SPECTRAL DECOMPOSITION OF LOCALLY STATIONARY RANDOM PROCESSES

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This paper deals with locally stationary random processes introduced by Silverman in [1]. The spectral representation of such processes is obtained; the results generalize those of Silverman.

The notion of a locally stationary process is introduced by Silverman in [1]. This is a new kind of a random process generalizing the notion of a weakly stationary process. Let \( \{x(t), t \in \mathbb{R}\} \) be a random process, generally complex, with vanishing mean value and finite covariance function \( R(s, t) = \mathbb{E}\{x(s)x(t)\} \) on \( \mathbb{R}_+ \times \mathbb{R}_+ \), where \( x(t) \) is the complex conjugate to \( x(t) \). The author of [1] says that the random process \( \{x(t), t \in \mathbb{R}\} \) is locally stationary in the wide sense, or has a locally stationary covariance, if its covariance can be written as

\[
R(s, t) = R_1 \left( \frac{s + t}{2} \right) R_2(s - t) \quad \text{for every pair } s, t \in \mathbb{R}_+.
\]

where \( R_1 \geq 0 \) and \( R_2 \) is a stationary covariance function. We can put \( R_1(0) = 1 \), \( R_2(0) = 1 \) without loss of generality. In case \( R_1 = \text{const} \neq 0 \) we obtain a weakly stationary covariance function. Some examples of locally stationary processes are exhibited in [1], too.

We need the following facts about the harmonic analysis of nonstationary random processes. Following [2], we say that \( x(t) \) is harmonizable if \( x(t) \) can be expressed in the form

\[
x(t) = \int_{-\infty}^{+\infty} e^{i\lambda t} d\xi(\lambda),
\]

where the integral is meant in the quadratic mean and \( \xi(\lambda) \) is a second order random process with zero mean value and covariance function \( \gamma(\lambda, \mu) = \mathbb{E}[\xi(\lambda)\xi(\mu)] \) of bounded variation on \( \mathbb{R}_+ \times \mathbb{R}_+ \). It is proved in [2] that \( x(t) \) is harmonizable if and
only if its covariance function \( R(s, t) \) has the spectral representation

\[
R(s, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(s\lambda - t\mu)} \, dy(\lambda, \mu)
\]

where \( y(\lambda, \mu) \) is a covariance function of bounded variation on \( \mathbb{R}_1 \times \mathbb{R}_1 \). The last integral is understood in the Riemann-Stieltjes sense. In such a case we shall say that \( R(s, t) \) is harmonizable, too. When

\[
dy(\lambda, \mu) = f(\lambda, \mu) \, d\lambda \, d\mu,
\]

then \( f(\lambda, \mu) \) is called the spectral density function of \( x(t) \). If \( x(t) \) is locally stationary and harmonizable with spectral density function then, as it is proved in [1], its covariance function has a spectral representation

\[
R(s, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(s\lambda - t\lambda)} f(\lambda, \mu) \, d\lambda \, d\mu,
\]

where \( f(\lambda, \mu) \) is a locally stationary covariance, too, i.e.

\[
f(\lambda, \mu) = f_1\left(\frac{\lambda + \mu}{2}\right) f_2(\lambda - \mu),
\]

\( f_1 \geq 0 \) and \( f_2 \) is a stationary covariance.

This relation can be understood as a generalization of the famous Wiener-Khintchine relation for the case of locally stationary random process.

The following Theorem 1 and Theorem 2 are opposite assertions with respect to the generalized Wiener-Khintchine relation.

**Theorem 1.** Let \( x(t) \) be a harmonizable random process with the spectral density function \( f(\lambda, \mu) \) of the form

\[
f(\lambda, \mu) = f_1\left(\frac{\lambda + \mu}{2}\right) f_2(\lambda - \mu),
\]

where \( f_1 \geq 0 \) and \( f_2 \) is a stationary covariance function. Then \( x(t) \) is a locally stationary random process.

