MULTIDIMENSIONAL RANDOM PROCESSES
WITH NORMAL COVARIANCES

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The definition and basic properties of multidimensional locally stationary and normal covariance functions are given. Necessary and sufficient conditions characterizing these covariance functions are presented and a close connection with normal operators is shown too.

1. INTRODUCTION

Let \( \{x(t), t \in \mathbb{R}_1\} \) be a second-order random process with vanishing mean and a covariance function \( R(\cdot, \cdot) \). Silverman suggested in [8] a generalization of weak stationarity, named by local stationarity, in the following way. A covariance function \( R(\cdot, \cdot) \) is called locally stationary if for every pair \( s, t \) of reals \( (s, t \in \mathbb{R}_1) \)

\[
R(s, t) = R(2) \left( \frac{s + t}{2} \right) R(1)(s - t)
\]

where \( R(2)(\cdot) \) is a weakly stationary covariance. Thanks to the facts that \( R(s, s) \geq 0 \) for every \( s \in \mathbb{R}_1 \) and \( R(2)(0) \geq 0 \) this definition yields \( R(1)(s) \geq 0 \) for every \( s \in \mathbb{R}_1 \). The definition of local stationarity for random sequences is given in [4]. In this case a covariance function \( R(\cdot, \cdot) \), defined on \( \mathbb{Z} \times \mathbb{Z} \) (Cartesian product of integers), can be expressed as

\[
R(n, m) = R(1)(n + m) R(2)(n - m)
\]

where \( R(2)(\cdot) \) is a stationary covariance. Here, the function \( R(1)(\cdot) \) need not be nonnegative.

Under assumption of continuity of \( R(1)(\cdot) \), \( R(2)(\cdot) \) and nonnegative-definite property of \( R(1)(\cdot) \), in the case of a random process, the corresponding locally stationary covariance function can be written in the form

\[
R(s, t) = \int_{-\infty}^{\infty} e^{iz + i\bar{z}} dF_1(\lambda) dF_2(\mu), \quad (z = \lambda + i\mu, \quad \bar{z} = \lambda - i\mu),
\]

as it is shown in [5]. This expression is a special case of a normal covariance func-
tion introduced and investigated in [5], [6]. For completeness, we present the definition here.

**Definition 1.** A covariance function $R(\cdot, \cdot)$ defined on the plane is said to be normal if for every $s, t$ of reals

$$R(s, t) = \int_{-\infty}^{\infty} e^{ixz} \, dF(x, \mu), \quad z = \lambda + i\mu,$$

where $F(\cdot, \cdot)$ is the distribution function corresponding to a bounded nonnegative measure on the Borel sets in the plane.

The definition of a normal covariance function due to a random sequence is given in [4]. The main aim of this paper is to give the definition of multidimensional locally stationary and normal covariance functions together with presenting necessary and sufficient conditions describing these classes. A close connection with groups of normal operators in a Hilbert space is also given.

2. MULTIDIMENSIONAL LOCAL STATIONARITY

Let $x^T(t) = \{x_1(t), x_2(t), \ldots, x_N(t), \quad t \in \mathbb{R}_1 \}$ be a multidimensional second order random process with vanishing mean value. Let

$$R(s, t) = \text{E}\{x(s) x^T(t)\}$$

be the corresponding covariance function.

**Definition 2.** We say the process $x^T(\cdot)$ is locally stationary (or its covariance function $R(\cdot, \cdot)$ is locally stationary) if for every $N$-tuple $z^T = (z_1, z_2, \ldots, z_N)$ of complex numbers the random process

$$\xi_z(t) = \sum_{i=1}^{N} z_i x_i(t), \quad t \in \mathbb{R}_1$$

has a locally stationary covariance function.

**Lemma 1.** If an $N$-dimensional covariance function $R(\cdot, \cdot)$ is locally stationary then for every $u \in \mathbb{R}_1$ the matrix $R(u, u)$ is positive semidefinite and the matrix $R(s - t, t - s)$ is an $N$-dimensional stationary covariance function.

**Proof.** If $R(\cdot, \cdot)$ is locally stationary then $z^T R(s, t) z$ is for every $z^T = (z_1, \ldots, z_N)$ a one-dimensional local stationary covariance. Then, according to the definition of local stationarity,

$$z^T R(s, t) z = R_z^{(1)}\left(\frac{s + t}{2}\right) R_z^{(2)}(s - t).$$

This fact means $z^T R(u, u) z = R_z^{(1)}(u) R_z^{(2)}(0)$ and

$$z^T R\left(\frac{v}{2}, -\frac{v}{2}\right) z = R_z^{(1)}(0) R_z^{(2)}(v).$$

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Hence,
\[ z^T R(0, 0) z z^T R(s, t) z = z^T R \left( \frac{s + t}{2}, \frac{s + t}{2} \right) z z^T R \left( \frac{s - t}{2}, \frac{s - t}{2} \right) z \]
where \( R_z^{(1)}(0) R_z^{(2)}(0) = z^T R(0, 0) z \). As local stationarity demands \( R_z^{(1)}(u) \geq 0 \) for every \( u \in \mathbb{R} \), and \( R_z^{(2)}(v) \) must be a stationary covariance, we obtain that for every \( z \)
\[ z^T R(u, u) z \geq 0 \]
and
\[ R \left( \frac{s - t}{2}, \frac{t - s}{2} \right) \]
is an \( N \)-dimensional stationary covariance function.

**Theorem 1.** An \( N \)-dimensional covariance function \( R(\cdot, \cdot) \) is locally stationary if and only if for every \( s, t \in \mathbb{R} \) and every multiindex \( x = (i, j, k, l) \in \{1, 2, 3, \ldots, N\}^4 \)
\[ R_{ij}(0, 0) R_{kl}(s, t) + R_{il}(0, 0) R_{kj}(s, t) + R_{ki}(0, 0) R_{ij}(s, t) + \]
\[ + R_{kl}(0, 0) R_{ij}(s, t) = \]
\[ = R_{ij} \left( \frac{s + t}{2}, \frac{s + t}{2} \right) R_{kl} \left( \frac{s - t}{2}, \frac{s - t}{2} \right) + R_{il} \left( \frac{s + t}{2}, \frac{s + t}{2} \right) R_{kj} \left( \frac{s - t}{2}, \frac{s - t}{2} \right) + \]
\[ + R_{kj} \left( \frac{s + t}{2}, \frac{s + t}{2} \right) R_{il} \left( \frac{s - t}{2}, \frac{s - t}{2} \right) \]
where \( R(\cdot, \cdot) = \{R_{ij}(\cdot, \cdot)\}_{i,j=1}^N \).

Before proving Theorem 1 it is suitable to introduce the following

**Lemma 2.** Let \( V_n \) be an \( n \)-dimensional complex vector modul and

\[ \Phi: V_n \times V_n \times V_n \times V_n \rightarrow \mathbb{C} \]
be a mapping having the form

\[ \Phi(u, v, x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} u_i v_j x_k y_l \Phi_{ijkl} \]
where \( u = \sum_{i=1}^{n} u_i e_i, v = \sum_{j=1}^{n} v_j e_j, x = \sum_{k=1}^{n} x_k e_k, y = \sum_{l=1}^{n} y_l e_l \), and \( \Phi_{ijkl} = \Phi(e_i, e_j, e_k, e_l) \) for a fixed basis \( \{e_i\}_{i=1}^n \) of \( V_n \). Then \( \Phi \) is vanishing on the principal diagonal (i.e. \( \Phi(u, u, u, u) = 0 \) for every \( u \in V_n \)) if and only if

\[ \Phi_{ijkl} + \Phi_{jikl} + \Phi_{ijlk} + \Phi_{jilk} = 0 \]
for every \( i, j, k, l = 1, 2, \ldots, n \).

