KYBERNETIKA ČÍSLO 1, ROČNÍK 2/1966

A Generalized Approach to Fault-Finding Procedures*

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In this paper a generalized formulation of fault-finding procedures is given. It is shown that the optimization of the fault-finding procedures leads to the optimum control of corresponding Markov chain of "informations".

INTRODUCTION AND SOME DEFINITIONS

The problem of fault-finding lies in the optimization – according to some given criterion – of the procedure of determining all defective elements in the checked system. We shall assume that the system containing n elements does not operate if at least one of its elements is defective and that the used measurement equipment allows to determine all defective elements. Further we shall assume that all probabilities of occurence of failures are known.

Let us denote $\xi_i = 0$ (1) if the i-th element of the checked system is good (or defective resp.). Then $(\xi_1, \xi_2, \ldots, \xi_n)$ is a n-dimensional random variable with values in the Cartesian product $X = \{0, 1\}^n$ where a probability measure P on the σ-algebra of all subsets of X is given. Let us denote by S the system of all non-void subsets of X. Any element of S will be called the *information* about $(\xi_1, \xi_2, \ldots, \xi_n)$. The information is $s \in S$ if $(\xi, \xi_2, \ldots, \xi_n) \in s$. Let us denote by S^* the set of all one-point subsets of S. Any element of S^* will be called the *complete information* about $(\xi_1, \xi_2, \ldots, \xi_n)$.

Let D be the set of all possible decisions (i.e. measurements or checking which can be made on the given system containing n elements) and D^* the set of all admisible decisions. We shall assume that D^* contanins the decision $\tilde{d}$ which denotes that further measurement is not necessary. For every $d \in D$ and every $s \in S$ the symbol $A(d, s)$

* Presented on the *Summer School on Information Theory and Statistical Methods of Control Theory* held in Prague from May 25 to June 4, 1965.

denotes the partition of the set s into sets $A_1(d, s)$, $A_2(d, s)$, ..., $A_m(d, s)$ where of course $A_j(d, s) \in S$.

Let $\delta = (\delta_1, \delta_2, \ldots)$ be a sequence of transformations of the set S into D^*; δ will be called the *decision procedure.*

OPTIMIZATION OF THE DECISION PROCEDURE

Let σ_0 be a given element of the system S which gives the initial information about $(\xi_1, \xi_2, \ldots, \xi_n)$ and let δ be a given dicision procedure. We shall define for every $i = 1, 2, \ldots$ the sequence

$$\sigma_i = A_j(\delta_i(\sigma_{i-1}), \sigma_{i-1})$$

if

$$(\xi_1, \xi_2, \ldots, \xi_n) \in A_j(\delta_i(\sigma_{i-1}), \sigma_{i-1}) \, .$$

Then $\sigma_0, \sigma_1, \ldots$ is a Markov chain. Let w be a non-negative real function defined on the set $D^* \times S$ such that for every $s \in S^*$ we have $w(\tilde{d}, s) = 0$ and for $s \in S - S^*$ we have $w(\tilde{d}, s) > 0$. For every initial information σ_0 let us set

$$\varrho(\sigma_0) = \inf_{\delta} \mathsf{E}_{\delta,\sigma_0} \sum_{i=1}^{\infty} w(\delta_i(\sigma_{i-1}), \sigma_{i-1}) \, ,$$

where $\mathsf{E}_{\delta,\sigma_0}$ denotes the mean value according to the used decision procedure δ and to the initial information σ_0.

By means of dynamic programming technique it can be shown that

$$\varrho(\sigma_0) = \inf_{\delta} \mathsf{E}_{\delta,\sigma_0}[w(\delta_1(\sigma_0), \sigma_0) + \sum_{i=2}^{\infty} w(\delta_i(\sigma_{i-1}), \sigma_{i-1})] =$$

$$= \inf_{d_1} [w(d_1, \sigma_0) + \sum_{\sigma \in A(d_1,\sigma_0)} \inf_{\delta'} \mathsf{E}_{\delta',\sigma} \sum_{i=2}^{\infty} w(\delta_i(\sigma_{i-1}), \sigma_{i-1}) \, P(\sigma|\sigma_0)] =$$

