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ON UPPER BOUNDS FOR TOTAL K-DOMINATION NUMBER VIA THE PROBABILISTIC METHOD

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For a fixed positive integer k and $G = (V, E)$ a connected graph of order n, whose minimum vertex degree is at least k, a set $S \subseteq V$ is a total k-dominating set, also known as a k-tuple total dominating set, if every vertex $v \in V$ has at least k neighbors in S. The minimum size of a total k-dominating set for G is called the total k-domination number of G, denoted by $\gamma_{kt}(G)$. The total k-domination problem is to determine a minimum total k-dominating set of G. Since the exact problem is in general quite difficult to solve, it is also of interest to have good upper bounds on the total k-domination number. In this paper, we present a probabilistic approach to computing an upper bound for the total k-domination number that improves on some previous results.

Keywords: domination, k-tuple total domination, probabilistic method

Classification: 05C30, 05C65, 05C69

1. INTRODUCTION

We start by giving some notations and terminologies. All graphs in this paper are simple graphs G with vertex set V and edge set E. For a graph G the neighborhood $\{u \in V : uv \in E\}$ of a vertex $v \in V$, the degree $|\{u \in V : uv \in E\}|$ of a vertex $v \in V$, the minimum degree and the maximum degree are denoted by $N(v)$, $deg(v)$, δ and Δ, respectively.

This article deals with domination, a well-studied topic in graph theory and combinatorial optimization. A detailed summary of the literature on this topic can be found in [5] and [8]. A dominating set of a graph G is a set S of vertices of G with $|N(v) \cap S| \geq 1$ for all $v \notin S$, and the domination number $\gamma(G)$ of a graph is the cardinality of a smallest dominating set. The term domination was first defined as a graph theoretical concept in 1958, in connection to various chessboard problems. Two decades later Cockayne, Dawes, and Hedetniemi [4] worked on the idea of dominating all vertices of the graph, rather than merely dominating vertices outside the set, and established the following concept. A *total dominating set* of a graph G is a set S of vertices of G with $|N(v) \cap S| \geq 1$ for all $v \in V$, and similarly the *total domination number* $\gamma_t(G)$ of a graph is the cardinality of a smaller total dominating set. Then in recent years, Henning and Kazemi [6] motivated by other variations of domination, considered the

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next generalization of total domination. Given a positive integer k, a set $S \subseteq V$ is a *total k-dominating set* if every vertex $v \in V$ has at least k neighbors in S. In such a case, it is necessary to have $k \leq \delta$ and $|S| \geq k+1$. The *total k-domination number* $\gamma_{kt}(G)$ is the minimum cardinality among all total k-dominating sets.

The concept of dominance and its variations form a rich area of graph theory, with many useful concepts and much practical interest in various fields ([9, 11]). In particular, due to the complexity of the algorithmic aspects associated with the problem of determining a minimal total k-dominant set of G [10], many studies focus on proposing upper bounds for $\gamma_{kt}(G)$ with respect to various graph parameters. For this reason, in this manuscript we present a new probabilistic upper bound for the total k-domination number of a graph, which improves some results presented in [1]. This confirms that the probabilistic method is a powerful tool for solving many problems in graph theory.

2. A PROBABILISTIC UPPER BOUND ON TOTAL K-DOMINATION NUMBER

This section presents the main results of the paper.

Theorem 2.1. For a fixed positive integer k, if $G=(V,E)$ is a connected graph of order n, minimum degree $\delta \geq k$ and $p \in (0,1)$, then

$$\gamma_{kt}(G) \leq np + \sum_{i=0}^{k-1} \sum_{v \in V} (k-i) \binom{deg(v)}{i} p^i (1-p)^{deg(v)-i}. \tag{1}$$

P r o o f. Let $p \in (0,1)$, we form a set A by picking every vertex v of G independently at random with $\mathbb{P}(v \in A) = p$. Let us denote $C_{1,i} = \{v \in V : v \in A \text{ and } |N(v) \cap A| = i\}$ with $i = 0, 1, \ldots, k-1$ and for every $v \in C_{1,i}$ we chose $u_{(v,1)}, \ldots, u_{(v,k-i)} \in N(v) \cap A^c$, and with this rule we define a new set $S_{1,i} = \{u_{(v,1)}, \ldots, u_{(v,k-i)} : v \in C_{1,i}\} \subseteq A^c$. Next for $i = 0, 1, \ldots, k-1$ we introduce $C_{2,i} = \{v \in V : v \notin A \text{ and } |N(v) \cap A| = i\}$ and in the same way we propose the set $S_{2,i}$=$\{u_{(v,1)}, \ldots, u_{(v,k-i)} : v \in C_{2,i}\} \subseteq A^c$. It is obvious that the set $D = A \cup (\cup_{i=0}^{k-1} S_{1,i}) \cup (\cup_{i=0}^{k-1} S_{2,i})$ is a total k-dominating set of the graph G.

