Generalization of the Non-additive Measures of Uncertainty and Information and their Axiomatic Characterizations*

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The object of this paper is to define generalized non-additive (i) entropy of order \( a \) and type \( \beta \) and (ii) information of order \( a \) and type \( \beta \) and to give their axiomatic characterizations. Further generalizations are indicated towards the end of the paper.

1. INTRODUCTION AND THE GENERALIZATIONS

Let \( P = (p_1, \ldots, p_n) \), \( n \geq 1 \) be a finite discrete probability distribution with \( p_i > 0, W(P) = \sum_{i=1}^{n} p_i = 1 \). \( W(P) \) is called the weight of the distribution \( P \). Let \( \mathcal{A} \) denote the set of all finite discrete generalized probability distributions. Introducing a parameter \( \beta \), we call \( W(P; \beta) = \sum_{i=1}^{n} p_i^{\beta} \leq 1, \beta > 0 \), as the generalized weight of the distribution \( P \). Clearly, \( W(P; 1) = W(P) \).

In what follows, \( \sum \) will stand for the sum \( \sum_{i=1}^{n} \) unless otherwise specified.

Now we introduce a new generalization of the non-additive entropy \([2,4]\) as

\[
H_a(P; \beta) = (1 - \sum p_i^{a+\beta-1})^{(1 - 2^{a-1})},
\]

where \( a \neq 1, \beta > 0, a + \beta - 1 > 0 \); which we shall call as the generalized non-additive entropy of order \( a \) and type \( \beta \).

Let \( P = (p_1, \ldots, p_n) \in \mathcal{A} \) and \( Q = (q_1, \ldots, q_n) \in \mathcal{A} \) be the two generalized probability distributions, the correspondence between the elements of \( P \) and \( Q \) is that given by their subscripts. Then we define a new generalized non-additive information of

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order $\alpha$ and type $\beta$ as

\begin{equation}
I_\alpha(P; \beta | Q) = (1 - \sum P_i^{\alpha + \beta - 1} Q_i^{1-\alpha}) \sum P_i^n (1 - 2^{\alpha - 1}),
\end{equation}

$\alpha + 1, \beta > 0, \alpha + \beta - 1 > 0$.

For $\beta = 1$, (1.2) reduces to the non-additive measure of information of order $\alpha$ which has recently been characterized by means of a functional inequality by the author [3].

The additive entropy of order $\alpha$ and type $\beta$ [5,6] is defined by the expression,

\begin{equation}
H_\alpha(P) = (1 - \alpha)^{-1} \log_2 \left( \sum P_i^{\alpha + \beta - 1} \right),
\end{equation}

$\alpha + 1, \beta > 0, \alpha + \beta - 1 > 0$;

where as the additive information of order $\alpha$ and type $\beta$ [7] is defined as,

\begin{equation}
I_\alpha(P; Q) = (\alpha - 1)^{-1} \log_2 \left( \sum P_i^{\alpha + \beta - 1} Q_i^{1-\alpha} \right),
\end{equation}

$\alpha + 1, \beta > 0, \alpha + \beta - 1 > 0$.

It is easy to find from (1.1) and (1.3) that* 

\begin{equation}
H_\beta(P; \alpha) = (1 - 2^{1-\alpha} H_\alpha(P)) / (1 - 2^{1-\alpha});
\end{equation}

and from (1.2) and (1.4), we get

\begin{equation}
I_\beta(P; \alpha | Q) = (1 - 2^{(\alpha-1)H_\alpha(P)}) / (1 - 2^{1-\alpha}).
\end{equation}

The conditions $\beta > 0$ and $\alpha + \beta - 1 > 0$ are put so that some of the $p$'s may be allowed to take zero values.

The object of this paper is to prove some characterization theorems for the generalized non-additive measures of uncertainty (1.1) and information (1.2) respectively by assuming certain sets of postulates. On specializing the parameter $\beta$ (i.e., $\beta = 1$), one can easily obtain similar results for the ordinary non-additive measures of uncertainty and information.

2. CHARACTERIZATION OF THE GENERALIZED UNCERTAINTY

This section deals with the characterizations of the generalized non-additive measures of uncertainty, $H_{\alpha}(P; \beta)$ by two sets of postulates. The axiomatic characterizations are given below in the form of two theorems which generalize the recent results of [4].

