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On Mathematical Models and the Role of the Mathematics in Knowledge of Reality

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In many branches of knowledge (e.g. economy, linguistics, biology etc.) we use the term "model" in the sense of a mathematical (that means symbolic) description of investigated facts and not in the sense of a purposely constructed device, by means of which we imitate this reality (e.g. the models of damlakes, bridges, various technical devices, etc.). In both cases however the *models of reality* are concerned. On the other hand, the term "model" has been for decades used in mathematics and logic in the sense of an example of a considered mathematical field, or a considered axiomatic theory (an example of a lattice or a group is nothing else than a *model of the theory* of lattices or groups, etc.). These two conceptions of the model are compared, both from the point of view of mathematics and of our knowledge, specified for the case of *finite models*, given by the *enumeration* of their elements.

1. APLICATION OF MATHEMATICAL FIELDS

The mostly prevailing part of examples for the use of mathematics confines itself to two basic domains of mathematical research – to *analysis* (i.e. differential and integral calculus in the large sense of the word) and *probability* with *statistics*.

If we understand under the word "*model*" *the mathematical description of an investigated reality*, then even the Newton's laws are the model of gravitation-field or the Maxwell's equations are the model of electromagnetic field, etc., so that the construction of mathematical models of different parts or aspects of reality is not the latest device of Cybernetics, but a very old and very well tested mathematical mean of our knowledge. Similarly, there is possible to introduce examples of statistical description. The models of all these types are sufficiently known and we are not going to deal with them. It is typical to them that they are mostly expressed by *equations* and that there is made substantial use of the *continuity*. However, the fact that in the last years the term "model" has become used not only in physics but also in economy, linguistics, biology, etc., is due to another specificity of models used there.

This specificity issues from the fact that in these fields there are not applied the notions of analysis, probability or statistics, but the basic notions of the *set theory*, or more special, the notions of *abstract algebra* and *mathematical logic*. Just the set-thinking has been immensely developped in the course of the last hundred years. It is characterized by the notion of *complicatedness* or *atomicity* (i.e. more simple elements compose more complicated ones). The set-thinking is making full use of symbolic notation and is yielding quite new means to our knowledge. Think of applications of the *set theory* (set-models in linguistics), *mathematical logic* (logical nets in electrical enginerring, automatic diagnostics in medicine, experiments with Brunnel's cards in psychology), the *theory of graphs* (connection of logical circuits, structure of sentences in linguistics, connecting nets and transport and distribution nets in economy), the *theory of semi-groups*, of *partly-ordering*, of *lattices*, of *Boolean algebras*, etc., and sometimes of more specialized mathematical fields not specially investigated and nameless till now.

2. THE PROCESS OF KNOWLEDGE OF REALITY

If we bear in mind the examples just mentioned and those to similar them, we can describe the procedure how to get acquainted with the corresponding part of reality as follows. First of all we must eliminate and *discern* single *objects*, their groups, *properties* of single objects and different *relations* among them. In our example we may suppose that this stage of knowledge is finished, i.e. that we know the objects, the relations and the properties we are interested in, because not sooner than under this supposition we can express questions concerning these objects, properties and relations (in many cases of the knowledge this stage is far the most important and difficult, because it covers all discoveries of new phenomena, too).

Then, naturally on the basis of our experience (either by natural observation or artificial experimentation) we learn out the *basic data* concerning the properties of objects (what properties they have or have not) and mutual relations (in which mutual relations they are or are not).

At the end we put the corresponding *question*. The aim of the whole knowledge process is to find the answer.

Thus, the model of the investigated part of reality is defined by the *enumeration* of considered *objects*, by the enumeration of considered *properties* and *relations* and by the enumeration of all *considered facts* (i.e. of basic data and eventually from elsehere known dependences and connections among the investigated properties and relations).

When putting a model we say clearly in which objects, properties and relations we are interested and thereby in which we are not interested, that means from which we are *abstracting* during the knowledge process. Thus, there is also stressed that at knowledge we always are concerned in some *part or aspects* of *reality*, not in the whole reality (i.e. on the whole in all objects and properties and relations) the way the knowledge is talked about in philosophy.

3. MATHEMATICAL MODEL OF REALITY

Mathematical model as the description of the investigated part of reality is naturally the description within the frame of some language. This language can be our natural language (eventually enriched with further means of expression), or some other artificially constructed language e.g. a mathematical language. For our purposes, there has been created an universal language of mathematical logic.