**Proof of Lemma 2.** Let the condition (1) hold. Then for every \( x \in V_n \),
\[ x = \sum_{i=1}^{n} x_i e_i, \]
\[ 4\Phi(x, x, x, x) = \sum_{i,j,k,l} x_i x_j x_k x_l [\Phi_{ijkl} + \Phi_{jikl} + \Phi_{ijlk} + \Phi_{jilk}] = 0. \]
Hence, \( \Phi \) is vanishing on the principal diagonal. Now, assume \( \Phi(x, x, x, x) = 0 \)
for every $x \in V_n$. Then $\Phi(x + t e^{i\omega} y, x + t e^{i\omega} y, x + t e^{i\omega} y, x + t e^{i\omega} y)$ for $x, y \in V_n$ and real $t, \omega$ presents a polynomial function of the 4th degree in $t$ having complex coefficients and vanishing everywhere. The coefficient standing by $t$ must satisfy

$$
[\Phi(y, x, x, x) + \Phi(x, y, x, x)] e^{i\omega} + [\Phi(x, x, y, x) + \Phi(x, x, x, y)] e^{-i\omega} = 0.
$$

Hence, $\Phi'(x) = \Phi(y, x, x, x) + \Phi(x, y, x, x) = 0$ for every $x, y \in V_n$. Now, we shall repeat this consideration twice. First, the coefficient by $t$ in the term $\Phi'(x + t e^{i\omega} y)$ equals

$$
[\Phi(y, z, x, x) + \Phi(z, y, x, x)] e^{i\omega} + [\Phi(y, x, z, x) + \Phi(y, x, z, x)] e^{-i\omega} = 0.
$$

This fact gives

$$\Phi(y, z, x, x) + \Phi(z, y, x, x) = \Phi''(x) = 0$$

for every $x, y, z \in V_n$. Finally, the expression of $\Phi''(x + t e^{i\omega} u)$ yields immediately for every $x, y, z, u \in V_n$

$$\Phi(y, z, x, u) + \Phi(y, z, u, x) + \Phi(z, y, x, u) + \Phi(z, y, u, x) = 0.$$ 

This implies easily condition (1). \qed

Now, the proof of Theorem 1 is an easy matter.

Proof of Theorem 1. Let an $N$-dimensional random process $\{x(t), t \in \mathbb{R}_1\}$ be locally stationary. It means that for every $z^T = (z_1, \ldots, z_N)$, an $N$-couple of complex numbers, and every $s, t \in \mathbb{R}_1$

$$z^T R(0, 0) z = z^T R\left(\frac{s + t}{2}, \frac{s + t}{2}\right) z.$$ 

This equality may be rewritten into the following form

$$0 = \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N z_iz_jz_kz_l \left( R_{ij}(0, 0) R_{kl}(s, t) - R_{ij}\left(\frac{s + t}{2}, \frac{s + t}{2}\right) R_{kl}\left(\frac{s - t}{2}, \frac{t - s}{2}\right)\right).$$

At this moment, we can apply Lemma 2 to the function

$$\Phi(u, v, x, y) = \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N u_i v_j x_k y_l \left( R_{ij}(0, 0) R_{kl}(s, t) - R_{ij}\left(\frac{s + t}{2}, \frac{s + t}{2}\right) R_{kl}\left(\frac{s - t}{2}, \frac{t - s}{2}\right)\right).$$ 

Silverman in [8] proved an assertion dealing with harmonizable locally stationary random processes. This result can be generalized to the multidimensional case.

Theorem 2. Let $\{x(t), t \in \mathbb{R}_1\}$ be an $N$-dimensional random process with harmoniz-
able (in the strong sense) locally stationary covariance function having a spectral density function. Then, this spectral density function is locally stationary and vice versa.

**Proof.** Being strongly harmonizable \( \{x(t), t \in \mathbb{R}_1\} \) can be expressed in the following form

\[
x(t) = \int_{-\infty}^{+\infty} e^{it\lambda} \, d\xi(\lambda)
\]

where \( \{\xi(\lambda), \lambda \in \mathbb{R}_1\} \) is an \( N \)-dimensional second-order random process with covariance function

\[
F(\lambda, \mu) = \{E[\xi_i(\lambda) \xi_j(\mu)]\}_{i,j=1}^{N}
\]

possessing finite variation \( \sum_{i=1}^{N} \sum_k |\Delta F_{ij}(\lambda_k, \mu_k)| \leq C < \infty \) \( (F_{ij}(\lambda, \mu) = E[\xi_i(\lambda) \xi_j(\mu)]) \). Then, the covariance function of \( \{x(t), t \in \mathbb{R}_1\} \) can be written as

\[
R(s, t) = \int_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)} \, f(\lambda, \mu) \, d\lambda \, d\mu
\]

because we assume existence of \( \partial^2 F_{ij}(\lambda, \mu)/\partial \lambda \partial \mu = f_{ij}(\lambda, \mu) \). As \( \{x(t), t \in \mathbb{R}_1\} \) is locally stationary, then by definition, for every \( z^T = (z_1, z_2, \ldots, z_N) \)

\[
z^T R(s, t) z = \int_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)} \, f(\lambda, \mu) \, z \, d\lambda \, d\mu
\]

must be a locally stationary covariance function. The inverse formula, see [3], gives under local stationarity of \( R(\cdot, \cdot) \).

\[
z^T f(\lambda, \mu) z = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)} \, z^T R(s, t) z \, ds \, dt =
\]

\[
= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \exp \left(-i \left(\frac{s + t}{2}\right)(\lambda - \mu)\right) \exp \left(-i(s-t)\left(\frac{\lambda + \mu}{2}\right)\right) z^T R\left(\frac{s - t}{2}, \frac{t - s}{2}\right) z ds \, dt =
\]

\[
= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \exp \left(-i \left(\frac{\lambda + \mu}{2}\right)(s - t)\right) \times
\]

\[
\times \exp \left(-i(\lambda - \mu)\left(\frac{s + t}{2}\right)\right) \frac{z^T R\left(\frac{s + t}{2}, \frac{s + t}{2}\right)}{R_z^{(1)}(0)} \, ds \, st =
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp \left(-i \left(\frac{\lambda + \mu}{2}\right)v\right) \frac{z^T R\left(v, \frac{v}{2}\right)}{R_z^{(1)}(0)} \, dv \times \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i(\lambda - \mu)u} \frac{z^T R(u, u)}{R_z^{(2)}(0)} \, du.
\]

This means

\[
z^T f(\lambda, \mu) z = f_z^{(1)}\left(\frac{\lambda + \mu}{2}\right) f_z^{(2)}(\lambda - \mu)
\]
where
\[ f_{x}^{(1)}(v) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega v} \frac{z^T R(v, v) z}{R_{x}^{(1)}(0)} \, dv \]
and
\[ f_{x}^{(2)}(v) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega v} \frac{z^T R(u, u) z}{R_{x}^{(2)}(0)} \, du . \]