**Proof.** We assume that \( x(t) \) is harmonizable having a spectral density, i.e. its covariance function \( R(s, t) \) has representation

\[
R(s, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(s\lambda - t\mu)} f_1\left(\frac{\lambda + \mu}{2}\right) f_2(\lambda - \mu) \, d\lambda \, d\mu,
\]

where \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f_1((\lambda + \mu)/2) f_2(\lambda - \mu)| \, d\lambda \, d\mu \) exists.

Let us consider the transformation \( T(\lambda, \mu) = (u, v) \), where \( u = (\lambda + \mu)/2, v = \lambda - \mu \). Using this transformation, the integral (1) can be expressed as

\[
R(s, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(su - tw)} e^{i\frac{(s+1)t}{2}} f_1(u) f_2(v) \, du \, dv,
\]

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hence

\[ R(s, t) = R_1 \left( \frac{s + t}{2} \right) R_2(s - t) \]

where

\[ R_1(x) = \int_{-\infty}^{+\infty} e^{iux} f_1(u) du, \quad R_2(x) = \int_{-\infty}^{+\infty} e^{ivx} f_2(v) dv. \]

As \( f_1 \geq 0 \), \( R_2 \) is a stationary covariance and as \( f_2 \) is a stationary covariance, it is \( R_1 \geq 0 \). It means that \( R(s, t) \) is a locally stationary covariance function.

We need the following Lemma 1 and Lemma 2 for proofs of further results.

**Lemma 1.** Let \( f, g \) be complex functions on \( (-\infty, +\infty) \), let \( \xi(\cdot), \eta(\cdot) \) be second order stochastic processes with \( E[\xi(s) \eta(t)] = \Gamma(s, t) \). If \( \int_{-\infty}^{+\infty} f \, d\xi, \int_{-\infty}^{+\infty} g \, d\eta \) exist in the quadratic mean then

\[ E\left\{ \int_{-\infty}^{+\infty} f \, d\xi \right\} \int_{-\infty}^{+\infty} g \, d\eta = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f \, d\xi \, g \, d\eta. \]

**Proof.** As \( \int_{-\infty}^{+\infty} f \, d\xi = \lim_{A \to +\infty} \int_{-A}^{A} f \, d\xi \) it is sufficient to prove Lemma 1 for bounded intervals only, i.e.

\[ E\left\{ \int_{-A}^{A} f \, d\xi \right\} \int_{-A}^{A} g \, d\eta = \int_{-A}^{A} \int_{-A}^{A} f \, g \, d\xi \, d\eta. \]

According to the definition of the stochastic integrals \( \int_{-A}^{A} f \, d\xi, \int_{-A}^{A} g \, d\eta \) there exists \( \delta > 0 \) for every \( \varepsilon > 0 \) such that for every subdivisions \( \mathcal{D}_1, \mathcal{D}_2 \) of \( [A, B] \) with the norms \( \|\mathcal{D}_1\| < \delta, \|\mathcal{D}_2\| < \delta \)

\[ E\left\{ \int_{-A}^{B} f \, d\xi \left( \sum_{t} f(l_t) \Delta \xi(l_t) \right) \right\} < \varepsilon^2 \]

\[ E\left\{ \int_{-A}^{B} g \, d\eta \left( \sum_{t} g(l_t) \Delta \eta(l_t) \right) \right\} < \varepsilon^2. \]

This fact gives, further, that

\[ E\left\{ \int_{-A}^{B} f \, d\xi \int_{-A}^{B} g \, d\eta \right\} - \sum_{t} f(l_t) \Delta \xi(l_t) \sum_{t} g(l_t) \Delta \eta(l_t) \leq \]

\[ \leq \left[ E\left\{ \sum_{t} g(l_t) \Delta \eta(l_t) \right\} \right]^{1/2} \left[ E\left\{ \sum_{t} f(l_t) \Delta \xi(l_t) \right\} \right]^{1/2} + \left( E\left\{ \int_{-A}^{B} f \, d\xi \right\} \right)^{1/2} \varepsilon \]

for every \( \mathcal{D}_1, \mathcal{D}_2 \) with \( \|\mathcal{D}_1\| < \delta, \|\mathcal{D}_2\| < \delta \). That proves firstly the existence of the Riemann-Stieltjes integral \( \int_{-A}^{B} f \, d\xi \) and secondly the equality

\[ E\left\{ \int_{-A}^{B} f \, d\xi \int_{-A}^{B} g \, d\eta \right\} = \int_{-A}^{B} \int_{-A}^{B} f \, g \, d\xi \, d\eta. \]