According to what the model is defined by (see paragraph 2) it is necessary for its expression in the language of mathematical logic to put, for example, the symbols denoting all considered objects, (they are so called *individual constants*; let for example J denote the set of all these constants) further the symbols, denoting all considered properties and relations (they are so called *predicate constants*) (while properties are denoted by one-place predicates, because they always refer to one object, whereas the two-membered relations are denoted by two-place predicates, because they refer to the pair of objects, three-membered relations by three-place predicates a.s.o.); for example let $\mathscr{P}_i$ denote the set of all considered i-place predicates, where $i = 1, 2, \ldots, n$) and at last the expressions expressing all basic data or basic facts (they are so called *individual formulas* usually of the form $P(a, b)$ when $P \in \mathscr{P}_2$ and $a, b \in J$, which we read that "individual constants a, b fulfil the two-place predicate P" or more concisely "it holds $P(a, b)$" meaning that "objects, denoted by the symbols a, b are – in this order – in a binary relationship denoted by the symbol P"; let, for instance, $\mathscr{F}$ denotes the set of all individual formulas).

In the all mentioned cases (see the end of the paragraph 1) we get the finite models, namely, models possessing a finite number of individual constants only, and a finite number of predicate constants only (then, they have, naturally, the finite number of individual formulas, too), and besides in most practical examples the sets J, $\mathscr{P}$ and $\mathscr{F}$ should be given directly by the enumeration of their elements (i.e. by means of different tables and similar).

The requirements for the model to be finite and to be given by an enumeration are also necessary and needed for the *machine processing*. On the other hand, it is convenient to put the models for direct human elaboration in another way. For example in fig. 1 there is given the model of a certain connecting net, when we know, namely, that single knots of a graph (i.e. circles) denote certain spots, its edges (i.e. lines connecting always two different circles) denote connecting traces, and numbers assigned to them denote lenghts of these traces. The question for this model is, how to find the minimal net, i.e. such a part of the given net that could connect (either directly or across other towns) every two towns in such a way that its lenght (which is the total of lenghts of all their traces) would be the smallest as possible.

According to fig. 1 we can easily construct the model that we have described at the beginning of this paragraph. Evidently, it will have the form $J = \{1, 2, 3, 4, 5, 6\}$, $P \in \mathscr{P}_2$ and P will denote "to be connected by the connecting trace" and Q $(Q \in \mathscr{P}_3)$ will denote "to be the lenght of a trace connecting both towns" so that $\mathscr{F}$ will run as follows:

$$P(1, 2),\ P(1, 4),\ P(1, 6),\ P(2, 3),\ P(2, 4),\ P(3, 4),\ P(3, 5),\ P(4, 5),$$
$$P(4, 6),\ P(5, 6);$$
$$Q(110, 1, 2),\ Q(150, 1, 4),\ Q(120, 1, 6),\ Q(120, 2, 3), \ldots,\ Q(140, 5, 6)\,.$$

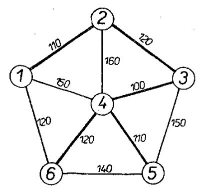

Fig. 1.

For the formulation of this question there is needed a further predicate $R \in \mathscr{P}_2$ having significance "to be interconnected either directly or across other towns" and that depends on the predicate P as follows: $R(x, y)$ holds just when there could be found such towns $r_1, r_2, \ldots, r_m$, where $1 < m \leq 6$ so that either $P(r_i, r_{i+1})$ or $P(r_{i+1}, r_i)$ holds for $i = 1, 2, 3, 4, 5$ and simultaneously $r_1 = x$ and $r_m = y$, when evidently $x, y, r_1, \ldots, r_m$ are *individual variables*, i.e. symbols by means of which we talk about arbitrary objects which is expressed in the form $x \in J$, $y \in J$, $r_i \in J$ for $1 \leq i \leq m$.

The net is called *continuous* just when each two its different knots are interconnected in it, i.e. when $R(x, y)$ holds for any $x \neq y$ where $x, y, \in J$.