We have proved that for every \( z^T = (z_1, z_2, \ldots, z_N) \) the covariance function \( z^T f(\cdot, \cdot) z \) is locally stationary because
\[ f_{x}^{(1)}(v) \geq 0 \]
for every \( x \in \mathbb{R} \) and \( f_{x}^{(2)}(\cdot) \) is a weakly stationary covariance function. We can summarize that the N-dimensional covariance function \( f(\cdot, \cdot) \) is locally stationary. Now, assume \( f(\cdot, \cdot) \) to be an N-dimensional locally stationary covariance function. Then, \( z^T f(\cdot, \cdot) z \) is locally stationary for every \( z^T = (z_1, z_2, \ldots, z_N) \), i.e.
\[ z^T f(\lambda + \mu, \mu + \lambda) z z^T f(\lambda - \mu, \mu - \lambda) z . \]

Hence,
\[ z^T R(s, t) z = \int f_{x}^{(1)}(v) \frac{z^T f(\lambda + \mu, \mu + \lambda) z z^T f(\lambda - \mu, \mu - \lambda) z}{f_{x}^{(2)}(0)} \, dv . \]

It is easy to see that \( R_{x}^{(1)}(\cdot) \geq 0 \) and \( R_{x}^{(2)}(\cdot) \) is a weakly stationary covariance. We proved local stationarity of \( z^T R(\cdot, \cdot) z \) hence, the process \( \{x(t), t \in \mathbb{R} \} \) is locally stationary.

Theorem 2 affirms, roughly speaking, that the Fourier transform of a locally stationary process is a locally stationary one again.

2. MULTIDIMENSIONAL NORMAL COVARIANCES

Let us suppose that an N-dimensional covariance function \( R(\cdot, \cdot) \) is locally stationary, i.e. one can write
\[ z^T R(0, 0) z z^T R(s, t) z = z^T R \left( \frac{s + t}{2}, \frac{s + t}{2} \right) z z^T R \left( \frac{s - t}{2}, \frac{t - s}{2} \right) z . \]
for every \( z^T = (z_1, z_2, \ldots, z_N) \) of complex numbers and every \( s, t \in \mathbb{R}_1 \). In general, \( R(\frac{1}{2}(s + t), \frac{1}{2}(s + t)) \) need not be an \( N \)-dimensional covariance function in \( s, t \), it is a positive semidefinite matrix for every fixed \( s, t \) as it is proved in Lemma 1. Now, let \( R(\frac{1}{2}(s + t), \frac{1}{2}(s + t)) \) be a covariance function and \( R_{ij}(s, t), i, j = 1, 2, \ldots, N \) be continuous functions on the plane. Then, for every \( z \) the function \( z^T R(\frac{1}{2}(s + t), \frac{1}{2}(s + t)) z \) is a covariance with the kernel \((s + t), h \), hence,

\[
T \left( \frac{s + t}{2}, \frac{s + t}{2} \right) z = \int_{-\infty}^{+\infty} e^{i(s+t)t} \ dF_z(\lambda)
\]

where \( F_z(\cdot) \) is a nondecreasing function with finite variation, for detail see [9].

Analogously, by means of Bochner’s theorem

\[
z^T R \left( \frac{s - t}{2}, \frac{t - s}{2} \right) z = \int_{-\infty}^{+\infty} e^{\lambda(t-s)} \ dG_z(\mu).
\]

Let us denote \( e^T(j, k) = (0, 0, \ldots, 1, \ldots, 0) \) if 1 stands on the \( j \)th and \( k \)th place \((j < k)\); similarly, \( d^T(j, k) = (0, 0, \ldots, 1, \ldots, -i, \ldots, 0) \). Then,

\[
e^T(j, k) R(s, t) e(j, k) = R_{jj}(s, t) + R_{jk}(s, t) + R_{kj}(s, t) + R_{kk}(s, t),
\]

\[
d^T(j, k) R(s, t) d(j, k) = R_{jj}(s, t) - iR_{jk}(s, t) + iR_{kj}(s, t) + R_{kk}(s, t).
\]

The choice of \( z^T_j = (0, 0, 0, 1, 0, \ldots, 0) \), where 1 stands on the \( j \)th place, gives

\[
z^T_j R \left( \frac{s + t}{2}, \frac{s + t}{2} \right) z_j = \int_{-\infty}^{+\infty} e^{i(s+t)t} \ dF_j(\lambda)
\]

and

\[
z^T_j R \left( \frac{s - t}{2}, \frac{t - s}{2} \right) z_j = \int_{-\infty}^{+\infty} e^{\lambda(t-s)} \ dG_j(\mu).
\]

This means, of course, that

\[
R_{jj}(0, 0) R_{jj}(s, t) = \int_{-\infty}^{+\infty} e^{i(s+t)t} e^{i\mu(t-s)} \ dF_j(\lambda) G_j(\mu).
\]

Similarly, for every \( z^T = (z_1, z_2, \ldots, z_N) \) local stationarity yields

\[
z^T R(0, 0) z z^T R(s, t) z = \int_{-\infty}^{+\infty} e^{i(s+t)t} e^{i\mu(t-s)} \ dF_z(\lambda) G_z(\mu).
\]

Especially,

\[
e^T(j, k) R(0, 0) e(j, k) = \int_{-\infty}^{+\infty} e^{i(s+t)t} e^{i\mu(t-s)} \ dF_{e(j,k)}(\lambda) G_{e(j,k)}(\mu),
\]

\[
d^T(j, k) R(0, 0) d(j, k) = \int_{-\infty}^{+\infty} e^{i(s+t)t} e^{i\mu(t-s)} \ dF_{d(j,k)}(\lambda) G_{d(j,k)}(\mu).
\]

Assuming regularity of the matrix \( R(0, 0) \) and combining (2), (3), (4), (5) we obtain that

\[
R_{jk}(s, t) = \int_{-\infty}^{+\infty} e^{i(s+t)t} e^{i\mu(t-s)} \ d \left( \frac{1}{2} \ e^T(j, k) R(0, 0) e(j, k) \right) + \int_{-\infty}^{+\infty} \frac{F_{e(j,k)}(\lambda)}{2} \ dF_{e(j,k)}(\lambda) G_{e(j,k)}(\mu)
\]
We achieved a possibility to express the covariance function $R(\cdot, \cdot)$ in the form

$$R(s, t) = \int \int e^{i(s+it)} e^{i\mu (s-t)} \, dd F(\lambda, \mu)$$

where $F_{jk}(\lambda, \mu)$ is defined by the formula (6). Thanks to the fact that $z^T R(s, t) z$ is a one-dimensional normal covariance function, under our assumptions,

$$z^T R(s, t) z = \int \int e^{i(s+it)} e^{i\mu (s-t)} \, dd F(\lambda, \mu) \frac{F_z(\lambda) G_z(\mu)}{z^T R(0, 0) z},$$

and thanks to the one-to-one correspondence between a normal covariance function and its spectral measure, see Theorem 3, we can assert

$$z^T F(\lambda, \mu) z = \frac{F_z(\lambda) G_z(\mu)}{z^T R(0, 0) z}.$$

As for every $z^T = (z_1, z_2, \ldots, z_N)$ of complex numbers

$$\Delta_{n_1} \Delta_{n_2} F_z(\lambda) G_z(\mu) \geq 0$$

this inequality proves that $F(\cdot, \cdot)$ is a matrix spectral measure. $F(\cdot, \cdot) = \{F_{ij}(\cdot, \cdot)\}_{i,j=1}^N$ is a matrix spectral measure, see [7], if every component $F_{ij}(\cdot)$ is a complex measure defined on the Borel sets in the plane satisfying

1) $F_{ij}(\cdot) = F_{ji}(\cdot)$ for every $i, j = 1, 2, \ldots, N$

2) $\sum_{i=1}^N \sum_{j=1}^N c_i c_j F_{ij}(A) \geq 0$ for every $N$-tuple $c_1, c_2, \ldots, c_N$ of complex numbers and every Borel set $A$ in the plane $\mathbb{R}_2$.

