authors were fully aware of the fact that what we are trying to determine is, properly speaking, nothing else than the characteristics of the receiver (sense organ) own channel, where the term 'channel' is conceived in the sense of information theory (see section 1).

Percept can be described by means of an abstract or *n*-dimensional Euclidean space in which a point is assigned to each percept characterized by a set of *n* sensorial parameters. A set of *stimuli* can similarly be described by means of a signal space so that to each stimulus a certain point can be allotted according to the physical parameters used for its description. The points of the *signal* and *sensorial* spaces are assumed to be statistically related. By studying these statistical relations, general information-theoretic characteristics of sensorial channels can be found.

This paper deals with the mathematical analysis of one of the most important aspects of information transmission in human organism, i.e. with the quantitative expression of *information* which can pass under given conditions through the sensory perception channel. The general model is gradually specified for single-parameter stimulus and two-parameter stimulus (size and duration) in the region of Weber-Fechner law, and it is applied in the field of auditory sensorial communication, namely, in discerning a finite count of tonic signals.

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* See: Albert Perez: Mathematical Theory of Information. Aplikace Matematiky *3* (1958), pp. 1—21 and 81—105 (in Czech with summary in French; the remark above is in p. 9).

The paper is based on the presumption that the link between *stimulus* and *percept* is of statistical nature. In particular, the subject is tackled from the standpoint of the Shannon's theory of information: Let us denote by X and Y the respective sets of all stimuli, i.e. of all points in the *signal space*, or of all percepts, i.e. of all points in the *sensorial space*. Then if the characteristics of the statistical link between the elements of sets X and Y as well as the incidence frequency of the individual stimuli are known, the mean quantity of information supplied by the percept about the stimulus concerned can be determined.

Let us suppose that for each set E of a class $\mathfrak{X}$ of subsets of the set X (corresponding to discernible stimuli), the probability of the stimulus $x \in X$ to belong to E is known. Let us further assume that for each stimulus $x \in X$ as well as for each set F of a class $\mathfrak{Y}$ of subsets of the set Y (corresponding to observable phenomena), there is known the conditional probability of percept $y \in Y$ falling into F if the receiver was stimulated by stimulus x. The former of the numbers just defined will be denoted by $P_X(E)$, the latter by $P_{Y/x}(F)$ and it will be assumed that the classes $\mathfrak{X}$ and $\mathfrak{Y}$ are σ-algebras, i.e. that for each countable system $E_1, E_2, \ldots$ of sets, belonging to $\mathfrak{X}$ or $\mathfrak{Y}$ there applies the rule that also their union $E_1 \cup E_2 \cup \ldots$ as well as their complements belong to $\mathfrak{X}$ or $\mathfrak{Y}$. The probability distribution P_X defines the statistical properties of the source of stimuli and the set $\{P_{Y/x}, x \in X\}$ of conditional probability distributions defines the statistical properties of the receiver with regard to the given set of stimuli X. In terms of information theory, the triplet $(X, \mathfrak{X}, P_X)$ would be called the *(stimuli) source of information* and the triplet $(\mathfrak{X}, \{P_{Y|x}, x \in X\}, \mathfrak{Y})$ would be described as the *(sensory) communication channel*.

In further considerations, $(X, \mathfrak{X}, P_X)$ will be a mathematical representation of physical reality affecting the receiver (sensory organ), while $(\mathfrak{X}, \{P_{Y|x}, x \in X\}, \mathfrak{Y})$ will be a mathematical description of the sensory communication channel.

The general model being established, the above task of assessing the amount of information, using general considerations, will be tackled.

Let us denote by $\mathfrak{X} \otimes \mathfrak{Y}$ the smallest σ-algebra which contains all the sets of the type $E \otimes F$ where $E \in \mathfrak{X}$, $F \in \mathfrak{Y}$ and $E \otimes F$ represent the set of all the pairs (x, y) which are characterized by $x \in E$ and $y \in F$ (see [1], Chap. 1, § 4.2). Let then P_{XY} be the simultaneous probability distribution of pairs (x, y) of stimuli and percepts which means that $P_{XY}(G)$ for $G \in \mathfrak{X} \otimes \mathfrak{Y}$ represents the probability of $(x, y) \in G$. As known, the distribution P_{XY} can be obtained from the known distributions P_X and $\{P_{Y|x}, x \in X\}$ by means of equation:

$$P_{XY}(G) = \int_G P_{Y|x}(G_x)\, \mathrm{d}P_X, \quad G \in \mathfrak{X} \otimes \mathfrak{Y},$$

where G_x is the set of all $y \in Y$ for which $(x, y) \in G$ applies. Let us finally denote by symbol P_Y the marginal probability distribution of distribution P_{XY} on the σ-algebra $\mathfrak{Y}$

and by symbol $P_{X \otimes Y}$ the product distribution $P_X \otimes P_Y$ on the σ-algebra $\mathfrak{X} \otimes \mathfrak{Y}$ (see [1], Chap. 1).