Now, it is evident that to *find the minimal net* of a given connecting net means *to construct, in fact, the model of the minimal net on the basis of the model of the given net.* This model can be expressed by the set of individual constants J^* and predicates $P^* \in \mathscr{P}_2$, $Q^* \in \mathscr{P}_3$ and $R^* \in P_2$, necessarily fulfilling these conditions: $J^* = J$ and if there holds $P^*(x, y)$ or $Q^*(t, x, y)$ then also holds $P(x, y)$ or $Q(t, x, y)$ for every $x, y \in J^*$ and every number t (thereby it is told that the new net is a certain part of the given net and it is easy to see that from the validity $R^*(x, y)$ there always follows the validity $R(x, y)$; besides, it must hold $R^*(x, y)$ for every $x \neq y$, where $x, y \in J^*$, and simultaneously it is to be $\sum t$, where we add over all pairs $x \neq y$, $x, y \in J^*$ such that $Q^*(t, x, y)$ is the smallest as possible.

The whole series of different algorithms which solve problems of this type is well-known. The solution of our problem is to determine a model of the minimal net, of course, again by its enumeration, and it suffices the enumeration of all individual formulas for the predicate P^* (in fig. 1 there is one of possible solutions of the choice of connecting traces denoted by gross connecting lines), because for the given model there evidently holds $P(x, y)$ just when it is possible to find such a number t that $Q(t, x, y)$ holds and similar situation occurs for P^* and Q^*.

For that reason, at real solution of this problem on a computer, the data on the predicate P would not be put in the memory at all, because the data on the predicate Q would suffice. Besides, it is evident that instead of data $P(1, 2)$ it is possible to put in the memory the mere datum $(1, 2)$ naturally under the supposition that we always know, how to discern at the datum $(1, 2)$ that it meant that the towns denoted by symbols 1 and 2 were interconnected by a trace or that 1 and 2 fulfil the predicate P. For this purpose it is necessary to place suitable data concerning single predicates in the memory of the machine. Then, it is obvious that these predicates are given as abstract relations (so many-termed how many-place predicates are concerned).

From this point of view the model of reality is properly defined by the *set of abstract individuals* (that is the set of individual constant J) and by certain *abstract subsets and relations* defined in the set of individuals (*n-termed abstract relation* is the set of ordered n-tuple elements). But the set, together with some subsets and relations defined in it, is called the *mathematical field* (according to A. Grzegorczyk [5]) or the *mathematical structure* (according to N. Bourbaki [1]).

Thus, from the mathematical point of view the model of reality is, first of all, certain mathematical field or mathematical structure. This conception is more illustrative because here to objects of reality correspond in an one-to-one manner abstract individuals (i.e. certain symbols) and to properties and relations from the reality correspond abstract subsets and relations, so that the object has some property, eventually two objects are in some relation just when the corresponding individuum is the element of the corresponding subset eventually the corresponding pair of individuals belongs to the corresponding relation. Such one-to-one assignment of mathematical structure are called in mathematics *isomorphisms*.

Decisive in the *model of reality* is that the *significances* of all symbols and expressions from $J, \mathscr{P}_i, \mathscr{F}$ are known and that we know what they mean or denote, although this important circumstance is not apparently expressed or put down in the mathematical notation itself. This is concealed in the fact that the whole *model is an expression of a certain language* and this one always has its own significance, i.e. it always *referes to the reality*, to certain objects, properties, relations a.s.o. Such references of language expressions to their significance is, in fact, a (many-valued) function assigning to the expressions their significances and it is called the *semantics* of the corresponding language. Thus, decisive in the model of reality is that we know besides its language expressions its semantic, too, so that the model is never the mere mathematical description keeping only the corresponding rules of syntax.

4. MATHEMATICAL THEORY AND ITS MODEL

Contrary to the model of reality in the axioms of a *mathematical theory* there is not given the set of individual constants but only the set of individual variables so that, in fact, certain actual objects are not taken into consideration but only *symbols that are able to denote these objects.* In addition, there are not even given the significances of single predicate constants so that even they remain mere *symbols which would denote some properties or relations.*

As there are no individual constants, the basic data on single predicates cannot be given by the enumeration of individual formulas. Single predicates are characterized by so called closed formulas — the *axioms of the theory* — that can be described in this way.