The spectral decomposition of $R(\cdot, \cdot)$ in the form (7) leads us to the following

**Definition 3.** An $N$-dimensional covariance function $R(\cdot, \cdot)$ will be called normal if it can be expressed in the form

$$R(s, t) = \int \int e^{i(s+it)} e^{i\mu (s-t)} \, dd F(\lambda, \mu)$$

where $F(\cdot, \cdot) = \{F_{ij}(\cdot, \cdot)\}_{i,j=1}^N$ is a matrix spectral measure.

**Properties of Normal Covariances**

The existence of $R(s, t)$ for every pair $(s, t) \in \mathbb{R}_2$ implies

$$|R_{ij}(s, t)| \leq \int \int e^{i(s+it)} \, dd |F_{ij}(\lambda, \mu)|, \quad i, j = 1, 2, \ldots, N,$$

where $|F_{ij}(\cdot)|$ is absolute variation of the complex measure $F_{ij}(\cdot)$ because the spectral measure $F$ satisfies the evident relation

$$|F_{ij}(A)| \leq F_{ii}^{1/2}(A) F_{jj}^{1/2}(A)$$

thanks to positive semidefiniteness of $F(\cdot)$. As $F_{ii}(A) \geq 0$ for every $i = 1, 2, \ldots, N$
and every Borel set $A$ in $\mathbb{R}^2$ we see that every integral

$$\int_{-\infty}^{+\infty} e^{i(s+t)} e^{is-t} \, ddF_{ij}(\lambda, \mu), \quad i, j = 1, 2, \ldots, N$$

is absolutely convergent. The above relation (8) gives that every component $F_{ij}(\cdot)$ is of finite absolute variation because

$$\text{Var} F_{ii}(\cdot) = F_{ii}(\mathbb{R}^2) = R_{ii}(0, 0)$$

is finite for every $i = 1, 2, \ldots, N$. Every component $F_{ij}(\cdot)$ can be expressed as the sum $\text{Re} F_{ij}(\cdot) + i \text{Im} F_{ij}(\cdot)$ where both the signed measures are of finite absolute variation. This fact implies that for every $i, j = 1, 2, \ldots, N$

$$F_{ij}(\Delta) = \text{Re} F_{ij}^+(\Delta) - \text{Re} F_{ij}^-(\Delta) + i(\text{Im} F_{ij}^+(\Delta) - \text{Im} F_{ij}^-(\Delta))$$

where all the terms are measures with finite variations. As every one-dimensional normal covariance function is continuous at every point in the plane $\mathbb{R}^2$, see [6], $R_{ij}(\cdot, \cdot)$, which is a sum of normal covariances, cf. (9), must be a continuous function. We can state that every $N$-dimensional normal covariance function is continuous. Further, every normal covariance can be expressed as

$$R_{ij}(s, t) = S_{ij}(s + t, s - t)$$

where $S(\cdot, \cdot) = \{S_{ij}(\cdot, \cdot)\}_{i,j=1}^N$ and

$$S_{ij}(u, v) = \int_{-\infty}^{+\infty} e^{iu} e^{iuv} \, ddF_{ij}(\lambda, \mu).$$

Every function $S_{ij}(\cdot, \cdot)$ is continuous and $S(u, -v) = S^T(u, v)$ where $T$ means the transposed matrix.

**Theorem 3.** Every normal covariance function $R(\cdot, \cdot)$ determines unambiguously a matrix spectral measure $F(\cdot, \cdot)$.

**Proof.** Let the covariance function $R(\cdot, \cdot)$ be normal and let

$$R_{ij}(s, t) = \int_{-\infty}^{+\infty} e^{i(s+t)} e^{is-t} \, ddF_{ij}(\lambda, \mu), \quad i, j = 1, 2, \ldots, N.$$ 

The covariance function $R(\cdot, \cdot)$ determines unambiguously the matrix spectral measure $F(\cdot, \cdot)$ if and only if every component $R_{ij}(\cdot, \cdot)$ determines unambiguously the corresponding complex measure $F_{ij}(\cdot)$. We begin with the diagonal elements $R_{ii}(\cdot, \cdot)$, $i = 1, 2, \ldots, N$. Then the corresponding spectral measure $F_{ii}(\cdot)$ is non-negative as follows from positive semidefiniteness of $F(\cdot)$. The element $R_{ii}(\cdot, \cdot)$ defines in the unique way $S_{ii}(\cdot, \cdot)$ because

$$S_{ii}(u, v) = R_{ii} \left( \frac{u + v}{2}, \frac{u - v}{2} \right).$$

The integral $S_{ii}(u, v) = \int_{-\infty}^{+\infty} e^{iu} e^{iuv} \, ddF_{ii}(\lambda, \mu)$ is absolutely convergent because

$$\int_{-\infty}^{+\infty} \left| e^{iu} e^{iuv} \right| \, ddF_{ii}(\lambda, \mu) = \int_{-\infty}^{+\infty} e^{iu} \, ddF_{ii}(\lambda, \mu) = S_{ii}(u, 0)$$

exists for every pair $(u, v) \in \mathbb{R}_2$. Now, let us consider a complex number $u = u_1 + iu_2$. Then, the integral

$$\int_{-\infty}^{+\infty} e^{iu_1} e^{iu_2} \, ddF_{ii}(\lambda, \mu)$$
is also absolutely convergent. In this way we can extend the function \( S_{il}(\cdot, \cdot) \) for every \( v \in \mathbb{R}_1 \) into the complex plane

\[
S_{il}(u_1 + iu_2, v) = \int \int_{-\infty}^{+\infty} e^{\lambda u_1} e^{i\mu u_2} e^{i\nu v} \, dF_{il}(\lambda, \mu).
\]

Let us prove that the function \( S_{il}(u, v) \) is for every \( v \in \mathbb{R}_1 \) a holomorphic function on the complex plane. We introduce, for this purpose, a complex measure \( \mathcal{G}_v(\cdot, \cdot) \) defined by the relation

\[
\mathcal{G}_v(\lambda, \mu) = \int \int_{-\infty}^{+\infty} e^{i\beta} \, dF_{il}(\alpha, \beta).
\]