If $P_{XY} \ll P_{X \otimes Y}$, i.e. if $P_{XY}(G) = 0$ applies equally for each set $G \in \mathfrak{X} \otimes \mathfrak{Y}$ characterized by $P_{X \otimes Y}(G) = 0$ then the (average) *information* I on stimulus x contained in percept y is given by the expression

$$(1) \qquad I = \int_{X \otimes Y} \log f(x, y) \, dP_{XY},$$

where f is the Radon-Nikodym density of the distribution P_{XY} with respect to $P_{X \otimes Y}$ and where the logarithm is here and in the sequel taken to the base 2. If the condition $P_{XY} \ll P_{X \otimes Y}$ is not satisfied, then I is put equal to infinity, $I = \infty$. (This definition of information is in the discrete case in agreement with the known definition which considers information as the difference between the a priori and a posteriori entropy (equivocation) of the parameter space X (see (4) in § 2).

There is a series of intuitive and logical reasons (acceptable even in our case of sensory perception) which have in the past given rise to this general definition of information (see [2]). This question shall not be examined here, the paper shall concentrate only on some very important properties of the number I (for further details and references see [2]).

1. I ranges between 0 and $+\infty$; it is equal to 0 if and only if the random quantities x and y are statistically independent, i.e. if $P_{XY} = P_{X \otimes Y}$.

2. No transformation of the X (or Y) set into itself or into any other set can increase the amount of the information. In case of *one-to-one* (or, more generally, of so called in mathematical statistics *sufficient*) *transformation*, the amount of the information is the same before and after the transformation. If the transformation is however non-sufficient (for instance, in case of technical reproductions of complex acoustic or visual stimuli), the amount of the information is reduced after transformation.

3. $$I = \sup \sum_{i,j} P_{XY}(E_i \times F_j) \log \frac{P_{XY}(E_i \times F_j)}{P_X(E_i) \cdot P_Y(F_j)},$$

where the least upper bound (sup) is taken over the class of all finite disjoint decompositions $E_1, E_2, \ldots, E_n$ of the set X with $E_i \in \mathfrak{X}$, $i = 1, 2, \ldots, n$, and for all finite disjoint decompositions $F_1, F_2, \ldots, F_m$ of the set Y with $F_j \in \mathfrak{Y}$, $j = 1, 2, \ldots, m$.

The first two properties of information are in full agreement with the requirements of an adequate definition of information. The third property makes it possible to determine the number I, at least approximately, in those cases when the integral (1) cannot be calculated directly.

It can be supposed, without loss of generality, that a positive number $t(x)$ can be ascribed to each stimulus to characterize its duration.

The channel of sensorial perception in relation to the source of stimuli can be sometimes characterized by the so called *transmission rate*

$$(2) \qquad\qquad\qquad i = I/\bar{t} ,$$

where $\bar{t}$ is the so called *mean duration* of stimuli

$$\bar{t} = \int_X t(x) \, dP_X .$$

In principle, a distribution $P_X^{(m)}$ could be found which would maximize i representing thus in some sense the optimum use of the given sensorial channel.

The incidence of significant outside stimuli in nature may be considered as given by a probability distribution $P_X^{(n)}$. It can be expected that natural sensorial systems are made so that the information rate through sensorial channels obtains its maximum value just for the probability distribution $P_X^{(n)}$ of stimuli, i.e.

$$P_X^{(n)} = P_X^{(m)} .$$

2. MODEL WITH SINGLE-PARAMETER STIMULUS

In this section we shall describe a special case of sensorial communication in which stimuli and percepts are uniquely given by a single numerical parameter, e.g. by frequency, intensity etc. This is the case of the so called *single-parameter stimulus model*. All the stimuli which can be psychophysically described by means of primary sensorial quantities belong to the set of single-parameter stimuli.

(I) Let, first, $X = \{1, 2, \ldots, n)\}$, $Y = \{1, 2, \ldots, m\}$ be *finite* sets. In that case the distributions P_X, $P_{Y|x}$ are given by non-negative numbers

$$P_i = P_X(i) , \quad P_{ij} = P_{Y|i}(j), \quad i \in X , \quad j \in Y ,$$

where $\sum_i P_i = 1$, $\sum_j P_{ij} = 1$ for each $i \in X$. It can be easily seen that the Radon-Nikodym density considered in section 1, has here the following form:

$$f(i, j) = \frac{P_{ij}}{\sum_k P_k P_{kj}}$$

so that

$$(3) \qquad\qquad I = \sum_{i, j} P_i P_{ij} \log \frac{P_{ij}}{\sum_k P_k P_{kj}} .$$

This result is in agreement with the definition of information in the finite case as the difference between the *a priori* and *a posteriori entropy*, i.e.

$$(4) \qquad\qquad\qquad I = H(X) - H(X|Y) ,$$

where

$$H(X) = -\sum_i P_i \log P_i ,$$

$$H(X/Y) = -\sum_{i,j} P_i P_{ij} \log \frac{P_i P_{ij}}{\sum_k P_k P_{kj}} .$$

If $\hat{\imath} = \sum t(i) P_i$ is substituted into (2), a special expression is obtained for i.