$$= \inf_{d_1} [w(d_1, \sigma_0) + \sum_{\sigma \in A(d_1,\sigma_0)} \varrho(\sigma) \, P(\sigma|\sigma_0)] \, ,$$

where $\delta' = (\delta_2, \delta_3, \ldots)$. From this it follows that the optimal decison procedure is *homogeneous,* i.e. the optimum decision procedure is given by $\delta_1 = \delta_2 = \ldots = \delta_0$, where δ_0 is determined by the equation

$$\varrho(\sigma_0) = \inf_{\delta_0(\sigma_0)} [w(\delta_0(\sigma_0), \sigma_0) + \sum_{\sigma \in A(\delta_0(\sigma_0),\sigma_0)} \varrho(\sigma) \, P(\sigma|\sigma_0)]$$

for every $\sigma_0 \in S$. If $\sigma_0 \in S^*$ then obviously for every $d \in D^*$ $(A(d, \sigma_0)$ contains only the set σ_0 and therefore – according to the definition of w – δ_0 characterizing the optimum decision procedure has the following property:

$$\delta_0(\sigma_0) = \tilde{d} \, .$$

Therefore $\varrho(\sigma_0) = 0$ holds for every $\sigma_0 \in S^*$, too.

The above considerations and formulae yield the following conclusion: *the optimum procedure of determining all defective elements in a non-operating system is equivalent to the solution of the optimum control of corresponding Markov chain of obtaining informations about* $(\xi_1, \xi_2, \ldots, \xi_n)$.

A SIMPLE EXAMPLE

Let the non-operating system consist of three elements ($n = 3$) numbered 1, 2, 3 connected as shown in Fig. 1. Possible measurements are the measurements on the outputs of individual elements. Thus, the set of all admissible measurements (decisions) is $D^* = \{d_i, d_2, d_3, \tilde{d}\}$, where d_1, d_2, d_3 is measurement on the output of element 1, 2, 3, respectively. Let the elements of the system be statistically independent. The state when the element is defective will be symbolized by 1, and that one when the element is operating by 0. Then the a priori probability p_i that the i-th element is defective can be expressed in the following way:

$$P(\xi_i = 1) = p_i \, ; \quad i = 1, 2, 3 \, .$$

The informations about the system are expressed as triples of symbols P, Q, 1, and 0. The group of P's denotes that at least one of the corresponding elements is defective; the symbol Q denotes that there is not any information about the corresponding element (i.e. this element is either defective or operating); the symbols 1 and 0 denote that the corresponding element is defective or in operating state, respectively. Thus, e.g. the triple (0PP) means that the element 1 is operating and at least one of the elements 2 and 3 is defective. Using this notation the initial information σ_0 can be written as (PPP). All complete informations (i.e. elements of the set s^*) are represented by triples containing 1's and 0's, only.

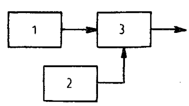

Fig. 1.

All possible informations and the decisions (measurements) giving the transitions from one information to another are given in Tab. 1.

Let the costs of measurement be independent on the state of the system, i.e.

$$w(d_{i,\cdot}) = w_i \, , \quad i = 1, 2, 3 \, .$$

Table 1.

The decisions for the transitions from one information to another

From \ to	PPP	0PP	1QQ	P0P	Q1Q	01Q	10Q	11Q	1PP	P1P	001	010	011	100	101	110	111
PPP	d_3	d_1	d_1	d_2	d_2	—	—	—	—	—	—	—	—	—	—	—	—
0PP	—	d_1, d_3	—	—	—	d_2	—	—	—	—	d_2	—	—	—	—	—	—
1QQ	—	—	d_1	—	—	—	d_2	d_2	d_3	—	—	—	—	d_3	—	—	—
P0P	—	—	—	d_2, d_3	—	—	d_1	—	—	—	d_1	—	—	—	—	—	—
Q1Q	—	—	—	—	d_2	d_1	—	d_1	—	d_3	—	d_3	—	—	—	—	—
01Q	—	—	—	—	—	d_1, d_2	—	—	—	—	—	d_3	d_3	—	—	—	—
10Q	—	—	—	—	—	—	d_1, d_2	—	—	—	—	—	—	d_3	d_3	—	—
11Q	—	—	—	—	—	—	—	d_1, d_2	—	—	—	—	—	—	—	d_3	d_3
1PP	—	—	—	—	—	—	—	d_2	d_1, d_3	—	—	—	—	—	d_2	—	—
P1P	—	—	—	—	—	—	—	d_1	—	d_2, d_3	—	—	d_1	—	—	—	—
001	—	—	—	—	—	—	—	—	—	—	$\tilde{d}$	—	—	—	—	—	—
010	—	—	—	—	—	—	—	—	—	—	—	$\tilde{d}$	—	—	—	—	—
011	—	—	—	—	—	—	—	—	—	—	—	—	$\tilde{d}$	—	—	—	—
100	—	—	—	—	—	—	—	—	—	—	—	—	—	$\tilde{d}$	—	—	—
101	—	—	—	—	—	—	—	—	—	—	—	—	—	—	$\tilde{d}$	—	—
110	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	$\tilde{d}$	—
111	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	$\tilde{d}$