**Lemma 2.** Let \( \xi(t) \) be a second order stochastic process having the derivative \( \xi' \)
in the quadratic mean with continuous covariance function. If $\int_{-\infty}^{+\infty} f \, d\xi$ exists in the quadratic mean then

$$\int_{-\infty}^{+\infty} f(t) \, d\xi(t) = \int_{-\infty}^{+\infty} f(t) \, \xi'(t) \, dt.$$  

Proof. Let $\Gamma$ be the covariance function of $\xi(\cdot)$. The existence of $\xi'(\cdot)$ implies the existence of the second derivative $\partial^2 \Gamma(s, t) / \partial s \, \partial t$ because $E[\xi(s) \, \xi'(t)] = -\partial^2 \Gamma(s, t) / \partial s \, \partial t$. Now, let $\int_{-\infty}^{+\infty} f \, d\xi$ exist. Then the integral $\int_{-\infty}^{+\infty} f \, d\xi$ exists as the limit of $\int_{A}^{B} f \, d\xi$ when $A \to -\infty$, $B \to +\infty$. This integral can be approximated by sums of the form

$$\sum_{j} \sum_{i} f(s_j^*) \, f(t_i^*) \, \Delta \Gamma(s_j, t_j)$$

where $s_i \leq s_i^* \leq s_{i+1}$, $t_j \leq t_j^* \leq t_{j+1}$. The existence and continuity of $\partial^2 \Gamma(s, t) / \partial s \, \partial t$ yield that $\Delta \Gamma(s_j, t_j) = (\partial^2 \Gamma(s, t) / \partial s \, \partial t)|_{(s_j^*, t_j^*)} \Delta s_j \, \Delta t_j$ with $0 < \theta_1 < 1$, $0 < \theta_2 < 1$. That proves the existence of the integral

$$\int_{A}^{B} \int_{A}^{B} f(s) \, f(t) \, \partial^2 \Gamma(s, t) / \partial s \, \partial t \, ds \, dt$$

and at the same time the equality

$$\int_{A}^{B} \int_{A}^{B} f(s) \, f(t) \, d\Gamma(s, t) = \int_{A}^{B} \int_{A}^{B} f(s) \, f(t) \, \partial^2 \Gamma(s, t) / \partial s \, \partial t \, ds \, dt.$$  

As $\partial^2 \Gamma(s, t) / \partial s \, \partial t$ is the covariance function of $\xi'$ the existence of

$$\int_{A}^{B} \int_{A}^{B} f(s) \, f(t) \, \partial^2 \Gamma(s, t) / \partial s \, \partial t \, ds \, dt$$

insures the existence of the stochastic integral $\int_{A}^{B} f(t) \, \xi'(t) \, dt$. It remains to prove the equality $\int_{A}^{B} f(t) \, d\xi(t) = \int_{A}^{B} f(t) \, \xi'(t) \, dt$. Let us prove that $E[|\int f \, d\xi - \int f \, d\xi'|^2] = 0$. We have

$$E \left( \int_{A}^{B} f \, d\xi - \int_{A}^{B} f \, d\xi' \right)^2 = \int_{A}^{B} \int_{A}^{B} f(s) \, f(t) \, d\Gamma(s, t) - E \left( \int_{A}^{B} f \, d\xi \int_{A}^{B} f \, d\xi' \right) - E \left( \int_{A}^{B} f \, d\xi \int_{A}^{B} f \, d\xi' \right) + \int_{A}^{B} \int_{A}^{B} \partial^2 \Gamma(s, t) / \partial s \, \partial t.$$  

According to Lemma 1

$$E \left( \int_{A}^{B} f \, d\xi \int_{A}^{B} f \, d\xi' \right) = \int_{A}^{B} \int_{A}^{B} f(s) \, f(t) \, E[\xi(s) \, \xi'(t) \, dt] =$$

$$\int_{A}^{B} \int_{A}^{B} f(s) \, f(t) \, \partial^2 \Gamma(s, t) / \partial s \, \partial t \, ds \, dt$$

because $E[\xi(s) \, \xi'(t)] = \partial \Gamma(s, t) / \partial t$.  