(II) Let again $X = \{1, 2, \ldots, n\}$ and let Y be either an *interval* or the *real line*. Then the distribution P_X is the same as above and it can be assumed that the conditional distributions $P_{Y/i}$ are given by probability densities, i.e. that

$$P_{Y/i}(E) = \int_E \psi_i(y)\, dy$$

for each Borel set $E \subset Y$. Under these conditions it applies that

$$f(i, y) = \frac{\psi_i(y)}{\sum_k P_k \psi_k(y)}$$

so that, according to (1),

(5)
$$I = \sum_{i \in X} \int_Y P_i \psi_i(y) \log \frac{\psi_i(y)}{\sum_{k \in X} P_k \psi_k(y)}\, dy .$$

The computation of this integral is often complicated and it may therefore be of advantage to use the following approximation (see property 3 in section 1):

(6)
$$\sum_{i \in X} \sum_{j=1}^m P_i P_{ij} \log \frac{P_{ij}}{\sum_{k \in X} P_k P_{kj}} \leqq I \leqq -\sum_{i \in X} P_i \log P_i ,$$

(7)
$$P_{ij} = \int_{F_j} \psi_i(y)\, dy , \quad i \in X , \quad j = 1, 2, \ldots, m$$

and where $F_1, F_2, \ldots, F_m$ is any finite system of disjoint Borel sets whose union is Y.

(III) Let X and Y be *intervals* or *real lines*. It can then be presumed that the distributions P_X and $P_{Y|x}$ are given by Borel densities $\varphi(x)$ and $\psi(y|x)$, $x \in X$, $y \in Y$, respectively. In that case:

$$f(x, y) = \frac{\psi(y \mid x)}{\gamma(y)} ,$$

where

$$\gamma(y) = \int_Y \varphi(x)\, \psi(y \mid x)\, dx ,$$

so that, according to (1),

$$(8) \qquad I = \int_X \int_Y \varphi(x)\, \psi(y \mid x) \log \frac{\psi(y \mid x)}{\gamma(y)} \, \mathrm{d}x \, \mathrm{d}y \, .$$

The mean duration $\tilde{t}$ in relation (2) can be found from equation:

$$(9) \qquad \tilde{t} = \int_X t(x)\, \varphi(x)\, \mathrm{d}x \, .$$

3. TWO-PARAMETER MODEL FOR THE REGION OF WEBER-FECHNER LAW

In this section a special model with two-parameter stimulus (ϑ, t) will be described, in which the real numbers ϑ and t represent the *size* and *duration* of the considered
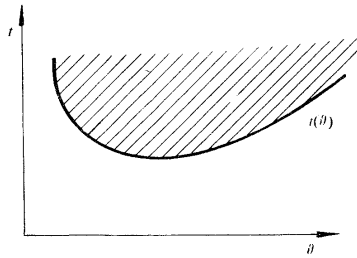

Fig. 1.

stimulus, respectively. It will be assumed that only the size parameter ϑ (in the sequel only: stimulus ϑ) and not the duration parameter t of stimulus is to be detected and that the resulting percept can be characterized by a single real parameter y, i.e. by the corresponding sensorial quantity. The set of all values ϑ and y will be denoted by Θ and Y and it shall be assumed that Θ and Y are *real lines*. We can then write $(\vartheta, t) \in \Theta \otimes T$ where T is obviously the real half line of positive numbers. Consequently, in the notation of section 1 or 2, $\Theta \otimes T = X$.

It will be further assumed that such pairs (ϑ, t) with t smaller than the *threshold duration* $t(\vartheta)$ of stimulus ϑ can not appear, i.e. that the distribution $P_{\Theta \otimes T} = P_X$ is given by a density $\varphi(\vartheta, t)$ which will be taken as different from zero only in the dashed section of Fig. 1, i.e. $\varphi(\vartheta, t) > 0$ only for $t \geq t(\vartheta)$.

For a broad class of communication problems here considered it may be assumed that, according to psychophysical measurements, the system of probability distribu-

tions $\{P_{Y|\vartheta,t}\}_{\vartheta\in\Theta,t>t(\vartheta)}$ is given by a system of Gaussian probability densities $\{\psi(y \mid \vartheta, t)\}$ on Y with mean value ϑ i.e.

$$(10) \qquad \psi(y \mid \vartheta, t) = \frac{1}{\sqrt{(2\pi)}\,\sigma(\vartheta, t)} \exp\left[- \frac{(y - \vartheta)^2}{2\sigma^2(\vartheta, t)} \right]$$

for each $\vartheta \in \Theta$, $t > t(\vartheta)$, the standard deviation $\sigma(\vartheta, t)$ being generally dependent on ϑ and t.