Postulate 1. \( \lim_{p \to 0} H_{\alpha}(1 - p; \beta) = A, \ p \in A. \)

* The author thanks I. Vajda, the reviewer of this paper, for suggesting the relationship between $H_\alpha(P; \beta)$ and $H_\beta(P)$. 
Postulate 2. $H_a(1; \beta) = 1.$

Postulate 3. If $p, q \in A$, then

$$H_a(pq; \beta) = H_a(p; \beta) + H_a(q; \beta) + (2^{1-x} - 1) H_a(p; \beta) H_a(q; \beta).$$

Postulate 4. If $P = (p_1, \ldots, p_n) \in A, Q = (q_1, \ldots, q_m) \in A$ and $W(P; \beta) + W(Q; \beta) \leq 1$, then

$$H_a(P \cup Q; \beta) = \frac{W(P; \beta) H_a(P; \beta) + W(Q; \beta) H_a(Q; \beta)}{W(P; \beta) + W(Q; \beta)},$$

where $P \cup Q = (p_1, \ldots, p_n, q_1, \ldots, q_m)$.

It is sufficient to assume postulate 4 for $n = m = 1$, the result for the general case follows by induction.

Theorem 1. A function $H_a(P; \beta)$ satisfying the postulates 1, 2, 3 and 4 is given by (1.1) for $n \geq 2$.

Proof. For $p = 1$ the postulate 3 takes the following form,

$$H_a(1; \beta) [1 + (2^{1-x} - 1) H_a(q; \beta)] = 0.$$

Taking $q = \frac{1}{2}$ and using the postulate 2, we find that

$$H_a(1; \beta) = 0.$$

Now with $q = 1 - \delta p/p$, the postulate 3 takes the form,

$$H_a(p; \beta) - H_a(p - \delta p; \beta) = H_a(1 - \delta p/p; \beta) [(1 - 2^{1-x}) H_a(p; \beta) - 1].$$

Dividing (2.3) by $\delta p$ and taking limits as $\delta p \to 0$, we get

$$dH_a(p; \beta)/dp = (A/p) [(1 - 2^{1-x}) H_a(p; \beta) - 1].$$

by using the postulate 1.

Solving the differential equation (2.4) under the boundary conditions given in the postulate 2 and (2.2), we arrive at

$$H_a(p; \beta) = (p^{x-1} - 1)/(2^{1-x} - 1).$$

Hence using (2.5) in postulate 4 proves theorem 1.

Postulate 1 implies that $H_a(p; \beta)$ is differentiable. We can weaken this postulate by assuming the following postulate of continuity:

**Postulate 1'.** $H_a(p; \beta)$ is a continuous function of $p \in (0,1]$. 

Now we prove the following theorem:

**Theorem 2.** A function $H_{\alpha}(P; \beta)$ satisfying the postulates 1, 2, 3 and 4 is given by (1.1) for $n \geq 2$.

**Proof.** Let

\[(2.6)\]
\[g_{\alpha}(p; \beta) = 1 + (2^{1-\alpha} - 1) H_{\alpha}(p; \beta),\]
then from postulate 3, we have

\[(2.7)\]
\[g_{\alpha}(pq; \beta) = g_{\alpha}(p; \beta) g_{\alpha}(q; \beta).\]

Since $H_{\alpha}(p; \beta)$, by postulate 1, is continuous in $(0,1]$ and therefore $g_{\alpha}(p; \beta)$ is also continuous. Hence the only non-zero continuous solutions \[1, p. 41\] of (2.7) are given by

\[(2.8)\]
\[g_{\alpha}(p; \beta) = \alpha^p,\]
where $\alpha$ is a real arbitrary constant which may depend on $\alpha$ and $\beta$.

Now the use of postulate 2 yields $\alpha = \alpha - 1$ giving (2.5). Hence as before, the postulate 4 proves the theorem.

3. CHARACTERIZATION OF THE GENERALIZED INFORMATION

In this section we characterize the generalized non-additive measure of information of order $\alpha$ and type $\beta$. We start by assuming the following postulates.