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In this way we obtained that $\int_0^\infty f d\xi = \int_0^\infty \xi d\zeta$. This equality holding for every bounded interval $[A, B]$ gives immediately that $\int_{-\infty}^{\infty} f(t)\,d\xi(t) = \int_{-\infty}^{\infty} \xi(t)\,d\zeta(t)$.\qed

**Theorem 2.** Let $x(t)$ be a harmonizable random process with spectral density function $f(\lambda, \mu)$ of the form

$$f(\lambda, \mu) = f_1 \left( \frac{\lambda + \mu}{2} \right) f_2(\lambda - \mu)$$

where $f_1$ is continuous and nonnegative, $f_2$ is a continuous stationary covariance. Then $x(t)$ has the following spectral representation

$$x(t) = \int_{-\infty}^{\infty} e^{i\omega t} \xi(\omega)\,d\omega,$$

where $\xi(\omega)$ is a locally stationary random process, too.

**Proof.** Theorem 1 says that $x(t)$ is locally stationary. Being harmonizable $x(t)$ can be expressed as

$$x(t) = \int_{-\infty}^{\infty} e^{i\omega t} \xi(\omega)\,d\omega,$$

where $\xi(\omega)$ is a second order random process with covariance function $\gamma(\lambda, \mu)$. We assume the existence of spectral density function of $x(t)$, i.e.

$$\gamma(\lambda, \mu) = \int_{-\infty}^{\infty} f_1 \left( \frac{\lambda + \mu}{2} \right) f_2(\lambda - \mu)\,d\lambda\,d\mu.$$

It follows from the existence of $\int_{-\infty}^{\infty} |f(\lambda, \mu)|\,d\lambda\,d\mu$ that

$$\gamma(\lambda, \mu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1 \left( \frac{\lambda + \mu}{2} \right) f_2(\lambda - \mu)\,d\lambda\,d\mu.$$

Let us prove that $\xi(\lambda)$ has the derivative in the quadratic mean. As familiarly known, such a derivative exists if and only if $\gamma(\lambda, \mu)$ has the generalized second derivative on the diagonal $(\lambda, \lambda)$

$$\lim_{h \to 0, h' \to 0} \frac{\Delta \lambda \Delta \mu \gamma(\lambda, \lambda)}{hh'} = \lim_{h \to 0, h' \to 0} \frac{1}{hh'} \int_{-\infty}^{\lambda + h} \int_{-\infty}^{\lambda + h'} f_1 \left( \frac{\lambda + \mu}{2} \right) f_2(\lambda - \mu)\,d\mu\,d\lambda.$$

As $f_1$ is assumed to be continuous there exists the quadratic mean derivative $\xi'(\lambda)$ of $\xi(\lambda)$ and its covariance function

$$\mathbb{E}[\xi'(\lambda)\xi'(\mu)] = f_1 \left( \frac{\lambda + \mu}{2} \right) f_2(\lambda - \mu)$$

is, as we see, locally stationary. Now, we can apply Lemma 2 and we immediately obtain

$$x(t) = \int_{-\infty}^{\infty} e^{i\omega t} \xi(\omega)\,d\omega.$$\qed

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The following Theorem 3 expresses the inverse formula to the formula (4).