Weber-Fechner law is widely applicable in psychology and, as modified by G. A. Miller [3], it has for fixed duration the form:

$$(11) \qquad \Delta\vartheta = k(\vartheta + \vartheta_r),$$

where $\Delta\vartheta$ is the DL* for a certain physical or chemical parameter ϑ of outside stimulus, ϑ_r is a constant and k is a constant of proportionality. For non-stationary outside stimuli, k and, thus, $\Delta\vartheta$ depends also on the duration of stimulus t. This last relation can often be approximated by a function [4] of the type:

$$(12) \qquad \Delta\vartheta = \frac{B(\vartheta)}{ct + d}.$$

Combining (11) and (12) we get

$$(13) \qquad \Delta\vartheta = \frac{a\vartheta + b}{ct + d}$$

where a, b, c, d are real constants depending only on the receiver and on the type of physical parameter.

For $\Delta\vartheta$ obtained by the so called *matching psychophysical method* it is found that

$$\Delta\vartheta = \sigma(\vartheta, t)$$

so that it results from (13)

$$(14) \qquad \sigma(\vartheta, t) = \frac{a\vartheta + b}{ct + d}$$

for (ϑ, t) belonging to the region W of validity of the Weber-Fechner law.

Hence, according to (10), it follows for $(\vartheta, t) \in W$ that

$$\psi(y \mid \vartheta, t) = \frac{ct + d}{\sqrt{(2\pi)}\,(a\vartheta + b)} \exp\left(- \left[\frac{(y - \vartheta)\,(ct + d)}{\sqrt{2}\,(a\vartheta + b)} \right]^2 \right).$$

Supposing that $\psi(\vartheta, t) > 0$ only for $(\vartheta, t) \in W$, i.e. if it is certain that under the

* DL = discrimination length.

given circumstances the receiver can be affected only by a stimulus from the region W,
it can then be derived from (8) that

(15) $\qquad I = \dfrac{1}{\sqrt{(2\pi)}} \displaystyle\int_{\Theta}\int_{T}\int_{Y} \varphi(\vartheta, t) \cdot Q \exp\left\{ - \left[\dfrac{(y - \vartheta)\, Q}{\sqrt{2}} \right]^2 \right\} \cdot$

$\qquad\qquad \cdot \log\left[\dfrac{1}{\sqrt{(2\pi)}} Q \dfrac{\exp\{-[(y - \vartheta)\, Q/\sqrt{2}]^2\}}{\gamma(y)} \right] \mathrm{d}\vartheta \,\mathrm{d}t \,\mathrm{d}y ,$

where

$\qquad Q = \dfrac{ct + d}{a\vartheta + b}$ and $\gamma(y) = \dfrac{1}{\sqrt{(2\pi)}} \displaystyle\int_{\Theta}\int_{T} \varphi(\vartheta, t)\, Q \exp\left\{ - \left[\dfrac{(y - \vartheta)\cdot Q}{\sqrt{2}} \right]^2 \right\} \mathrm{d}\vartheta \,\mathrm{d}t .$
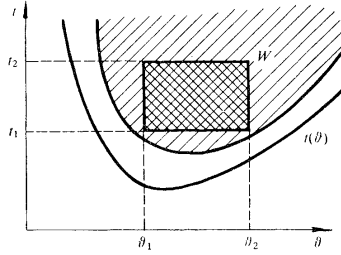

Fig. 2.

Similarly, according to (2), $i = I/\tilde\imath$ where

(16) $\qquad\qquad\qquad \tilde\imath = \displaystyle\int_{\Theta}\int_{T} t\varphi(\vartheta, t)\, \mathrm{d}\vartheta \,\mathrm{d}t .$

Supposing that $\varphi(\vartheta, t)$ is a uniform distribution on the rectangle in Fig. 2 which is assumed to be contained completely in region W, it then follows from (16)

(17) $\qquad\qquad\qquad \tilde\imath = \dfrac{t_1 + t_2}{2}$

and according to (15)

(18) $\qquad\qquad I = \dfrac{1}{\sqrt{(2\pi)}\,(\vartheta_2 - \vartheta_1)\,(t_2 - t_1)} \displaystyle\int_{-\infty}^{\infty}\int_{t_1}^{t_2}\int_{\vartheta_1}^{\vartheta_2} Q \cdot$

$\qquad \cdot \exp\left\{ - \left[\dfrac{(y - \vartheta)\, Q}{\sqrt{2}} \right]^2 \right\} \log\left[\dfrac{Q \exp\{-[(y - \vartheta)\, Q/\sqrt{2}]^2\}}{\sqrt{(2\pi)}\,\gamma(y)} \right] \mathrm{d}\vartheta \,\mathrm{d}t \,\mathrm{d}y ,$

where

$$\gamma(y) = \frac{1}{\sqrt{(2\pi)}\,(t_2 - t_1)\,(\vartheta_2 - \vartheta_1)} \int_{\vartheta_1}^{\vartheta_2} \int_{t_1}^{t_2} Q \exp\left\{ -\left[\frac{(y - \vartheta)\,Q}{\sqrt{2}} \right]^2 \right\} d\vartheta\, dt\,.$$

It results from (17) and (18) that in this case also I and i depend solely on constants $t_1, t_2, \vartheta_1, \vartheta_2, a, b, c, d$. Integral (18) is too complicated for direct computation. But it can be calculated by means of a computer, using a standard program. It could be calculated under any circumstances as a function of the concrete values $t_1, t_2, \vartheta_1, \vartheta_2, a, b, c, d$.