If it is given the set of indiviual variables and for instance $P \in \mathscr{P}_3$ then we call the *primitive formula* the expression $P(x, y, z)$ where x, y, z are individual variables. Similar situation comes up with the other variables and predicates. If we denote the *logical conjuction "and"* by the symbol $\wedge$, the *logical conjuction "or"* by the symbol $\vee$, the *logical negation* of the formula $P(x, y, z)$ by the stripe over the whole formula or over the symbol of predicate $\overline{P(x, y, z)}$ only, the *logical idiom "for each x"* by the expression $\bigwedge x$ and the *logical idiom "is possible to find x"* by the expression $\bigvee x$, then under the *formula* we understand either primitive formulas or expression in the form

$$(\overline{\Phi}) \quad \text{or} \quad \bigvee x(\Phi) \quad \text{or} \quad \bigwedge x(\Phi) \quad \text{or} \quad (\Phi) \vee (\Psi) \quad \text{or} \quad (\Phi) \wedge (\Psi),$$

where Φ and Ψ are some formulas and x can be an arbitrary individual variable. Thus, it is shown in what way there can be constructed from primitive formulas complicated-ones. The variable x occurring on a certain place of the formula Δ is called *bound* if it occurs immediately after the symbol $\bigvee$ or $\bigwedge$, or if it occurs in the formula Φ and $\Delta = \bigvee x(\Phi)$ or $\Delta = \bigwedge x(\Phi)$, eventually it has been bound in the formula Φ or Ψ when for the given formula Δ there holds either $\Delta = (\overline{\Phi})$ or $\Delta = (\Phi) \vee (\Psi)$ or $\Delta = (\Phi) \wedge (\Psi)$. At last, the *closed formula* we call such a formula that every variable occuring in it on any place is bound.

For example $\bigwedge x(\bigwedge y(R(x, y))$ or $\bigwedge x(\bigwedge y(\bigvee t(P(x, y \vee Q(t, x, y))))$ are closed formulas where as primitive formulas $P(x, y, z)$ are not closed. It is easy to transscribe into the form of logical formulas even all expressions and theorems from the preceeding paragraph (in the case of individual formulas the use of idioms "for every x" or "is possible to find x" is naturally superfluous, because there are here only the individual constants and in no way variables which could be bound by these idioms).

A simple example is the theory of partial ordering having one two-place predicate $P \in \mathscr{P}_2$. If x, y, z are individual variables then there are prescribed two axioms of this theory. We often use (besides of the mentioned *logical conjunctions* yet the logical connection given by the phrase *"if"* ... *"then"*, ..." which is denoted by a symbol $\rightarrow$. Simultaneously, there holds that $(\Phi) \rightarrow (\Psi)$ does not mean anything else than $(\overline{\Phi}) \vee (\Psi)$. Our axioms can be expressed by means of the symbol in this way:

$$\text{(As)} \qquad \bigwedge x \bigwedge y (P(x, y) \rightarrow \overline{P}(y, x))$$

$$\text{(Tr)} \qquad \bigwedge x \bigwedge y \bigwedge z ((P(x, y) \wedge P(y, z) \wedge (x \neq z)) \rightarrow P(x, z)) .$$

The model of the theory with predicate constants $P_1, P_2, \ldots, P_k$ *and axioms* $A_1, \ldots, A_l$ is called such a mathematical field with a given set M and given subsets and relations $\varrho_1, \varrho_2, \ldots, \varrho_k$ that every predicate P_i can be taken as denotation of a relation ϱ_i (i.e. of how many-place the predicate is, of so many-place must be the rela-

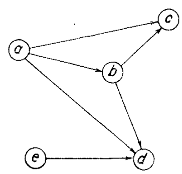

Fig. 2.

tion) $i = 1, 2, \ldots, k$ and when we consider the individual variables to be again an arbitrary denotation of objects from M, all the axioms $A_1, \ldots, A_l$ are fulfilled i.e. they are true sentences, when we have given both to the predicate constants and the individual variables their significances in ouf field.

A simple example of the model of our theory of partial ordering is illustrated on the fig. 2, where the knots of the graph are elements from M and the fact that there holds $P(x, y)$ is presented by the arrow-head, leading from the knot x to the knot y.

According to the fig. 2 we can easily see that $M = \{a, b, c, d, e\}$ and the abstract binary relation $\varrho = \{(a, b), (a, c), (a, d), (b, c), (b, d), (e, d)\}$. To prove that this mathematical field or this mathematical structure is really the model of the mentioned theory is simple but it takes up too much of time. Evidently, the predicate P will denote the relation ϱ. To prove that there is fulfilled the axiom (As) means to prove that on the fig. 2 the arrow-head never leads from y into x when leading from x into y, namely, for every $x, y \in M$. In a similar way can be checked the fulfilling of the axiom (Tr).