**Postulate 1.** $\lim_{q \to 0^+} I_{\alpha}(1; \beta | 1 - q)/q = A, \ q \in \Delta.$

**Postulate 2.** $I_{\alpha}(p; \beta | 1)$ is a continuous function of $p \in (0,1]$.

**Postulate 3.** $I_{\alpha}(1; \beta | \frac{1}{2}) = 1.$

**Postulate 4.** $I_{\alpha}(-\frac{1}{2}; \beta | \frac{1}{2}) = 0.$

**Postulate 5.** If $p_1, p_2, q_1, q_2 \in \Delta$, then

\[I_{\alpha}(p_1 p_2; \beta | q_1 q_2) = I_{\alpha}(p_1; \beta | q_1) + I_{\alpha}(p_2; \beta | q_2) +
(2^{1-\alpha} - 1) I_{\alpha}(p_1; \beta | q_2) I_{\alpha}(p_2; \beta | q_1).\]

**Postulate 6.** If $P, Q \in \Delta$, then

\[I_{\alpha}(p; \beta | Q) = \frac{W(P_1; \beta) I_{\alpha}(p_1; \beta | Q_1) + W(P_2; \beta) I_{\alpha}(p_2; \beta | Q_2)}{W(P_1; \beta) + W(P_2; \beta)},\]

where $P = P_1 \cup P_2$ and $Q = Q_1 \cup Q_2$. 

Theorem 3. A function $I_a(p;\beta|q)$ satisfying the postulates 1, 2, 3, 4, 5 and 6 is given by (1.2) for $n \geq 2$.

Proof. Taking $p_1 = p, p_2 = q_1 = 1$ and $q_2 = q$ in postulate 5, we have

$$I_a(p;\beta|q) = I_a(p;\beta|1) + I_a(1;\beta|q) + (2^{-1} - 1)I_a(p;\beta|1)I_a(1;\beta|q)$$

(3.1)

Postulate 5 for $p_1 = p_2 = 1$ gives

$$I_a(1;\beta|q_1q_2) = I_a(1;\beta|q_1) + I_a(1;\beta|q_2) +$$

$$+ (2^{-1} - 1)I_a(1;\beta|q_1)I_a(1;\beta|q_2).$$

(3.2)

Now for $q_2 = 1$, (3.2) yields

$$I_a(1;\beta|1)[1 + (2^{-1} - 1)I_a(1;\beta|q_1)] = 0.$$

(3.3)

Taking $q_1 = \frac{1}{2}$ and using the postulate 3, we have

$$I_a(1;\beta|1) = 0.$$

(3.4)

Again taking $q_1 = q, q_2 = 1 - \delta q/q$ in (3.2), we get

$$I_a(1;\beta|q) - I_a(1;\beta|q - \delta q) = I_a(1;\beta|1 - \delta q/q) \left[(1 - 2^{-1})I_a(1;\beta|q) - 1\right];$$

which on dividing by $\delta q$, taking limits as $\delta q \to 0$ and using the postulate 1 gives the following differential equation

$$dI_a(1;\beta|q)/dq = (A/q) \left[(1 - 2^{-1})I_a(1;\beta|q) - 1\right].$$

(3.5)

Solving the differential equation (3.5) under the boundary conditions given in (3.4) and the postulate 3, we have

$$I_a(1;\beta|q) = (q^{1-z} - 1)/(2^{z-1} - 1).$$

(3.6)

Taking $q_1 = q_2 = 1$ in postulate 5, we get

$$I_a(p_1,p_2;\beta|1) = I_a(p_1;\beta|1) + I_a(p_2;\beta|1) +$$

$$+ (2^{-1} - 1)I_a(p_1;\beta|1)I_a(p_2;\beta|1).$$

(3.7)

Let

$$g_a(p;\beta|1) = 1 + (2^{-1} - 1)I_a(p;\beta|1),$$

then from (3.7) we have

$$g_a(p_1,p_2;\beta|1) = g_a(p_1;\beta|1)g_a(p_2;\beta|1).$$

(3.8)

By postulate 2 the continuity of $I_a(p;\beta|1)$ implies the continuity of $g_a(p;\beta|1)$ and hence the non-zero continuous solutions of (3.9) are given by [1, p. 41],

$$g_a(p;\beta|1) = p^z,$$

(3.10)
where $a$ is a real arbitrary constant. Hence

\begin{equation}
I_a(p; \beta | 1) = (p^a - 1)(2^{a-1} - 1).
\end{equation}

Thus (3.1) on using (3.6) and (3.11) gives

\begin{equation}
I_a(p; \beta | q) = (p^a q^{-a} - 1)(2^{a-1} - 1).
\end{equation}

The use of postulate 4 yields $a = \alpha - 1$ giving

\begin{equation}
I_a(p; \beta | q) = (p^{\alpha-1} q^{-\alpha - 1} - 1)(2^{\alpha-1} - 1).
\end{equation}

Theorem 3 can now be obtained on using (3.13) and the postulate 6.