Surely, \( |\mathcal{G}_v(\lambda, \mu)| \leq F_{il}(\lambda, \mu) \). Hence, absolute variations of \( \{\mathcal{G}_v(\cdot, \cdot), v \in \mathbb{R}_1\} \), are uniformly bounded and

\[
(10) \quad S_{il}(u, v) = \int \int_{-\infty}^{+\infty} e^{\lambda u} \, d\mathcal{G}_v(\lambda, \mu) = \int \int_{-\infty}^{+\infty} e^{\lambda u} \, d\mathcal{G}_v^{(1)}(\lambda)
\]

where \( \mathcal{G}_v^{(1)}(\cdot) \) is the first marginal measure of \( \mathcal{G}_v(\cdot, \cdot) \). We see that, by (10), the function \( S_{il}(u, v) \) is for every \( v \in \mathbb{R}_1 \) the bilateral Laplace transform of \( \mathcal{G}_v(\cdot, \cdot) \), and hence, it is a holomorphic function of the variable \( u \). The subset \((-\infty, +\infty) \times \{0\}\) is not isolated in the complex plane. This fact implies that \( S_{il}(u_1 + iu_2, v) \) is the unique holomorphic extension that is determined by the values of \( S_{il}(w_1, v), \, u_1 \in (-\infty, +\infty) \). Now, let \( u_1 \) be chosen quite arbitrarily. Then,

\[
S_{il}(u_1 + iu_2, v_2) = \int \int_{-\infty}^{+\infty} e^{\lambda u_1} e^{i\mu u_2} e^{i\nu v_2} \, dH_{u_1}(\lambda, \mu),
\]

\[
dH_{u_1}(\lambda, \mu) = \int \int_{-\infty}^{+\infty} e^{i\alpha u_1} \, dF_{il}(\alpha, \beta).
\]

We see that for every fixed \( u_1 \in (-\infty, +\infty) \) the function \( S_{il}(u_1 + iu_2, v) \) is in the variables \( u_2, v \) the two-dimensional Fourier transform of \( H_{u_1}(\cdot, \cdot) \). Thanks to properties of the Fourier transform the measure \( H_{u_1}(\cdot, \cdot) \) is determined unambiguously. As the function \( e^{su_1} \) is the Radon-Nikodym derivative of \( H_{u_1}(\cdot, \cdot) \) with respect to \( F_{il}(\cdot, \cdot) \), the measure \( F_{il}(\cdot, \cdot) \) is determined by \( H_{u_1}(\cdot, \cdot) \) and \( e^{su_1} \) in the unique way. We have proved a one-to-one correspondence between \( R_{il}(\cdot, \cdot) \) and \( F_{il}(\cdot, \cdot) \).

In the case of a complex measure \( F_{ij}(\cdot, \cdot) \) for \( i + j \) we shall proceed in the following way. Let exist two complex measures such that

\[
R_{ij}(s, t) = \int \int_{-\infty}^{+\infty} e^{i(\lambda s + \mu t)} \, dF_{ij}(\lambda, \mu) = \int \int_{-\infty}^{+\infty} e^{i(\lambda s + \mu t)} \, dG_{ij}(\lambda, \mu)
\]

for every \( s, t \in \mathbb{R}_1 \). Then,

\[
\int \int_{-\infty}^{+\infty} e^{i(\lambda s + \mu t)} \, d(F_{ij}(\lambda, \mu) - G_{ij}(\lambda, \mu)) = 0
\]

for every \( s, t \in \mathbb{R}_1 \). This means, we have to prove that the only complex measure satisfying for every \( u, v \in \mathbb{R}_1 \)

\[
\int \int_{-\infty}^{+\infty} e^{i\lambda u} e^{i\mu v} \, dH(\lambda, \mu) = 0
\]

is zero.

Writing \( H(\cdot, \cdot) = H_1(\cdot, \cdot) + iH_2(\cdot, \cdot) \) we obtain that

\[
\int \int_{-\infty}^{+\infty} e^{i\lambda u} \cos \mu v \, dH_1(\lambda, \mu) = \int \int_{-\infty}^{+\infty} e^{i\lambda u} \sin \mu v \, dH_2(\lambda, \mu)
\]

\[
\int \int_{-\infty}^{+\infty} e^{i\lambda u} \cos \mu v \, dH_2(\lambda, \mu) = - \int \int_{-\infty}^{+\infty} e^{i\lambda u} \sin \mu v \, dH_1(\lambda, \mu)
\]

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This fact yields
\[ \int_{-\infty}^{+\infty} e^{\lambda u} e^{i\mu v} \, ddH_1(\lambda, \mu) = 0, \quad \int_{-\infty}^{+\infty} e^{\lambda u} e^{i\mu v} \, ddH_2(\lambda, \mu) = 0. \]
As we consider measures with finite variations we can decompose
\[
H_1(\cdot, \cdot) = H_1^+(\cdot, \cdot) - H_1^-(\cdot, \cdot)
\]
\[
H_2(\cdot, \cdot) = H_2^+(\cdot, \cdot) - H_2^- (\cdot, \cdot)
\]
by means of the Jordan decomposition. Then, we have for every \( u, v \in \mathbb{R} \)
\[
\int_{-\infty}^{+\infty} e^{\lambda u} e^{i\mu v} \, ddH_1^+(\lambda, \mu) = \int_{-\infty}^{+\infty} e^{\lambda u} e^{i\mu v} \, ddH_1^-(\lambda, \mu),
\]
and similarly
\[
\int_{-\infty}^{+\infty} e^{\lambda u} e^{i\mu v} \, ddH_2^+(\lambda, \mu) = \int_{-\infty}^{+\infty} e^{\lambda u} e^{i\mu v} \, ddH_2^- (\lambda, \mu).
\]
The one-to-one correspondence between one-dimensional normal covariance and spectral measure proved above gives that
\[
H_1^+(\cdot) = H_1^-(\cdot), \quad H_2^+(\cdot) = H_2^- (\cdot).
\]
This fact completes the proof of the theorem. \( \square \)

Necessary and sufficient conditions given in the following theorem describe the class of multidimensional normal covariances.

**Theorem 4.** An \( N \)-dimensional covariance function \( R(\cdot, \cdot) \) defined on the plane \( \mathbb{R}^2 \) is a normal covariance if and only if there exists a continuous matrix function \( S(\cdot, \cdot) \) defined on the plane such that
\[
R(s, t) = S(s + t, s - t)
\]
and
\[
\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{n} \sum_{l=1}^{n} \alpha_k \overline{\alpha_l} S_{ij}(u_k + u_l, v_k - v_l) \geq 0
\]
for the every \( 2n \)-tuple of real numbers \( u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n \) and every \( n \times N \)-matrix of complex numbers \( \{\alpha_k\}_{k=1,2,\ldots,n} \).

**Proof.** The proof of this theorem is transformed into the one-dimensional case. Let \( e^T = (c_1, c_2, \ldots, c_N) \) be any \( N \)-dimensional vector of complex numbers and let us consider the function \( R_e(\cdot, \cdot) = e^T R(\cdot, \cdot) e \). We shall prove that \( R_e(\cdot, \cdot) \) is a one-dimensional normal covariance function. At the first sight, \( R_e(\cdot, \cdot) \) is defined on the plane and is continuous here. Further \( \overline{R_e(s, t)} = R_e(t, s) \) because
\[
\overline{R_e(s, t)} = \sum_{i=1}^{N} \sum_{j=1}^{N} \overline{c_i \bar{c}_j} R_{ij}(s, t) = \sum_{i=1}^{N} \sum_{j=1}^{N} \overline{c_i \bar{c}_j} \overline{R_{ij}(s, t)} =
\]
\[
= \sum_{i=1}^{N} \sum_{j=1}^{N} c_j \bar{c}_i R_{ji}(t, s) = R_e(t, s).
\]
\( R_e(\cdot, \cdot) \) is a covariance function because it is positive semidefinite as follows from
the assumptions of the theorem

\[ \sum_{k=1}^{n} \sum_{i=1}^{n} \alpha_k \tilde{a}_i R_e(s_k, s_l) = \sum_{k=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{N} \alpha_k \tilde{c}_i \tilde{c}_j R_{ij}(s_k, s_l) = \]

\[ = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{n} \sum_{l=1}^{n} \alpha_k \tilde{c}_i (\tilde{c}_j \tilde{c}_j) S_{ij}(s_k + s_l, s_k - s_l) \geq 0 \]

if we put \( \alpha_k c_i = \alpha_k \) and \( s_k = u_k = v_k \).