**Theorem 3.** Let $z(X)$ be a locally stationary random process with the continuous covariance function $f(\lambda, \mu) = f_1[(\lambda + \mu)/2]f_2(\lambda - \mu)$. If $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(\lambda, \mu)| < \infty$ then there exists

$$x(t) = \int_{-\infty}^{t} e^{i \lambda \lambda} z(\lambda) d\lambda$$

in the quadratic mean sense and $x(t)$ is locally stationary, too. If $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |E[x(s)] - z(t)|^2 ds dt < \infty$ then

$$z(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i \lambda t} x(t) dt .$$

**Proof.** The integral (5) exists if and only if the integral $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\lambda - \mu)t} f(\lambda, \mu) \lambda d\lambda d\mu$ exists, where $f(\lambda, \mu)$ is the covariance function of $z(t)$. As we assume the existence of $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(\lambda, \mu)| \lambda d\lambda d\mu$ and $|e^{i\lambda t}| \leq 1$ then the integral (5) exists in the quadratic mean sense. It follows that $x(t)$ is harmonizable with covariance function

$$R(s, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\lambda - \mu)s} f_1(u) e^{i(\mu + \lambda)/2} f_2(\lambda - \mu) \lambda d\lambda d\mu$$

according to Lemma 1. By use of transformation $T(\lambda, \mu) = (u, v), u = (\lambda + \mu)/2, v = \lambda - \mu$ we can write

$$R(s, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\lambda - \mu)s} f_1(u) e^{i(\mu + \lambda)/2} f_2(v) dv = R_1 \left( \frac{s + t}{2} \right) R_2(s - t) .$$

Surely, $R_1 \geq 0$ and $R_2$ is a stationary covariance. It means that $x(t)$ is a locally stationary process, too. We have proved that

$$R_1(x) = \int_{-\infty}^{\infty} e^{ixs} f_2(v) dv \quad R_2(y) = \int_{-\infty}^{\infty} e^{iyu} f_1(u) du .$$

The integral

$$w(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i \lambda t} x(t) dt$$

exists in the quadratic mean sense because according to Lemma 1

$$E[w(\lambda) \bar{w}(\mu)] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i \lambda \lambda} E[x(s) \bar{x}(t)] ds dt =$$

$$\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i \lambda \lambda} \int_{-\infty}^{\infty} R_1 \left( \frac{s + t}{2} \right) R_2(s - t) ds dt =$$

$$\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i \lambda \lambda} R_1(u) e^{-i \left( \frac{s + t}{2} \right)} R_2(s) du dv =$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i \lambda \lambda} R_1(u) du \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i \left( \frac{s + t}{2} \right)} R_2(s) dv = f_1 \left( \frac{\lambda + \mu}{2} \right) f_2(\lambda - \mu) .$$
exists for every pair \((\lambda, \mu)\). Now, we must prove that \(w(\lambda) = z(\lambda)\) for every \(\lambda \in R_1\). For this reason we estimate

\[
E \left[ \left| \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\lambda x(t)} dt \right|^2 \right].
\]

Surely, \(z(\lambda) = \text{i.m.} (1/h) \int_{-\infty}^{+\infty} z(u) du\), where the integral is understood in the quadratic mean sense. The integral \(\int_{-\infty}^{+\infty} z(u) du\) exists in the quadratic mean because the process \(z(\cdot)\) has the continuous covariance function and therefore the integral \(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E[z(u) \overline{z}(v)] du dv\) exists. We can express

\[
E \left[ \frac{1}{h} \int_{-\lambda}^{+\lambda} z(u) du - z(\lambda) \right]^2 = E \left[ \frac{1}{h} \int_{-\lambda}^{+\lambda} [z(u) - z(\lambda)] du \right]^2 =
\]

\[
= \frac{1}{h^2} \int_{-\lambda}^{+\lambda} \int_{-\lambda}^{+\lambda} E[[z(u) - z(\lambda)], [z(v) - z(\lambda)]] du dv
\]

according to Lemma 1. The continuity of \(E[z(u) \overline{z}(v)]\) at the point \((\lambda, \lambda)\) implies that \(\lim_{h \to 0} \int_{-\lambda}^{+\lambda} \int_{-\lambda}^{+\lambda} E[z(u) \overline{z}(v)] du dv = 0\). With respect to the inverse formula for harmonizable processes

\[
\text{i.m.} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\lambda x(t)} dt = \int_{-\lambda}^{+\lambda} z(u) du
\]

for \(h > 0\) and every \(\lambda \in R_1\). Then it is possible to write

\[
E \left\{ \frac{1}{h} \int_{-\lambda}^{+\lambda} z(u) du - \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix(t)} \bar{z}(t) dt \right\} =
\]