At the end of this paragraph, let us remind that the theory, as it has been introduced, is only a very special case and is usually called an *elementary theory* (within the frame of the so called *predicate logic of the first order*); logics of higher orders admit as variables not only individuals but also predicates so that we are able to express in them even the properties of properties, or relations, or the relations among properties a.s.o.). It would be impossible to construct by the mentioned means a nonelementary theory, the model of which would be the mathematical field considered in the preceeding paragraph (as example of the model of reality), because the condition referring to the minimality comprises in itself the demand that the predicate Q^* fulfills a certain condition with respect to all other predicates Q'. And just the idiom "for all predicates Q'" we have not introduced. There exists another circumstance that hampers the logicians at building up the theory, namely, that we would suppose knowledge of numbers and numbers themselves.

5. EXPERIENCE AND THEORY

In the preceeding two paragraphs it has been shown that from a pure mathematical point of view the models of theories as well as the models of reality (of the type, we have confined ourselves) always are some mathematical structure and this-one always refers to something. The difference between these two conceptions consists in the fact that first refers to reality, and second to mathematical theory.

Thus, it is necessary to discern three domains: *reality*, its parts or aspects, further *mathematical structure* and at last *mathematical theories*. In what relation are theories to reality?

If there occurs the case that we have a mathematical structure which on the one hand is a model of reality and on the other hand a model of some theory, then evidently such a theory is connected with the reality very closely, and we know quite exactly in what way, because it is obviously possible to indicate its real semantic and thus, formal theorems proved in this theory, of course, hold even in the reality. This is, after all, the sense and aim of any theory. From this point of view we could say that also the given theory is the model of reality (when, namely, under the model of reality we understand its mathematical description), of course, by adding the corresponding semantic (which, some times, is not introduced when it is supposed that it is taken for granted).

On the other hand the considered mathematical structure is the model of reality so that objects, relations, properties of this reality (when being abstracted from other things) form, in fact, an "isomorphic structure". Then, even this "isomorph real structure" is the model of a given theory. Thus, in the considered case the reality represents a model of the theory as well as, on the contrary, this theory represents the model of reality. This formulation is, of course, enabled by means of some logical inaccuracies, but in spite of this, it clearly shows that the use of the term "model" in the mentioned two senses is not too convenient. (In fact, there is no danger of misunderstanding, because they are mathematicians only who speak of the model of theory, whereas on the contrary of the model of reality speak only non-mathematicians – mathematicians reject to speak of it because it exceeds the frame of mathematics. Anxiety that there could occur a non-mathematician who would have a good command of mathematics or, on the contrary, a mathematician who would master some non-mathematical branch are quite unsubstantial at nowadays very strict specialization of science).

As an example of the introduced situation can serve the oldest mathematical theory, namely, the *Euclidian geometry*, the axioms of which has formulated D. Hilbert [6]. Its model is the best known Cartesian model which is the well-known mathematical structure of the analytical right-angle geometry in the threedimensed space. The fact the analytic geometry is the model of real space what we are living in is also by experience sufficiently proved (of course in other way than it has been mentioned above, because it matters the infinite structure and really we do not know enough well, how the triples of coordinates would be assigned to points in our space; far more relevant would be analogous objections if we asked what would correspond with the force in reality within the frame of the Newton's laws).

The example of the Euclidian geometry which was built up in the course of 2,500 years and in the construction of which many people took share shows sufficiently clearly that the matter was to built up the *theory that completely describes the space which we are living in*, i.e. the theory that would be its model (in the sense of the model of reality); in the Hilbert's axioms of geometry there is no trace that it ought to refer to the model of reality and thus, that it would be necessary to give for axioms and basic predicates (to be a point, a line, a plane, incide among points, lines and planes, and coincide among straight lines and angles) their significance in the reality. This need is but very urgently felt in Foundations [4] by Euclid who was trying there to define a point and a line a.s.o. The mathematician of nowadays has for it in store an indulgent smile only, but an experimental physicist, or an architect of bridges or tunnels, perhaps has not.