In the following part of the paper examples will be given of application of the above theoretical results in the field of auditory sensorial communication. It will deal with the so called *tonic signals*, i.e. with tone signals the duration t of which is equal to the threshold duration $t(\vartheta)$ of the tone-pitch ϑ. The *tone-pitch duration threshold* is in psychoacoustics defined as the minimum duration of tone signals necessary for perceiving them as a sound of tonal character [5], [6].

4. DISCERNING A FINITE COUNT OF TONIC SIGNALS

Let us suppose that a person is faced with the problem of detecting one of n tonic signals of frequency $x_1, x_2, \ldots, x_n$ (cycles per sec.) and of duration $t(x_1), t(x_2), \ldots \ldots, t(x_n)$ (ms), the probability of the i-th tone being P_i. The question is: how much information will the person receive under these conditions about the tonic signal affecting his hearing and what is the transmission rate (bits per sec.).

This single-parameter stimulus, characterized by a single real parameter which is its *frequency*, can be described by means of model (II) of section 2. The frequency of the stimulus can have any of the values $x_1, x_2, \ldots, x_n$ and the percept y can exhibit any pitch within the range of audibility.

It can be expected within psychoacoustics that the density ψ_i, of the conditional probability distribution on the set of detected frequencies Y, has approximately Gaussian shape with mean value x_i, on condition that the hearing organ received a tonic signal of frequency $10^2 \leqq x_i \leqq 10^4$ and of pitch duration threshold $t(x_i)$. Its standard deviation $\sigma(x_i, t(x_i))$ is related with x_i by (cf. [6])

$$\sigma\big(x_i, t(x_i)\big) = 4{\cdot}4 \,.\, 10^{-2} x_i\,,$$

so that:

$$\psi_i(y) \doteq \frac{9{\cdot}10}{x_i} \exp\left(-\left[16{\cdot}36\, \frac{y - x_i}{x_i} \right]^2 \right).$$

In this informative part it will be assumed, for the sake of simplicity, that ψ_i are approximately triangular distributions according to Fig. 3 with mean value x_i and standard deviation $\sigma_i = 4{\cdot}4 \,.\, 10^{-2} x_i$. (In the following paragraph the Gaussian

distributions will be approximated even more roughly by means of uniform distribu-
tions with the same mean value and standard deviation, to make numerical calcula-
tions more easy.)

The values $t(x_i)$ can be found from the empirical curve representing the relation
between the average tone-pitch duration threshold and the frequency according to
results in [6], [7], [8] and [9].

The standard deviation of a triangular distribution of height α being approximately
equal to $0.815/\alpha$, the density $\psi_i(y)$ has triangular shape with triangle height $\alpha_i =$
$= 20.4/x_i$ and base $x_i/10.8$.

For a concrete numerical calculation let us assume $n = 4$, $P_1 = 2/3$, $P_2 = P_3 =$
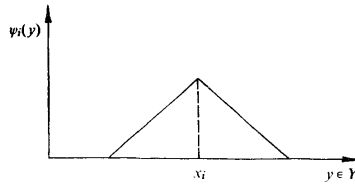$= P_4 = 1/9$, $x_1 = 1000$ c/s, $x_2 = 1100$ c/s, $x_3 = 1210$ c/s and $x_4 = 1320$ c/s.

Fig. 3.

In order to give an example of the approximate determination of information I by
means of (6), let $F_1 = (-\infty, 1050)$, $F_2 = \langle 1050, 1155 \rangle$, $F_3 = \langle 1155, 1270 \rangle$, $F_4 =$
$= \langle 1270, +\infty \rangle$.

In that case it can be easily found according to (7) that the matrix $P = (P_{ij})$ of
conditional probabilities is given as follows:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.004 & 0.996 & 0 & 0 \\ 0 & 0.004 & 0.996 & 0 \\ 0 & 0 & 0.031 & 0.969 \end{pmatrix}$$

which substituted in the left-hand side of inequality (6) gives $1.38 \leq I$. Similarly,
from the right-hand side of the inequality (6) the following is found

$$I \leq H(X) = 3/9 \log 9 + 2/3 \log 3/2 = 1.45$$

and therefore

(19) $$1.38 \leq I \leq 1.45 \quad [\text{bits}]$$

It will be easily found from Fig. 4 that the mean tone duration is $\bar{t} = \sum t(x_i) P_i =$
$= 9.5$ ms so that the transmission rate is in the given case within the limits