\[
= \lim_{h \to 0} E \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix(t)} [e^{-ish} - 1] x(s) ds \right\} =
\]

\[
= \lim_{h \to 0} \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i\lambda (e^{-ish} - 1)} e^{itx} E[x(s) \bar{z}(t)] ds dt =
\]

\[
= \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i\lambda (e^{-ish} - 1)} e^{itx} R_1 \left( \frac{s + t}{2} \right) R_2(s - t) ds dt
\]

because \(|(e^{-ish} - 1)(e^{itx})| \leq 1\) and \(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |R(s, t)| ds dt = \text{finite}\). Now, we use the triangular inequality

\[
(6) \quad \left( E \left[ \frac{1}{h} \int_{-\lambda}^{+\lambda} z(u) du \right]^2 \right)^{1/2} \leq
\]

\[
\leq \left( E \left[ \frac{1}{h} \int_{-\lambda}^{+\lambda} z(u) du \right]^2 \right)^{1/2} + \left( E \left[ \frac{1}{h} \int_{-\lambda}^{+\lambda} z(u) du - \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\lambda x(t)} dt \right]^2 \right)^{1/2}
\]

which holds for every \(h > 0\). The first term in (6) tends to zero for \(h \to 0\), the second
term can be calculated by aim of Lemma 1 as follows

\[
E\left\{ \left[ \frac{1}{h} \int_{-\infty}^{+\infty} z(u) du - \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ist} x(t) dt \right]^2 \right\} = \frac{1}{h^2} \int_{-\infty}^{+\infty} f'(u/v) f^2(u-v) du + \\
+ \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i(s-t)} R_1 \left( \frac{s+t}{2} \right) R_2(s-t) ds dt - \\
- \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i(s-t)} R_1 \left( \frac{s+t}{2} \right) R_2(s-t) ds dt - \\
- \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i(s-t)} R_1 \left( \frac{s+t}{2} \right) R_2(s-t) ds dt .
\]

As \(|e^{ish} - 1|/(ish)| \leq 1 and \lim (e^{ish} - 1)/(ish) = 1, it is

\[
\lim_{h \to 0} E\left[ \left[ \frac{1}{h} \int_{-\infty}^{+\infty} z(u) du - \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ist} x(t) dt \right]^2 \right] = 0 ,
\]

which proves that

\[
z(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ist} x(t) dt .
\]

We have so far assumed the existence of the spectral density function of \(x(t)\). At this moment we omit this assumption and let \(x(t)\) be generally locally stationary and harmonizable. It means that there exists a spectral decomposition

\[
x(t) = \int_{-\infty}^{+\infty} e^{ist} d\xi(\lambda) ,
\]

where \(E[d\xi(\lambda) d\xi(\mu)] = ddy(\lambda, \mu)\) and \((\lambda, \mu)\) is a covariance function with bounded variation on \(R \times R\). The covariance function of \(x(t)\) \(R(s, t)\) can be expressed as

\[
R(s, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(s-t)\lambda} ddy(\lambda, \mu) = R_1 \left( \frac{s+t}{2} \right) R_2(s-t) ,
\]

where \(R_1 \geq 0\) and \(R_2\) is a stationary covariance function. When we put \(s = t\), then

\[
R(s, s) = R_1(s) . R_2(0) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(s-t)\lambda} ddy(\lambda, \mu) ,
\]

similarly when \(s = -t\), then

\[
R \left( \frac{s}{2} , -\frac{s}{2} \right) = R_1(0) R_2(s) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(s-t)\lambda} ddy(\lambda, \mu) .
\]