Since the time of D. Hilbert it has been explicitly said and argued that it is of no use to ask what is a point, a line etc. Similar situation occurs in other theories at different mathematic-logical reasonings. However, in the case of Euclidian geometry there is clear that its significance and *value* is just in the fact that this is the *geometry of space we are living in*, that this is the model of our real space. It is paradoxical that some mathematicians are not interested in this fact just for the reason that the question, whether the Euclidian geometry is the geometry of our space, is no mathematical question i.e. that it is impossible to give a theory where the theorem giving a positive or a negative answer would be proved. However, it is true that N. Lobačevskij was anxious to solve just this question with respect to the axiom of paralel lines by measuring the great angle defined by stars. It is also obvious that all physicists and mathematicians dealing with geometry who took share in building up the geometry, all the time had in front of their eys our space and that they described thisone (even if in this case there is not quite easy to describe in a lucid way the wholeprocess of our knowledge), as long as we, of course, advocate the empiric and not apriori point of view with respect to our knowledge.

The relation of theories to the reality is obvious in such cases where theories were built up for the description of different phenomena similar like Euclidian geometry.

Besides, in old times these theories (even Newton, Maxwell, Einstein etc.) were not built up by pure mathematicians, but by physicists having a good command of mathematics. Here is self-evident, too, that in the equations single parametres and coefficients must correspond with certain quantity obtained by measuring, i.e. must have their significance and thereby also their semantics.

In mathematics there are also studied and built up even such theories that originated in quite another way without the direct respect to the reality or to another branches of knowledge. The most frequent case is that single theories are differently generalized and modified, some axioms are left, some others are added so that after several such fittings there remains all the same whether in the initial theory some important model of reality was known or not. It is not possible, naturally, to exclude totaly that once will be found a relevant model for such a theory. With respect to baseless fittings of axioms there is, however, very improbable, without any regard to the fact that this model could be constructed only by a man who knows the theory and this man could be again a mathematician. But this-one do not want to do it, because it is not mathematics. But not even physicist will construct it, because for him it is necessary to start with mathematical formulas. This is only to absurd consequences drawn specialization of single branches of knowledge which has been continuing uninterruptedly and against which we do not defend ourselves with enough energy and purposely. (See [2]). Besides, the construction of additional models of reality to given theories belongs to less hopeful and mere isolated mathematical applications (to so called aposteriori applications see [3]).

At last, if we compare both conceptions of models from the respect of our knowledge then it is evident that the models of reality have always a knowing value, both being mathematical structures or mathematical theories, whereas the models of theories have often their value within the frame of mathematics itself.

6. THEORETICAL SOLUTION OF QUESTIONS

Why, in general, do we construct the models of reality? In what way they are useful for the solution of given questions?

Consider as an example the model of reality from the paragraph 3, i.e. the model of connecting net and the corresponding question after finding the minimal net. This question can be answered without constructing the model and without all preceeding knowledge in the following way: in the reality itself there are succeedingly constructed all possible nets containing all considered towns, always continuous, and their total lenght must always be measured; "to construct" and "to measure" would practically signify to go or to travel from town to town, to record and measure (and, in addition, in recording there is again concealed the abstract modeling). The lenghtiness and the lack of economy of this empiric measuring is obvious when we introduce one of the well-known algorithms by J. B. Kruskal [8]: the minimal net can be constructed from single traces of the given net chosen in the following way: we always choose the trace having the smallest value (if there are traces with the same value, we can choose any of them), but such one- that will not result in closed circle after its adding to traces elected before it (we understand that in the corresponding diagramm). We do it as long as it is possible. When it is impossible to go on, the minimal net is defined by the chosen traces. On the whole, it is easy to prove that this algorithm really ends in demanded aim.

How properly it is possible to guarantee by a mathematical proof that our concrete case of the connecting net can be solved by means of the mentioned algorithm?

It is guaranteed because of the known fact that every theorem proved in some theory is a true theorem on every model of this theory (but the converse need not always hold, namely, there are true theorems on a given model – e.g. that it has 6 knots and 10 edges – which cannot be in the corresponding theory either proved or disproved) and that the considered model of reality is the mathematical structure that is "isomorph" with the structure of reality. Simultaneously, from an abstract point of view two isomorph structures are, in fact, indiscernable and for that reason if there holds anything for one, it holds for another, too. Thus, it is guaranteed by this fact more that we know – and this must be proved in a tedious way sometimes (see par. 4, fig. 2) – that the given model of reality is, on the contrary, at the same time the model of corresponding theory where the correctness of algorithm has been proved. This theory, however, has not been here introduced (reasons are in par. 4 ot the end).