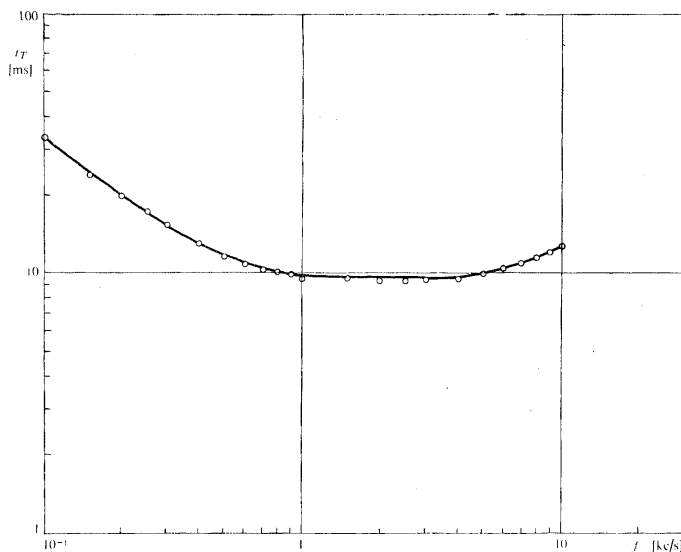$$146 \leq i \leq 153 \quad [\text{bits/sec}].$$

Fig. 4.

5. INFORMATION RATE OF TONIC SIGNALS

This paragraph deals with the question of the information rate in detecting tonic signals on the condition that all the tones from a frequency interval within the range of audibility are *equiprobable*. It is obvious that here it is appropriate to use model III (section 2) because both the stimuli and percepts can be represented by a real number — the tonic frequency. It can be therefore assumed that X and Y are real lines and that the density $\varphi(x)$ is *uniform* in the interval $10^2 \leqq x \leqq 10^4$ c/s, i.e. $\varphi(x) = 1/9900$ for x from the interval considered and elsewhere $\varphi(x) = 0$. It will be further assumed that the density $\psi(y \mid x)$ is also *uniform* for each $10^2 \leqq x \leqq 10^4$ and its mean value is x and the standard deviation is $\sigma_x = 4\cdot4 \,.\, 10^{-2}x$ (see section 4). As pointed out already in section 5, the latter assumptions are not quite in accordance with the experimental data but if the empirical distributions were approximately substituted by Gaussian distributions, too complicated integrals would be obtained from (8).

The standard deviation of uniform distribution in the interval of length α being equal to $\alpha/2\sqrt3$, the conditional probability density $\psi(y \mid x) = 1/0\cdot153x$ for

$x(1 - 0.0765) \leqq y \leqq x(1 + 0.0765)$ and $\psi(y \mid x) = 0$ for y out of this interval, i.e. $\psi(y \mid x)$ will be uniform in the interval of length $\alpha = 2 \cdot \sqrt{3} \cdot 4 \cdot 4 \cdot 10^{-2}x \doteq 0.153x$. All data required for model III (section 2) being defined, information I or transmission rate i will be found.

For brevity let $w(x, y)$ stand for the density $\varphi(x) \psi(y \mid x)$ on $X \otimes Y$.

It is obvious that $w(x, y) > 0$ only in the region Δ dashed in Fig. 5, where the equation of the straight line making the upper boundary of this region is $y =$
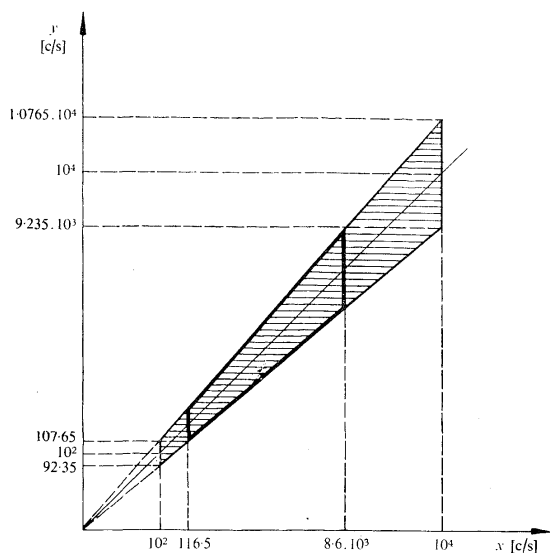

Fig. 5.

$= (1 + 0.0765) x$ and that of the lower boundary is $y = (1 - 0.0765) x$. We have $w(x, y) = (10^{-4} - 10^2) \, 0.153x$ for $(x, y) \in \Delta$.

Since $\gamma(y) = 10^{-4}$ for $107.65 \leqq y \leqq 9235$, i.e. it does not depends on y, it is then easy to calculate the integral

$$\int\int_{\Delta_1} w(x, y) \log \frac{\psi(y \mid x)}{10^{-4}} \, dx \, dy \doteq 3.5 \,,$$

where Δ_1 is the region framed by a thick line in Fig. 5. It is of course difficult to cal-

culate the integral

$$\iint_{A-A_1} w(x,\,y) \log \frac{\psi(y \mid x)}{10^{-4}}\, \mathrm{d}x\, \mathrm{d}y\,.$$

It is possible to prove that the expression is positive, so that

$$I = \iint_{A} w(x,\,y) \log \frac{\psi(y \mid x)}{\gamma(y)}\, \mathrm{d}x\, \mathrm{d}y > 3\cdot 5 \quad \text{bits.}$$

Next an attempt will be made to find an upper estimate of the number I.