Of course, for every $s^* \in S^*$ the equation

$$w(\tilde{d}, s^*) = 0$$

holds. In the following, the steps of the decision procedure with cost $w(\tilde{d}, s^*)$ will be ommitted.

Now, the following relations can be written (such steps of decision procedure, by which the information remain in the same state, are ommitted):

$$\begin{aligned}
\varrho(01Q) &= w_3 ;\\
\varrho(10Q) &= w_3 ;\\
\varrho(11Q) &= w_3 ;\\
\varrho(1PP) &= w_2 + \varrho(11Q)\, P(11Q \mid 1PP) = w_2 + w_3\, P(11Q \mid 1PP) ;\\
\varrho(P1P) &= w_1 + \varrho(11Q)\, P(11Q \mid P1P) = w_1 + w_3\, P(11Q \mid P1P) ;\\
\varrho(0PP) &= w_2 + \varrho(01Q)\, P(01Q \mid 0PP) = w_2 + w_3\, P(01Q \mid 0PP) ;\\
\varrho(P0P) &= w_1 + \varrho(10Q)\, P(10Q \mid P0P) = w_1 + w_3\, P(10Q \mid P0P) ;\\
\varrho(1QQ) &= \min[w_2 + \varrho(10Q)\, P(10Q \mid 1QQ) + \varrho(11Q)\, P(11Q \mid 1QQ),\\
&\qquad w_3 + \varrho(1PP)\, P(1PP \mid 1QQ)] =\\
&= \min[w_2 + w_3, w_3 + (w_2 + w_3\, P(11Q \mid 1PP))\, P(1PP \mid 1QQ)] ;\\
\varrho(Q1Q) &= \min[w_1 + \varrho(01Q)\, P(01Q \mid Q1Q) + \varrho(11Q)\, P(11Q \mid Q1Q),\\
&\qquad w_3 + \varrho(P1P)\, P\,(P1P | Q1Q)] =\\
&= \min[w_1 + w_3, w_3 + (w_1 + w_3\, P(11Q \mid P1P))\, P(P1P \mid Q1Q)] .
\end{aligned}$$

From the four last equations the values of $\varrho(0PP)$, $\varrho(P0P)$, $\varrho(1QQ)$, and $\varrho(Q1Q)$ can be determined according to the values of w_1, w_2, w_3 and p_1, p_2, p_3. Substituting thus obtained values into the equation

$$\begin{aligned}
\varrho(PPP) = \min[&w_1 + \varrho(0PP)\, P(0PP \mid PPP) + \varrho(1QQ)\, P(1QQ \mid PPP),\\
&w_2 + \varrho(P0P)\, P(P0P \mid PPP) + \varrho(Q1Q)\, P(Q1Q \mid PPP)]
\end{aligned}$$

the optimum procedure for our example of fault-finding will be obtained.

(Received July 26th, 1965.)

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Obecný přístup k vyhledávání poruch v systému

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V článku je podána obecná formulace určování optimálních procedur pro vyhledávání všech vadných elementů v nefungujícím systému. Je ukázáno, že optimalizace vzhledem k danému kritériu se dá formulovat jako optimální řízení odpovídajícího Markovova řetězce získávání informací o stavu systému.

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