As we assume that \( R_{ij}(s, t) = S_{ij}(s + t, s - t) \) then \( R_e(s, t) = e^T S(s + t, s - t) e = S(s + t, s - t) \) and the function \( R_e(\cdot, \cdot) \) is a function of \( s + t \) and \( s - t \).

There is no problem to prove that \( S(\cdot, \cdot) \) is positive semidefinite in the following sense

\[ \sum_{k=1}^{n} \sum_{i=1}^{n} \alpha_k \tilde{a}_i S(u_k + u_l, v_k - v_l) \geq 0 \]

\[ \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{n} \sum_{l=1}^{n} c_i \alpha_k (\tilde{c}_j \tilde{c}_j) S_{ij}(u_k + u_l, v_k - v_l) \geq 0 \]

for every matrix \( \{c_i \alpha_k\} \) of complex numbers and every \( 2n \)-tuple \( u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n \) of reals. Finally, we have

\[ R_e(0, 0) = \sum_{i=1}^{N} \sum_{j=1}^{N} c_i \tilde{c}_j R_{ij}(0, 0) \geq 0 \]

and by means of results given in [6] we can assert that the covariance function \( R_e(\cdot, \cdot) \) is normal. Hence, there exists a spectral representation of \( R_e(\cdot, \cdot) \) in the form

\[ R_e(s, t) = \int [e^{t \lambda (s-t)} e^{i \mu (s-t)}] ddF_e(\lambda, \mu) \]

where \( R_e(\cdot, \cdot) \) is a two-dimensional measure with finite variation equal to \( R_e(0, 0) \), see [6]. Let us consider now special cases of the vector \( e \). Let

\[ e^T(k,j) = (0, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0) \]

where 1 stands on the \( k \)th and \( j \)th places \( (k < j) \); similarly, \( a^T(k,j) = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0) \) \( (k < j) \).

Then,

\[ R_e(k,j)(\cdot, \cdot) = R_{kk}(\cdot, \cdot) + R_{kj}(\cdot, \cdot) + R_{jk}(\cdot, \cdot) + R_{jj}(\cdot, \cdot) \]

\[ R_d(k,j) = R_{kk}(\cdot, \cdot) + i R_{kj}(\cdot, \cdot) - i R_{jk}(\cdot, \cdot) + R_{jj}(\cdot, \cdot) \]

hence,

\[ R_{jk} = \frac{1}{2} (R_{e(k,j)} - i R_{d(k,j)} - (1 - i) (R_{kk} - R_{jj})) \]

and thanks to the one-to-one correspondence between \( R_e \) and \( F_e \) we can state that

\[ F_{kj} = \frac{1}{2} (F_e(k,j) - i F_d(k,j) - (1 - i) (F_{kk} - F_{jj})) \]

We obtain an expression of an off-diagonal component \( R_{kj}(\cdot, \cdot) \) in the form

\[ R_{kj}(s, t) = \int \int e^{t \lambda (s-t)} e^{i \mu (s-t)} ddF_{kj}(\lambda, \mu) \]

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We have constructed in this way a matrix complex measure \( F = \{ F_{k,j} \}_{k,j=1}^N \). We have to verify that \( F \) is a spectral measure. Surely,
\[
\overline{F}(\cdot, \cdot) = F^T(\cdot, \cdot)
\]
because
\[
\overline{R}(\cdot, \cdot) = R^T(\cdot, \cdot)
\]
The function \( F_\varepsilon(\cdot, \cdot) \) defines for every \( \varepsilon \) a measure, hence,
\[
\Delta_{h_1} \Delta_{h_2} F_\varepsilon(\lambda, \mu) \geq 0
\]
for every \((\lambda, \mu) \in \mathbb{R}_2\) and every \( h_1 \in \mathbb{R}_1, h_2 \in \mathbb{R}_1\). This means, for every vector \( \varepsilon \) of complex numbers
\[
\sum_{i=1}^N \sum_{j=1}^N c_i \bar{c}_j \Delta_{h_i} \Delta_{h_j} F_{ij}(\lambda, \mu) \geq 0
\]
We see, immediately, that the matrix \( F(\Delta) \) is positive semidefinite for every Borel subset \( \Delta \) in the plane \( \mathbb{R}_2 \). If \( F \) is a matrix spectral measure, then, every function
\[
R(s, t) = \int \int e^{i(s+t) \omega} e^{i(s-t) \mu} d\lambda d\mu
\]
is a normal covariance function, (we assume the existence for every pair \((s, t) \in \mathbb{R}_2\)). The function \( R(\cdot, \cdot) \) satisfies:
1) \[
\sum_{i=1}^N \sum_{j=1}^N \bar{\alpha}_i \alpha_j R_{ji}(t, t) = \int \int e^{2s \omega} d\lambda d\mu = \int \int e^{2t \omega} d\lambda d\mu \geq 0
\]
because \( \sum_{i=1}^N \sum_{j=1}^N \alpha_i \bar{\alpha}_j F_{ij}(\cdot, \cdot) \) defines a nonnegative measure (\( F \) is a matrix spectral measure)
2) \[
R(s, t) = \int \int e^{i(s+t) \omega} e^{i(s-t) \mu} d\lambda d\mu = \int \int e^{i(s+t) \omega} e^{i(s-t) \mu} d\lambda d\mu = R^T(t, s)
\]
3) \[
|R_{jk}(s, t)| = \left| \int \int e^{i(s+t) \omega} e^{i(s-t) \mu} d\lambda d\mu \right| \leq \left( \int \int e^{2s \omega} d\lambda d\mu \right)^{1/2} \left( \int \int e^{2t \omega} d\lambda d\mu \right)^{1/2}
\]
This fact follows from positive definiteness of \( F \) because for every complex \( \alpha \) the inequality
\[
F_{ij}(\Delta) + |\alpha|^2 F_{jj}(\Delta) + \bar{\alpha} F_{ij}(\Delta) + \alpha F_{ij}(\Delta) \geq 0
\]
holds. Then, put \( \alpha = F_{ij}(\Delta)|F_{jj}^{1/2}(\Delta) \) if \( F_{jj}(\Delta) \neq 0 \).
4) Let us consider the function \( S(u, v) = R(\frac{1}{2}(u + v), \frac{1}{2}(u - v)) \); then,
\[
S(u, v) = \int \int e^{itu} e^{ivu} d\lambda d\mu
\]
and \( R(s, t) = S(s + t, s - t) \). Let us prove that this function \( S(\cdot, \cdot) \) satisfies the assumption of the theorem.
For this purpose, we need the Karhunen theorem, see \cite{2}. By means of this theorem, we can express every random process \( \{x(t), t \in \mathbb{R}_1\} \) having a normal covariance as a stochastic integral understood in the quadratic mean sense

\[
x(t) = \int_{-\infty}^{+\infty} e^{it\xi} \, dd\xi(z)
\]

where \( z = \lambda + i\mu \) and \( E\{\xi(z_1) \xi^T(z_2)\} = F(\min(z_1, z_2)); \min(z_1, z_2) = (\min(\text{Re} \, z_1, \text{Re} \, z_2), \min(\text{Im} \, z_1, \text{Im} \, z_2)) \). At this moment, let us consider random variables

\[
y(u, v) = \int_{-\infty}^{+\infty} e^{i\lambda u_1} e^{i\mu u_2} \, dd\xi(z),
\]

where \( u = u_1 + iu_2, u_1, u_2 \in \mathbb{R}_1 \). These random variables exist because

\[
E\{ y(u, v) y^T(x, y) \} = \left| \int_{-\infty}^{+\infty} e^{i\lambda(u_1 + x_1)} e^{i\lambda(u_2 - x_2)} e^{i\mu(v-y)} \, ddF(\lambda, \mu) \right| \leq \left| \int_{-\infty}^{+\infty} e^{i\lambda(u_1 + x_1)} \, dd|F(\lambda, \mu)| \right| < \infty.
\]