From the system of probability distributions $\psi(y \mid x)$ corresponding to $10^2 \leqq$ $\leqq x \leqq 10^3$ the smallest standard deviation, equal to 4·4, corresponds to $\psi(y \mid 10^2)$, while from the system of probability distributions $\psi(y \mid x)$ corresponding to $10^3 \leqq$ $\leqq x \leqq 10^4$ the smallest standard deviation 44·0 corresponds to $\psi(y \mid 10^3)$. If for all $10^2 \leqq x \leqq 10^3$ the standard deviation was 4·4 and for $10^3 \leqq x \leqq 10^4$ was 44·0, then the discernibility and consequently the information would be higher. Let this information be $\check{I}$; then $I < \check{I}$. In this case $\check{\psi}(y \mid x) = 1/15\cdot3$ for $x - 7\cdot65 \leqq y \leqq$ $\leqq x + 7\cdot65$ and elsewhere $\check{\psi}(y \mid x) = 0$ if $10^2 \leqq x \leqq 10^3$, while $\check{\psi}(y \mid x) = 1/153$ for $x - 76\cdot5 \leqq y \leqq x + 76\cdot5$ and elsewhere $\check{\psi}(y \mid x) = 0$ if $10^3 \leqq x \leqq 10^4$, so that

$$\check{I} = \iint_{\check{A}} \check{w}(x,\,y) \log \frac{\psi(y \mid x)}{\check{\gamma}(y)}\, \mathrm{d}x\, \mathrm{d}y\,,$$

where $\check{\gamma}(y) = \int \check{w}(x,\,y)\, \mathrm{d}x$ and where $\check{w}(x,\,y) = \varphi(x)\, \check{\psi}(y \mid x)$ is different from zero only in the dashed region $\check{A}$ in Fig. 6.

Since, for $107\cdot65 \leqq y \leqq 923\cdot5$ or $1076\cdot5 \leqq y \leqq 9230\cdot5$, $\check{\gamma}(y) = 1\cdot01 \cdot 10^{-4}$, it can be proved that

$$\iint_{\check{A}_1 \cup \check{A}_2} \check{w}(x,\,y) \log \frac{\check{\psi}(y \mid x)}{\check{\gamma}(y)}\, \mathrm{d}x\, \mathrm{d}y \doteq 6\cdot06\,.$$

The regions $\check{A}_1$ and $\check{A}_2$ are marked in Fig. 6 as framed by a thick line. It can be shown that the integral in the region $\check{A} - (\check{A}_1 \cup \check{A}_2)$ is a quantity of the order of 10^{-2}, so that the relation $\check{I} \leqq 6\cdot2$ is obviously applicable here too, and we can conclude that

(20) $$3\cdot50 < I < 6\cdot20 \quad [\text{bits}]\,.$$

The quantity $\check{\imath}$ can be derived from equation (9)

(21) $$\check{\imath} = \int_{100}^{10,000} \frac{t(x)\, \mathrm{d}x}{9\cdot900}\,.$$
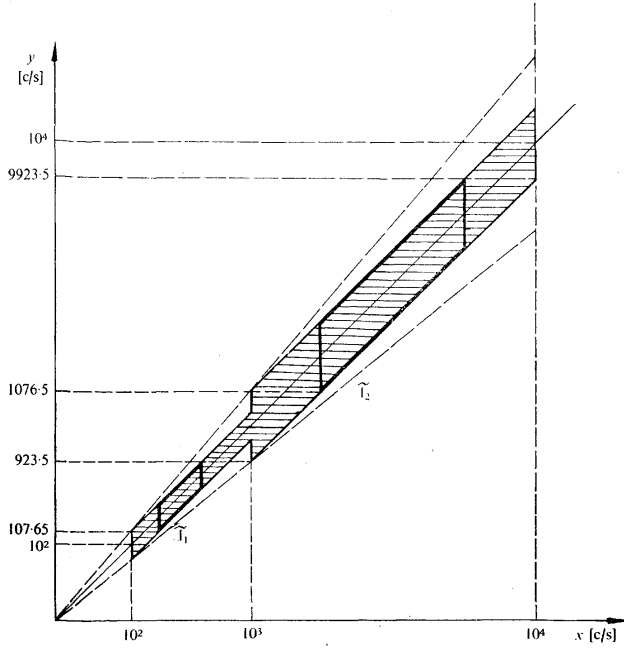
Fig. 6.