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These relations imply that
\[ R(u, v) = R\left(\frac{u + v}{2}\right) \quad R_2(u - v) = R\left(\frac{u - v}{2}, \frac{v - u}{2}\right) \]
for every pair \((u, v)\). Now, we shall put
\[ \frac{s + t}{2} = x, \quad s - t = y \]
into (7). Then \(s = \frac{x + y}{2}, t = \frac{x - y}{2}\) and
\[
R_1(x) R_2(y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x+\frac{y}{2})} e^{-(x+\frac{y}{2})^2} d\lambda d\mu = \\
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x-\frac{y}{2})} e^{-(x-\frac{y}{2})^2} d\lambda d\mu.
\]
We again apply the transformation \(T: u = (\lambda + \mu)/2, v = \lambda - \mu\) which implies a new measure \(\rho(u, v)\) in \(R_1 \times R_1\) by the relation
\[
E \times F \times T \mapsto E \times F
\]
where \(E \times F\) is a measurable rectangle in \(R_1 \times R_1\). In terms of \(\rho(u, v)\) we obtain
\[
R_1(x) R_2(y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\rho} \, d\rho(u, v).
\]
If we put \(y = 0\) then
\[
R_1(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\rho} \, d\rho(u, v) = \int_{-\infty}^{\infty} e^{i\rho} \, d\rho_1(u)
\]
where
\[
d\rho_1(u) = \int_{-\infty}^{\infty} \, d\rho(u, v),
\]
similarly for \(x = 0\) we get
\[
R_2(y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\rho} \, d\rho(u, v) = \int_{-\infty}^{\infty} e^{i\rho} \, d\rho_2(u)
\]
where
\[
d\rho_2(u) = \int_{-\infty}^{\infty} \, d\rho(u, v).
\]
But it means, together, that
\[
\int_{-\infty}^{\infty} e^{i\rho} \, d\rho_2(v) \int_{-\infty}^{\infty} e^{i\rho} \, d\rho_1(u) = \int_{-\infty}^{\infty} e^{i(u+\rho_1v)} \, d\rho(u, v)
\]
which yields that
\[
d\rho(u, v) = d\rho_1(v) \, d\rho_2(u).
\]
The process \( x(t) \) is locally stationary, it means that \( R_1 \geq 0 \) and \( R_2 \) is a stationary covariance; hence \( R_2 \) can be written as
\[
R_2(\lambda) = \int_{-\infty}^{+\infty} e^{i\lambda v} dF_2(v),
\]
where \( F_2 \) is a probability distribution function because we put \( R_2(0) = 1 \).
This fact implies that
\[
dr_1(u) = dF_1(u).
\]
As \( R_1 \geq 0 \) and hence \( R_1 \) is real, then
\[
R_1(x) = \int_{-\infty}^{+\infty} e^{i\lambda x} dr_2(v) = R_1(x) = \int_{-\infty}^{+\infty} e^{-i\lambda x} d\hat{r}_2(v) = \int_{-\infty}^{+\infty} e^{i\lambda v} d\hat{r}_2(v);
\]
It gives \( dr_2(v) = d\hat{r}_2(-v) \), which means that \( d \text{Re} \ r_2(\cdot) \) is symmetric, i.e.,
\[
\int_{-\infty}^{0} d \text{Re} \ r_2(v) = \int_{0}^{+\infty} d \text{Re} \ r_2(v)
\]
and \( d \text{Im} \ r_2(v) \) is antisymmetric, i.e.,
\[
\int_{-\infty}^{0} d \text{Im} \ r_2(v) = -\int_{0}^{+\infty} d \text{Im} \ r_2(v).
\]
Summarizing these facts we obtain conclusion that the transformation \( T \) associated with definition of locally stationary processes decomposes the induced measure \( \gamma(u,v) \) into two independent parts \( r_1(v), r_2(v) \), where \( r_1(\cdot) \) is a probability distribution function and the Fourier transform of \( r_2(\cdot) \) is nonnegative. On the contrary, if \( \xi(\lambda) \) is a random process whose covariance function \( \gamma(\lambda, \mu) \) has bounded variation on \( \mathbb{R}_1 \times \mathbb{R}_1 \) and if under the transformation \( T \) the induced measure \( \gamma T^{-1} \); i.e.,
\[
\gamma T^{-1}(u,v) = d\gamma(u,v),
\]
is decomposable into two independent parts
\[
d\gamma(u,v) = dr_1(v) dr_2(v)
\]
where \( r_1(v) \) is a probability distribution function and the Fourier transform of \( r_2(u) \) is nonnegative, then the Fourier transform of \( \xi(\lambda) \) (in the quadratic mean sense)
\[
\int_{-\infty}^{+\infty} e^{i\lambda \xi} d\gamma(\lambda)
\]
is a locally stationary random process.
The following Theorem 4 gives the answer when a locally stationary random process is harmonizable.

