Weighting Function and State Equations of Linear Discrete-Time-Varying System

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A linear nonstationary discrete-time system is considered in this work. The ways are presented for determination of system state equations if the weighting function is known.

State equations of a linear discrete-time system can be obtained from its input-output difference equation in stationary [1] as well as in nonstationary case [2]. It is the purpose of this paper to show the direct transformation of the system weighting function into the state space description.

I. FUNDAMENTAL RELATIONS

A single-input, single-output, linear discrete-time system can be described on definite time interval $N$ by the state equations

\begin{align}
\text{(1a)} \quad x(n + 1) &= A(n) x(n) + b(n) u(n), \\
\text{(1b)} \quad y(n) &= c(n) x(n) + d(n) u(n)
\end{align}

where a system input and output are denoted by $u(n)$ and $y(n)$ respectively, $x(n)$ is an $s$-vector of state variables; $A(n), b(n), c(n)$ and $d(n)$ are parameters of proper dimensions. The action period is assumed here to be $T = 1$ for simplicity, i.e., discrete values of time ranges over integers $n \in N = [n_0, n_1]$.

Solving the equation (1a) we get [1]

\begin{equation}
(2) \quad x(n) = \Phi(n, n_0) x(n_0) + \sum_{k=n_0}^{n-1} \Phi(n, k + 1) b(k) u(k)
\end{equation}

where the system transition (fundamental) matrix

\begin{equation}
(3) \quad \Phi(n, k) = A(n - 1) A(n - 2) \ldots A(k) \quad (n > k)
\end{equation}
satisfies the equation
\[ (4) \quad \Phi(n + 1, k) - A(n) \Phi(n, k) = 0 \]
under the initial condition
\[ (5) \quad \Phi(k, k) = I \]
(identity matrix).

The transition matrix possesses the following properties:
\[ (6) \quad a) \quad \Phi(n, n) = I ; \]
\[ (7) \quad b) \quad \Phi(n, k) = \Phi(n, l) \Phi(l, k) ; \quad n \geq l \geq k ; \]
\[ (8) \quad c) \quad \Phi(k, n) = \Phi^{-1}(n, k) = A^{-1}(k) A^{-1}(k + 1) \ldots A^{-1}(n - 1) \]
provided the inverses of $A(n)$ exist.

Using the equations (2) and (1b) the output can be expressed as
\[ (9) \quad y(n) = c(n) \Phi(n, n_0) x(n_0) + c(n) \sum_{k=n_0}^{n-1} \Phi(n, k + 1) b(k) u(k) + d(n) u(n) . \]

Let us now consider the system weighting function (weighting sequence, impulse response) $g(n, k)$. It can be defined as the response of initially relaxed system (1) to the discrete-time equivalent of Dirac impulse signal determined \[3\] by the relation
\[ (10) \quad \sigma(n - k) = \begin{cases} 0 ; & n \neq k , \\ 1 ; & n = k . \end{cases} \]

Obviously with respect to physical realizability of the system
\[ (11) \quad g(n, k) = 0 \quad \text{for} \quad n < k . \]

Assuming a system that is fully relaxed at $n < n_0$, the output $y(n)$ resulting from any input $u(n), n \geq n_0$, is determined by the summation
\[ (12) \quad y(n) = \sum_{k=n_0}^{n} g(n, k) u(k) . \]

Comparing the relations (12) and (9) with $x(n_0) = 0$ we have
\[ (13) \quad g(n, k) = c(n) \Phi(n, k + 1) b(k) ; \quad n_0 \leq k < n \]
and
\[ (14) \quad g(n, n) = d(n) . \]
The weighting function of linear, finite-dimensional, discrete-time system can always be written in the form

\[ g(n, k) = q(n) h(k) \]

where

\[ q(n) = [q_1(n) \ q_2(n) \ldots q_s(n)] \]

is an \((1 \times s)\) row vector and

\[ h(k) = [h_1(k) \ h_2(k) \ldots h_s(k)]^T \]

is an \((s \times 1)\) column vector.

The order \(s\) of minimal system realization results directly from the form (15) of the weighting function.

If we compare the equations (13) and (15) and respect the above properties of \((n, k)\), the following relations are valid:

\[ q(n) = c(n) \Phi(n, 0) \]

and

\[ h(k) = \Phi(0, k + 1) b(k). \]

II. DETERMINATION OF THE STATE EQUATIONS

Using the relations (14)–(19) the weighting function can be found from the state equations (1) provided that the transition matrix \(\Phi(n, k)\) is available.