The function $t(x)$ shown diagramatically in Fig. 4 can be expressed analytically by the following empirical equations

$$(22) \qquad t(x) = \frac{9 \cdot 5}{1 - \exp\left(-x/300\right)} \qquad\qquad \text{for} \quad x \in \langle 100; 4{,}000 \rangle \,,$$

$$t(x) = \frac{10^2}{12 \cdot 1 - 4 \cdot 21 \cdot 10^{-4} x} \qquad\qquad \text{for} \quad x \in \langle 4{,}000; 10{,}000 \rangle \,,$$

where t is expressed in ms and x in cps. Substituting (22) into (21) we get

$$\tilde{t} = \mathscr{I}_1 + \mathscr{I}_2$$

where

$$\mathscr{I}_1 = 9 \cdot 6 \, . \, 10^{-4} \int_{100}^{4.000} \frac{1}{1 - \exp(-x/300)} \, dx = 4 \cdot 1 \,,$$

$$\mathscr{I}_2 = 1 \cdot 01 \, . \, 10^{-1} \int_{4.000}^{10.000} \frac{1}{12 \cdot 1 - 4 \cdot 21 \, . \, 10^{-4} x} \, dx = 6 \cdot 9 \,,$$

so that $\bar{t} = 11$ ms.

From this result it is possible to find the transmission rate limits by means of (20),

$$318 < i < 565 \quad [\text{bits/sec}] \,.$$

In the conclusion a practical application of the above methods will be pointed out. It appears that people with some auditory defects have different resolution ability than people with normal hearing which is shown e.g. by curves $t(x)$ (see [9]). As a result of reduced resolution ability also the information rate changes. It is therefore necessary to adapt also the probability distribution of outside stimuli so as to make the maximum use of the human sensorial auditory channel. This can be in principle calculated for each individual case of auditory defect and thus there can be established the probability distribution of stimuli which provides the maximum rate of information. Further it is possible to use the rate of information as a mesure of the extent of the auditory defect, taking the hearing organ as an information transmission channel.

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Přenos informace smyslovými kanály s jednou aplikací ve sluchové sensorické komunikaci

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Práce vychází z předpokladu, že závislost mezi *podnětem* (stimulem) a *vjemem* má statistický charakter. K otázce sensorické komunikace se přistupuje z hlediska Shannonovské teorie informace.

V části 1 je zaveden obecný matematický model smyslové komunikace, který přiřazuje k danému zdroji podnětů abstraktní zdroj informace a k danému smyslovému kanálu abstraktní sdělovací kanál (tak jako se tyto pojmy definují v teorii informace), při čemž se předpokládá, že zdroj informace je přímo aplikovaný na vstup kanálu. Dále je zaveden pojem *informace* o podnětu, která je obsažena ve vjemu, jakož i pojem *rychlosti přenosu informace*, a jsou připomenuty některé ze základních vlastností pojmu informace, které jsou použity dále.

V části 2 se tento obecný model aplikuje na případ tzv. *jednoparametrového stimulu*, kdy v sensorické komunikaci jak popudy tak i vjemy jsou pokládány za jednoznačně určené jedním číselným parametrem (frekvencí, intenzitou apod.).

V části 3 se obecný model specializuje na *dvouparametrový model* pro oblast platnosti Weberova-Fechnerova zákona. Jde o model s dvouparametrovým popudem (ϑ, t), při čemž reálné číslo ϑ reprezentuje hodnotu určitého fyzikálního parametru a t reprezentuje trvání popudu. Přitom se předpokládá, že jde o problém detekce hodnoty parametru a nikoliv délky trvání popudu a že vzniklý vjem je možné reprezentovat jediným reálným parametrem y, tj. hodnotou příslušné sensorické veličiny. Dále se předpokládá, že takové dvojice (ϑ, t), pro které je t menší než prahová délka trvání $t(\vartheta)$ příslušného popudu, se nemohou vyskytovat a že jsme v oblasti platnosti Weberova-Fechnerova zákona.

V části 4 jsou výše dosažené výsledky aplikovány na případ rozlišování konečného počtu *tonalitních signálů*, tj. tonových signálů, jejichž délka trvání je rovná časovému prahu tonality. Jde o problém detekce jednoho z n tonalitních signálů frekvence $x_1, x_2, \ldots, x_n$ [c/s] a délky trvání $t(x_1), t(x_2), \ldots, t(x_n)$ [ms] respektive, z nichž i-tý signál může nastat s pravděpodobností p_i, a nás zajímá otázka odhadovat jaké množství informace člověk za těchto podmínek obdrží o tonalitním signálu, který působil na jeho sluch a jaké je při tom dosaženo rychlosti přenosu. Na základě experimentálních údajů a za předpokladu trojúhelníkových distribucí charakterizujících smyslový kanál je pak propočítán konkrétní numerický příklad.

V části 5 je sledována otázka, jaké rychlosti přenosu se dosahuje při detekci tonalitních signálů za předpokladu, že se stejnou pravděpodobností mohou nastat všechny

tóny z určitého frekvenčního intervalu uvnitř pásma slyšitelnosti. Konkrétní numerický příklad je propočítán zde za předpokladu obdélníkových distribucí charakterizujících sluchový kanál. Odhady se dobře shodují s experimentálními údaji.

Na závěr je poukázáno na jednu možnost praktického použití výše uvedených metod, a to u sluchově vadných, aby jejich sluchový kanál byl maximálně využit.

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