Then,

\[
0 \leq E\left\{ \sum_{i=1}^{N} \sum_{p=1}^{n} \alpha_p^i y_i(u_p, v_p) \right\}^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{p=1}^{n} \sum_{q=1}^{n} \alpha_p^i \alpha_q^j E\{ y_i(u_p, v_p) y_j(u_q, v_q) \} \leq \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{p=1}^{n} \sum_{q=1}^{n} \alpha_p^i \alpha_q^j \int_{-\infty}^{+\infty} e^{i\mu(v_p-v_q)} \, ddF_{ij}(\lambda, \mu).
\]

If we put \( u_p = \text{Re} \, u_p \), then, we obtain

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{p=1}^{n} \sum_{q=1}^{n} \alpha_p^i \alpha_q^j S_{ij}(u_p + u_q, v_p - v_q) \geq 0.
\]

5) Every component \( R_{ij}(\cdot, \cdot) \) of \( R(\cdot, \cdot) \) is a continuous function because all diagonal elements are one-dimensional normal covariances and off-diagonal elements can be expressed as a linear combinations of one-dimensional normal covariances. This completes the proof of the theorem. \( \square \)

3. NORMAL COVARIANCES AND NORMAL OPERATORS

In the multidimensional case we can show also a close connection between normal covariances and normal operators. Let a process \( x(\cdot) = \{x_i(\cdot)\}_{i=1}^{N} \) be a random process with a normal covariance function \( R(\cdot, \cdot) \), i.e.

\[
R(s, t) = \int_{-\infty}^{+\infty} e^{i\lambda(s+t)} \, e^{i\mu(s-t)} \, ddF(\lambda, \mu).
\]

As it was mentioned above such a process can be expressed in the form of a stochastic integral

\[
x(t) = \int_{-\infty}^{+\infty} e^{it\xi} \, dd\xi(z).
\]
Let \( L(\xi(\cdot)) \) be the linear set of all linear combinations

\[
\sum_{i=1}^{n} \alpha_i \xi_i(z_i)
\]

and let \( H(\xi(\cdot)) = \overline{L(\xi(\cdot))} \) be a closure of \( L(\xi(\cdot)) \) with respect to the convergence in the quadratic mean sense. Let us denote by \( H(z) \) the subspace of \( H(\xi(\cdot)) \) generated by all random variables

\[
\sum_{i=1}^{n} \alpha_i \xi_i(z_i), \quad z_i \leq z;
\]

let \( P_z \) be the orthogonal projector in \( H(\xi(\cdot)) \) on the subspace \( H(z) \). Thanks to properties of the spectral measure \( F \) one can easily prove that the family \( \{P_z; z \in \mathbb{C}\} \) forms a complex resolution of the identity in \( H(\xi(\cdot)) \). We can construct normal operators

\[
A_t = \int_{\mathbb{R}}^{+\infty} e^{it} dP_z, \quad t \in \mathbb{R}_1
\]

with the definition domain

\[
\mathcal{D}(A_t) = \{x \in H(\xi(\cdot)); \int_{-\infty}^{+\infty} e^{2it} \text{dd} \langle P_x x, x \rangle < \infty\}.
\]

As \( x(0) = \int_{-\infty}^{+\infty} \text{dd} \xi(z) = \text{l.i.m.} \xi(z) \) then \( x_i(0) \in H(\xi(\cdot)) \) for every \( i = 1, 2, \ldots, N \) and \( P_z x_i(0) = \xi_i(z) \). Then, we see that

\[
x_i(t) = \int_{-\infty}^{+\infty} e^{it} dP_z x_i(0), \quad i = 1, 2, \ldots, N
\]

because \( \text{dd} \langle P_z x_i(0), x_i(0) \rangle = \text{dd} \langle \xi_i(z), x_i(0) \rangle = \text{dd} \langle \xi_i(z), \xi_i(z) \rangle = \text{dd} F_i(\lambda, \mu) \) and the integral

\[
\int_{-\infty}^{+\infty} e^{2it} \text{dd} F_i(\lambda, \mu)
\]

exists for every \( t \in \mathbb{R}_1 \) and every \( i = 1, 2, \ldots, N \) as we assume. We obtained that

\[
x_i(t) = A_t x_i(0), \quad i = 1, 2, \ldots, N, \quad t \in \mathbb{R}_1.
\]

**Corollary to Theorem 4.** An \( N \)-dimensional covariance function \( R(\cdot, \cdot) \) is normal if and only if for every \( N \)-tuple \( z^T = (z_1, z_2, \ldots, z_N) \) of complex numbers \( z^T R(\cdot, \cdot) z \) is a one-dimensional normal covariance function.

Another connection between normal covariances and normal operators in a Hilbert space is shown in Theorem 5.

**Theorem 5.** Let a group \( \{T_s, s \in \mathbb{R}_1\} \) of normal, in general unbounded, operators be given in a Hilbert space \( (\mathcal{K}, \langle \cdot, \cdot \rangle) \). Let, for every \( x, y \in \mathcal{D} = \bigcap_{s \in \mathbb{R}_1} \mathcal{D}(T_s) \), \( \langle T_s x, T_s y \rangle \) be a continuous function on the plane. Then for every \( N \)-tuple \( x_1, x_2, x_3, \ldots, x_N \) of elements in \( \mathcal{K} \) belonging to the subset \( \mathcal{D} \)

\[
R(s, t) = \{\langle T_s x_i, T_x y_j \rangle\}_{i,j=1}^{N}
\]

is an \( N \)-dimensional normal covariance function \( (\mathcal{D}(T_s) \) is the definition domain of \( T_s \) in \( \mathcal{K} \).