We shall investigate the converted problem here, i.e., the formulation of state equations from given weighting function. A system described by its weighting function can be represented, of course, in variety of equivalent state-space forms. Only \(d(n)\) is given unambiguously by the equation (14) while we must always choose \(s^2\) elements to determine all other parameters.

Now several convenient ways will be given for obtaining the state equations of linear discrete-time-varying system represented by its weighting function \(g(n, k)\).

1. Writing \(g(n, k)\) in the form (15) and choosing

\[ \Phi(n, 0) = \begin{bmatrix} q_1(n) & 0 & \ldots & 0 \\ 0 & q_2(n) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & q_s(n) \end{bmatrix} \]

we have from the equation (18)

\[ c(n) = c = [1 \ 1 \ \ldots \ 1]. \]
According to (4) we determine

\[ A(n) = \Phi(n + 1, 0) \Phi^{-1}(n, 0) \]

having the diagonal structure

\[
A(n) = \begin{bmatrix}
q_1(n+1) & 0 & 0 & \ldots & 0 \\
q_1(n) & q_2(n+1) & 0 & \ldots & 0 \\
0 & q_2(n) & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & q_s(n+1)
\end{bmatrix}
\]

and at last from the equation (19) we get

\[ b(n) = \Phi^{-1}(0, n+1) h(n) = \Phi(n+1, 0) h(n) = \begin{bmatrix}
q_1(n+1) h_1(n) \\
q_2(n+1) h_2(n) \\
\vdots \\
q_s(n+1) h_s(n)
\end{bmatrix}.
\]

2. The matrix \( A(n) \) can be taken in the diagonal form as

\[
A = \begin{bmatrix}
\lambda_1 & 0 & 0 & \ldots & 0 \\
0 & \lambda_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \lambda_s
\end{bmatrix}
\]

where all \( \lambda_i \) are arbitrary real constants.

The according to the relations (3) and (6)

\[
\Phi(n, k) = A^{n-k} = \begin{bmatrix}
\lambda_1^{n-k} & 0 & 0 & \ldots & 0 \\
0 & \lambda_2^{n-k} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \lambda_s^{n-k}
\end{bmatrix}.
\]

Using the equations (18) and (19) we obtain

\[ c(n) = q(n) \Phi^{-1}(n, 0) = [q_1(n) \lambda_1^{n-1}; q_2(n) \lambda_2^{n-1}; \ldots; q_s(n) \lambda_s^{n-1}] \]

and

\[ b(n) = \Phi(n+1, 0) h(n) = \begin{bmatrix}
\lambda_1^{n+1} h_1(n) \\
\lambda_2^{n+1} h_2(n) \\
\vdots \\
\lambda_s^{n+1} h_s(n)
\end{bmatrix}
\]

respectively.
3. The special case of the previous way may be formed by putting

\[ A = I. \]

Then obviously

\[ \Phi(n, k) = I, \]
\[ c(n) = q(n) \]

and

\[ b(n) = h(n). \]

The following theorem summarizes these results.

**Theorem.** Every linear, finite-dimensional, discrete-time system, given by the weighting function \( g(n, k) = q(n) h(k) \), can be described in the state-space form

\[
\begin{align*}
    x(n + 1) &= x(n) + h(n) u(n), \\
    y(n) &= q(n) x(n) + g(n, n) u(n).
\end{align*}
\]

The system matrix \( A(n) \) and the transition matrix \( \Phi(n, k) \) are unit matrices.

**Note.** The analogous form with \( A(t) = 0 \) and \( \Phi(t, t) = I \) given for continuous-time system by Kalman [4] is called the normalized canonical form.

4. The widely used canonical form of state equations possesses the parameters

\[
A(n) = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
\alpha_0(n) & \alpha_1(n) & \alpha_2(n) & \cdots & \alpha_{r-1}(n)
\end{bmatrix}
\]

and

\[
b = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\]

Obviously just \( s^2 \) elements are fixed in advance by the relations (34) and (35) provided a vector

\[
\sigma(n) = [\alpha_0(n) \alpha_1(n) \cdots \alpha_{r-1}(n)]
\]

is required to be stated.
Substituting (34) into the equation (4) we get

\[
\Phi(n+1,0) = \begin{bmatrix}
\varphi_1(n+1) \\
\varphi_2(n+1) \\
\vdots \\
\varphi_s(n+1) \\
\varphi(n+1)
\end{bmatrix} = \begin{bmatrix}
\varphi_1(n) \\
\varphi_2(n) \\
\vdots \\
\varphi_s(n) \\
\varphi(n)
\end{bmatrix}
\]

where \( \varphi_j(n) \) is the \( j \)-th row of \( \Phi(n,0) \).