Theorem 4. Let \( x(t) \) be a locally stationary random process such that
\[
R_1(x) = \int_{-\infty}^{+\infty} e^{i\lambda x} dF_1(\lambda),
\]

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where $F_2$ is generally a complex measure with bounded variation. Then $x(t)$ is harmonizable.

Proof. We know that $E\{x(s)x(t)\} = R_x((s + t)/2)$ where $R_x(y) = \int_0^\infty e^{iy\lambda}dF_2(\lambda)$ with a probability distribution function $F_1$. This yields

$$R(s, t) = \int_{-\infty}^{\infty} e^{i(s-t)\lambda}dF_2(\lambda) = \int_{-\infty}^{\infty} e^{i(s-t)\lambda}dF_1(\mu)F_2(\lambda).$$

Now, let us consider the transformation $S: u = \mu + \lambda/2$, $v = \mu - \lambda/2$. Then $R(s, t) = -\int_{-\infty}^{\infty} e^{i(u-v)\lambda}d\gamma(u, v)$ where $\gamma(u, v)$ is induced from $F_1(\mu)F_2(\lambda)$ by the transformation $S$. As the function $e^{i(u-v)\lambda}$ is continuous we can assume that $\gamma(u, v)$ is normalized as usually assumed in the harmonic analysis. At this moment we must prove that $\gamma(u, v)$ is covariance function belonging to a random process $\xi(t)$. $F_1(\mu)F_2(\lambda)$ is of bounded variation so $\gamma(u, v)$ has bounded variation, too. It is easy to prove that at every point $(u, v)$ and for every pair $h, h' > 0$ there exists the limit

$$\lim_{t\to\infty} \int_{-T}^{T} \int_{-T}^{T} e^{-i(u-v)\lambda}e^{-i(h-h')\lambda}d\gamma(u, v) = \lim_{t\to\infty} R(t, s) dt ds$$

equal to $\Delta_0 \Delta_0 \gamma(u, v)$. Using this fact we immediately see that the sequence

$$\left\{ \int_{-\infty}^{\infty} e^{-i(u-v)\lambda}e^{-i(h-h')\lambda}d\gamma(u, v) \right\}_{t, s, h, h'}$$

is fundamental in the quadratic mean and hence there exists a random variable

$$z_h(u) = \lim_{t\to\infty} \int_{-\infty}^{\infty} e^{-i(u-v)\lambda}e^{-i(h-h')\lambda}d\gamma(u, v)$$

is possible to put $\gamma(-\infty, -\infty) = 0$. Formula (8) gives by elementary calculation the additive property of $z_h(u)$ in the following sense

$$z_h(u) = \sum_{j=0}^{n} z_h(u_j)$$

the last equality holds for every subdivision of $[-t, \lambda]$. Let us prove that there exists the limit

$$\lim_{t\to\infty} z_h(u) = \xi(t).$$

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it follows from that
\[ E[|z_{t+(-t)} - z_{v+(-t)}|^2] = \] \[ = \Delta_A y(-(t + r), -(t + r)) \rightarrow 0 \text{ as } t \rightarrow \infty \text{ for every } r > 0. \]

The continuity of scalar product with respect to convergence in the quadratic mean yields that
\[ \text{cov}(\zeta(u), \zeta(v)) = E[\zeta(u) \zeta(v)] = \lim_{t \to \infty} E[z_{t+(-t)} z_{v+(-t)}] = \] \[ = \lim_{t \to \infty} \Delta_{A,t} \Delta_{A,v} y(-t, -t) = y(u, v). \]

This assertion says that \( y(u, v) \) must be a covariance function. Using the theory of harmonizable random processes presented in [2] we obtain from these facts that \( x(t) \) is harmonizable and hence \( x(t) \) has a representation of the form
\[ x(t) = \int_{-\infty}^{+\infty} e^{itu} d\zeta(u). \]

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