**Proof.** The subset \( \mathcal{D} \) is not empty because \( 0 \in \mathcal{D} \) in every case. Let \( x_1, x_2, \ldots, x_N \)
belong to $\mathcal{D}$. First, we need to show that the matrix function $R(\cdot, \cdot)$ is a covariance function. Let $n$ be an arbitrary natural number, let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be an arbitrary $n$-tuple of complex numbers and $s_1, s_2, \ldots, s_n$ an arbitrary $n$-tuple of reals. We must prove that

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \alpha_k \overline{\alpha_l} \langle T_{s_k} x_{i_k}, T_{s_l} x_{i_l} \rangle \geq 0$$

where $x_{i_k} \in \{x_1, x_2, \ldots, x_N\}$ for every $k = 1, 2, \ldots, n$. This inequality holds evidently because

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \alpha_k \overline{\alpha_l} |T_{s_k} x_{i_k}|^2 \geq 0.$$

For next steps, it is suitable to introduce the function $S_{xy}(u, v)$, $x, y \in \mathcal{D}$, defined by the relation

$$S_{xy}(u, v) = \langle T_{(u+v)/2} x, T_{(u-v)/2} y \rangle.$$

We immediately see

$$R_{xy}(s, t) = S_{xy}(s + t, s - t);$$

hence, $S_{xy}(\cdot, \cdot)$ is continuous on the plane. Let $z^T = (z_1, z_2, \ldots, z_N)$ be an arbitrary $N$-tuple of complex numbers and we must prove that $z^T R(\cdot, \cdot) z$ is a normal covariance function. To prove this fact we need validity of the equality

$$T_t^* T_s = T_s T_t^*$$

on $\mathcal{D}$. As $\{T_s, s \in \mathbb{R}_1\}$ is a group then $T_{t+s} = T_t T_s = T_s T_t$, i.e. $\mathcal{D}(T_{t+s}) = \mathcal{D}(T_s T_t) = \mathcal{D}(T_t T_s)$ must hold too. Next, it follows $\mathcal{A}(T_t) \subset \mathcal{D}(T_t)$ and simultaneously $\mathcal{A}(T_s) \subset \mathcal{D}(T_s)$ ($\mathcal{A}(T_t)$ is the range of $T_t$). Let $n$ be an integer. Then,

$$(T_s)^n = T_n s$$

thanks to the group property holding for $\{T_s^*, s \in \mathbb{R}_1\}$ too. Now, let $t = n \cdot s$. Then

$$T_t^* T_t = T_{s}s T_{s} = (T_s^*)^n T_s = T_s (T_s^*)^n = T_s T_s^*$$

because $T_s^* T_s = T_s^* T_s$. Similarly, in case $t = s \cdot (p/q)$, where $p/q$ represents a rational number, we can prove

$$T_t^* T_s = T_s T_t^*$$

as

$$T_t^* T_s = T_{s/p} T_{s/q} = (T_{s/q})^p (T_{s/q})^q = (T_{s/q})^q (T_{s/q})^p = T_s T_t^*.$$

Finally, let $t$ be quite arbitrary. Then, there exists a sequence $\{t_n\}_{n=1}^{\infty}$ $t_n = s \cdot p_n/q_n \rightarrow t$ where $p_n/q_n$ are rational and continuity of the scalar product in $\mathcal{H}$ proves

$$T_t^* T_s = T_s T_t^*$$

for every pair $s, t$ of reals. If $x \in \mathcal{D}$ then $T_t x \in \mathcal{D}$ as well because $T_{t+s} x = T_s (T_t x)$ which implies $T_{t+s} x \in \mathcal{D}(T_t)$ for every real $s$. This proves that $T_t x \in \mathcal{D}$. If $T_t x \in \mathcal{D}$ then $T_t^* (T_t x)$ is well defined as $\mathcal{D}(T_t^*) = \mathcal{D}(T_t)$. In case $s = n \cdot t$, $n$ is an integer,

$$T_t^* T_s x = T_s T_t^* x$$

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as it is proved above and this gives $T_t^* x \in \mathcal{D}(T_t)$ for every $n$, $T_t^* x \in \mathcal{D}$ too. That means both the operators $T_t^* T_s$, $T_t^* T_t^*$ are well defined on the subset $\mathcal{D}$. Now, we are ready to prove the “nonnegative-definite” property of $z^T R(\cdot, \cdot) z$, see [6]. Let $n$ be a natural number, let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be an $n$-tuple of complex numbers, let $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ be a 2$n$-tuple of reals. Let us consider the sum
\[
\sum_{k=1}^{N} \sum_{i=1}^{n} \alpha_i \bar{\alpha}_j \sum_{k=1}^{N} \sum_{i=1}^{n} z_k \bar{z}_l S_{k\alpha_i} (u_i + u_j, v_i - v_j) = 
\]
\[
= \sum_{k=1}^{N} \sum_{i=1}^{n} z_k \bar{z}_l \sum_{i=1}^{n} \alpha_i \bar{\alpha}_j \langle T_{(u_i + u_j + v_i - v_j)/2} x_k, T_{(u_i + u_j + v_j - v_i)/2} x_l \rangle = 
\]
\[
= \sum_{k=1}^{N} \sum_{i=1}^{n} z_k \bar{z}_l \sum_{i=1}^{n} \alpha_i \bar{\alpha}_j \langle T_{(u_i - v_i)/2} T_{(u_i + v_i)/2} x_k, T_{(u_j - v_j)/2} T_{(u_j + v_j)/2} x_l \rangle = 
\]
\[
= \left| \sum_{k=1}^{N} \sum_{i=1}^{n} z_k \bar{z}_l T_{(u_i - v_i)/2} T_{(u_i + v_i)/2} x_k \right|^2 \geq 0.
\]
A necessary and sufficient condition characterizing normal covariances is proved, see [6]. This inequality, together with continuity of $R_{ij}(\cdot, \cdot)$, $i, j = 1, 2, \ldots, N$, show that the matrix covariance function $R(\cdot, \cdot)$ is normal.

4. CONCLUSION

In the literature, we can meet two types of generalization of the notion weak stationarity. First generalization, originated by Loève in [3], can be characterized as the nonorthogonal integral representation

\[
x(t) = \int_{-\infty}^{+\infty} \varphi(t, \lambda) \, d\xi(\lambda)
\]
in the quadratic mean sense where $\varphi(\cdot, \cdot)$ is a nonrandom complex function and $\xi(\cdot)$ is a second-order random process with covariance function having finite variation on the plane. The second generalization, originated by Karhunen, see [2], can be called the orthogonal integral representation

\[
x(t) = \int_{-\infty}^{+\infty} \varphi(t, \lambda) \, d\eta(\lambda)
\]
where $\varphi(\cdot, \cdot)$ is a nonrandom complex function and the process $\eta(\cdot)$ defines an orthogonally scattered random measure on the Borel field in reals. There is no problem to generalize the Karhunen representation in the following way: instead of the Borel sets with the Lebesgue representation we can consider a measure space $(\Theta, \sigma, m)$ and an orthogonally scattered measure $\eta(\cdot)$ satisfying

\[
E(\eta(A_1) \, \eta(A_2)) = m(A_1 \cap A_2)
\]
for every $A_1, A_2 \in \sigma$. Then, the corresponding covariance function of the process $\{x(t), t \in \mathbb{R}_1\}$ can be expressed as

\[
R(s, t) = \int_{\Theta} \varphi(s, \theta) \, \tilde{\varphi}(s, \theta) \, dm(\theta).
\]
Immediately, we see that a process with a normal covariance function belongs into the Karhunen class with $\Theta = \mathbb{R}_2$, $\sigma$ is the $\sigma$-algebra of Borel sets in the plane, $\varphi(s, \theta) = e^{st + ls^2}$, i.e. $\theta = (\lambda, \mu)$. The measure $m(\cdot)$ defined on the Borel sets is determined by a function $F(\cdot, \cdot)$, see Definition 1. In a similar way, we can handle with the multidimensional case.

As well known, the spectral decomposition of weakly stationary process is connected with groups of unitary shift-operators in the Hilbert space of random process values. Considering normal shift operators we reach, of course, the class of normal covariance functions. In general, if a random process possesses a Karhunen representation (11) then there exists a self-adjoint operator $A$ defined in the mentioned Hilbert space such that

$$x(t) = \varphi(t, A) x(0)$$

(see [1]). In case of the nonorthogonal integral representation, mainly in the harmonizable case, the question about the characterization of the corresponding shift operators, has so far been open.

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