Thereby, the last question still remains without any answer, namely, how really can be proved the universal validity of some theorem when we cannot prove it on the basis of some axioms. That is just the situation when we are anxious to prove that "theorem proved in the theory holds for every model of this theory" or in the special case when we want to prove that "the given structure is the model of the given theory" eventually only that "the given structure fulfils the given axiom of the theory" (what in other words means just the same as "the given axiom of the theory holds in the given structure").

From the viewpoint of our knowledge (and even from the viewpoint of the significance of the theory and mathematics for it) there is a particularly clear question – naturally non-mathematical – in what way axioms are proved or on what basis the axioms of theories themselves are chosen. A mathematician accepts them as axioms, i.e. makes no doubt of them and he does not prove them, but on the contrary he deduces from them the whole theory. His whole activity of deducing and defining belongs to deductive reasoning. To his reasoning, however, evidently preceeds the inductive reasoning of a non-mathematician who on the basis of single facts and data passes to the generally valid demand – to an axiom that remains a hypothese for a long time. For that reason, a mathematician ought naturally to know very well how to deduce different consequences from the proposed hypotheses in order to reveal easily by means of the conventiently chosen proofs the mistakes and then to state new hypotheses or, on the contrary, to back the original hypotheses by another results (here belongs theoretical anticipation of the existence of the planet Pluto, or the existence of at one time unknown elements according to the suggested table of Mendeleev and the analogous situation is repeating nowadays with discoveries and foreseeing of elementary particles in atomic physics, here belongs also the well-known experiment proposed by Einstein a.s.o.).

For a non-mathematician there is possible to change the axiom and these-ones that, for a long time, were unchanged, were carefully proved by experience in the course of time. A non-mathematician has the axioms justified by experience. On the contrary for a mathematician any axiom is untouchable and when he changes it, then he works in quite another theory, in quite another world where nothing from the preceeding time need not hold.

Since the time of Hilbert [7] in mathematics there has been more and more apreciated the factor of deductive reasoning which has been making full use in development of theories, i.e. in defining and deducing all what is possible to be defined and deduced. Rather different in mathematics is the situation when they are to be solved – and up to all consequences in order to be performed at a computer – the tasks concerning concrete problems (examples of branches are at the end of the par. 1) on the models of reality and the solution of concrete examples at all. Here the inductive reasoning asserts itself and in this domain there is no difference between a mathematician or another scientist. Both are obliged to guess and to put hypotheses, i.e. both must reason inductively and not deductively. On inductive reasoning and on the solution of various mathematical problems has written G. Polyai [9,10] outstanding and remarkable books.

Nowadays, when there are at disposal great and very quick computers with immense memories and high reliability, it stops to be self-evident that the quickiest, the most advantageous, the cheapest and the most reliable solution of questions concerning the models of reality is the way leading across building up the corresponding mathematical theory. In some complicated examples – as mathematics devotes itself exclusively to theories sufficiently simple and lucid – will be undoubtedly found far more passable way without any theory, the way of a direct testing of many or even all possibilities, because this work can do quite well a computer itself.

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VÝTAH

O matematických modelech a podílu matematiky na poznání skutečnosti

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V celé řadě vědních oborů (např. v ekonomii, lingvistice, biologii aj.) se užívá termínu model ve smyslu matematického (tj. symbolického) popisu zkoumané skutečnosti a nikoli ve smyslu záměrně sestrojeného zařízení, kterým zkoumanou skutečnost napodobujeme (např. modely přehrad, mostů, různých technických zařízení aj.). V obou případech ovšem jde o *modely skutečnosti*. Naproti tomu v matematické logice se již několik desítek let užívá termínu model ve smyslu příkladu uvažovaného matematického oboru či uvažované axiomatické teorie (příklad svazu nebo grupy není nic jiného než *model teorie* svazů či grup apod.). Tato dvě pojetí modelu jsou porovnána jednak z hlediska matematického, jednak z hlediska našeho poznání, a to pro zvláštní případ *konečných modelů*, které jsou zadány *výčtem svých prvků*.

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