Simply writing \( \varphi(n) \) instead of \( \varphi_j(n) \) it follows from (37)

\[
\Phi(n,0) = \begin{bmatrix}
\varphi(n) \\
\varphi(n+1) \\
\vdots \\
\varphi(n+s-1)
\end{bmatrix}
\]

and

\[
\varphi(n+s) = \sigma(n)\Phi(n,0).
\]

In accordance with (19) we can write

\[
\Phi(n+1+i,0) = b(n+i) = b
\]

where \( i = 0, 1, \ldots, s-1 \) and \( b \) stands in (35).

Then substituting (38) into (40) the following equations are valid for the rows of \( \Phi(n,0) \):

\[
\varphi(n+i) = [1 0 \ldots 0] H^{-1}(n-s+i),
\]

\( i = 0, 1, \ldots, s \).

The \((s \times s)\) matrix

\[
H(n) = [h(n); h(n+1); \ldots; h(n+s-1)]
\]

is always nonsingular if minimal \( s \) in (16) and (17) is taken.

Then \( \Phi(n,0) \) is stated and we obtain

\[
\sigma(n) = \varphi(n+s)\Phi^{-1}(n,0)
\]

according to (39) and using (18)

\[
c(n) = q(n)\Phi^{-1}(n,0).
\]

EXAMPLE

We want to formulate state equations of a system characterized by the weighting function

\[
g(n,k) = 1 - ne^{-(2n-k)}.
\]
Using (15)—(17) we put at first

\[ q_1(n) = 1, \quad q_2(n) = -ne^{-2n}, \]

\[ h_1(k) = 1, \quad h_2(k) = e^k. \]

According to (14) we have

\[ d(n) = 1 - ne^{-n}. \]

The other parameters will be gradually found by applying all above derived equivalent ways.

1. Substituting determined \( q_1 \) and \( h_1 \) into the relations (20)—(24) we get

\[ \Phi(n, 0) = \begin{bmatrix} 1 & 0 \\ 0 & -ne^{-2n} \end{bmatrix}, \]

\[ c(n) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \]

\[ A(n) = \begin{bmatrix} 1 & 0 \\ 0 & n + 1 - ne^{-2} \end{bmatrix} \]

and

\[ b(n) = \begin{bmatrix} 1 \\ -(n + 1) e^{-(n+2)} \end{bmatrix}. \]

2. If we choose

\[ A = \begin{bmatrix} e^{-1} & 0 \\ 0 & e^{-2} \end{bmatrix} \]

the remaining parameters are determined by (26)—(28) as

\[ \Phi(n, 0) = \begin{bmatrix} e^{-n} & 0 \\ 0 & e^{-2n} \end{bmatrix}, \]

\[ c(n) = \begin{bmatrix} e^n; & -n \end{bmatrix} \]

and

\[ b(n) = \begin{bmatrix} e^{-(n+1)} \\ e^{-(n+2)} \end{bmatrix}. \]

3. Choosing \( A = I \) the normalized canonical form of state equations is given by (33):

\[ x(n+1) = x(n) + \begin{bmatrix} 1 \\ e^n \end{bmatrix} u(n), \]

\[ y(n) = \begin{bmatrix} 1; & -ne^{-2n} \end{bmatrix} x(n) + (1 - ne^{-n}) u(n). \]

4. In accordance with (42)

\[ H(n) = \begin{bmatrix} 1 & 1 \\ e^n & e^{n+1} \end{bmatrix}. \]
and

\[ H^{-1}(n) = \frac{1}{e - 1} \begin{bmatrix} e^{-s} & -e^{-s} \\ -1 & e^{-s} \end{bmatrix} \]

Then using (41), (38), (43) and (44)

\[ \phi(n + 2) = \frac{1}{e - 1} \begin{bmatrix} e; -e^{-s} \end{bmatrix}, \]

\[ \Phi(n, 0) = \frac{1}{e - 1} \begin{bmatrix} e & -e^{-(s-2)} \\ e^{-s} & e^{-(s-1)} \end{bmatrix}, \]

\[ a(n) = a = \begin{bmatrix} -e^{-s} & 1 + e^{-s} \end{bmatrix} \]

and

\[ c(n) = [e^{-s}(ne^{-s} - 1); 1 - ne^{-(s+1)}] \]

respectively.

The results are completed by

\[ A(n) = A = \begin{bmatrix} 0 & 1 \\ -e^{-1} & 1 + e^{-1} \end{bmatrix} \]

and

\[ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]

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REFERENCES


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