Now we replace the postulate 1 by a weaker postulate assuming the continuity of $I_a(1; \beta | q)$.

**Postulate 1**: $I_a(1; \beta | q)$ is a continuous function of $q \in (0,1]$.

**Theorem 4**: A function $I_a(p; \beta | Q)$ satisfying the postulates 1', 2, 3, 4, 5 and 6 is given by (1.2) for $n \geq 2$.

**Proof.** As done in the later part of the proof of theorem 3, it is easy to prove in this case that

\begin{equation}
I_a(p; \beta | 1) = (p^a - 1)(2^{a-1} - 1)
\end{equation}

and

\begin{equation}
I_a(1; \beta | q) = (q^b - 1)(2^{b-1} - 1)
\end{equation}

giving

\begin{equation}
I_a(p; \beta | q) = (p^a q^b - 1)(2^{a+b} - 1).
\end{equation}

The use of postulate 3 and 4 yields $a = \alpha - 1$ and $b = 1 - \alpha$ giving (3.13) from which theorem 4 follows by postulate 6.

### 4. FURTHER GENERALIZATIONS

In this section we give some further generalizations of the non-additive measures of uncertainty and information. They are:

(i) The generalized non-additive entropy of order $\alpha$ and type $\{\beta_i\}$,

\begin{equation}
H_2(P; \beta_i | Q) = (1 - \sum p_i^{z+1} / \sum p_i^z)(1 - 2^{z-1}),
\end{equation}

$\alpha + 1, \ \beta_i > 0, \ \alpha + \beta_i - 1 > 0$.  

(ii) The generalized non-additive information of order $a$ and type $\{\beta_i\}$,

\[(5.2) \quad I_a(P; \beta_i | Q) = (1 - \sum p_i^{a-1} q_i^{1-a})/(1 - 2^{a-1}) \cdot \]

\[\alpha + 1, \quad \beta_i > 0, \quad \alpha + \beta_i - 1 > 0. \]

Clearly (5.1) and (5.2) yield (1.1) and (1.2) respectively for $\beta_i = \beta$ for all $i = 1, \ldots, n$. It is proposed to study (5.1) and (5.2) in subsequent papers.

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REFERENCES


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**VÝTAH**

Zobecnění neaditivních měr nejistoty a informace a jejich axiomatické charakteristiky

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Budíž $P = (p_1, \ldots, p_n)$ konečné diskrétní rozložení pravděpodobnosti pro $p_i > 0, \sum p_i \leq 1$. Nechť $A$ znamená množinu všech konečných diskrétních rozložení pravděpodobnosti. Pak zobecněná neaditivní entropie řádu $a$ a typu $\beta$ je definována vztahem

\[(1.1) \quad H_a(P; \beta) = (1 - \sum p_i^{a-1} q_i^{1-a})/(1 - 2^{a-1}) \cdot \]

\[\alpha + 1, \quad \beta > 0, \quad \alpha + \beta - 1 > 0. \]

Rovněž pro $P = (p_1, \ldots, p_n) \in A$ a $Q = (q_1, \ldots, q_n) \in A$ je definována zobecněná
nedělitelní informace řádu α a typu β vztahem

\[ L(P; β | Q) = (1 - \sum p_i^{x+β-1} q_i^{1-β} | \sum p_i^\beta)(1 - 2^{α-1}) , \]
\[ α + 1, \ β > 0, \ α + β - 1 > 0. \]

Pro (1.1) a (1.2) jsou dokázány čtyři charakterizační věty při uvážení určitých souborů postulátů. Je naznačeno další zobecnění (1.1) a (1.2). První dvě věty zobecnůjí výsledky získané I. Vajdou.

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