Then, we have

$$\begin{aligned} |D| &= |A \cup (\cup_{i=0}^{k-1} S_{1,i}) \cup (\cup_{i=0}^{k-1} S_{2,i})| \\ &= |A| + |(\cup_{i=0}^{k-1} S_{1,i}) \cup (\cup_{i=0}^{k-1} S_{2,i})|, \end{aligned}$$

since $A \cap ((\cup_{i=0}^{k-1} S_{1,i}) \cup (\cup_{i=0}^{k-1} S_{2,i})) = \emptyset$. Linearity of expectation establishes

$$\mathbb{E}(|D|) = \mathbb{E}(|A|) + \mathbb{E}(|(\cup_{i=0}^{k-1} S_{1,i}) \cup (\cup_{i=0}^{k-1} S_{2,i})|).$$

It is possible to demonstrate that $|A|$ is a $Bin(n,p)$ random variable, hence $\mathbb{E}(|D|) = np + \mathbb{E}(|(\cup_{i=0}^{k-1} S_{1,i}) \cup (\cup_{i=0}^{k-1} S_{2,i})|)$. Furthermore,

$$\begin{aligned} |(\cup_{i=0}^{k-1} S_{1,i}) \cup (\cup_{i=0}^{k-1} S_{2,i})| &\leq |\cup_{i=0}^{k-1} S_{1,i}| + |\cup_{i=0}^{k-1} S_{2,i}| \\ &\leq \sum_{i=0}^{k-1} |S_{1,i}| + \sum_{i=0}^{k-1} |S_{2,i}| \end{aligned}$$

and by construction $|S_{1,i}| \leq (k-i)|C_{1,i}|$ and $|S_{2,i}| \leq (k-i)|C_{2,i}|$ for $i = 0, 1, \ldots, k-1$, consequently

$$\mathbb{E}(|D|) \leq np + \sum_{i=0}^{k-1}(k-i)\mathbb{E}(|C_{1,i}|) + \sum_{i=0}^{k-1}(k-i)\mathbb{E}(|C_{2,i}|). \tag{2}$$

To find $\mathbb{E}(|C_{1,i}|)$ with $i = 0, 1, \ldots, k-1$ we write $V = \{v_1, v_2, \ldots, v_n\}$ and we decompose $|C_{1,i}| = \sum_{j=1}^{n} C^j_{1,i}$ where $C^j_{1,i}$ is the indicator random variable of the event $\{v_j \in C_{1,i}\}$, thus $\mathbb{E}(|C_{1,i}|) = \sum_{j=1}^{n} \mathbb{P}(C^j_{1,i})$. On the other hand,

$$\mathbb{P}(C^j_{1,i}) = \mathbb{P}(\{v_j \in A\} \cap \{|N(v_j) \cap A| = i\}) \tag{3}$$

then, by independence, $\mathbb{P}(C^j_{1,i}) = p\mathbb{P}(|N(v_j) \cap A| = i)$. Now, observe that $|N(v_j) \cap A|$ is a $Bin(deg(v_j), p)$ random variable, then $\mathbb{P}(C^j_{1,i}) = p\binom{deg(v_j)}{i}p^i(1-p)^{deg(v_j)-i}$ and

$$\mathbb{E}(|C_{1,i}|) = \sum_{v \in V} p\binom{deg(v)}{i}p^i(1-p)^{deg(v)-i}. \tag{4}$$

Similarly, it is possible to obtain that

$$\mathbb{E}(|C_{2,i}|) = \sum_{v \in V} (1-p)\binom{deg(v)}{i}p^i(1-p)^{deg(v)-i}. \tag{5}$$

Therefore, by replacing (4) and (5) in (2), it follows that

$$\mathbb{E}(|D|) \leq np + \sum_{i=0}^{k-1}\sum_{v \in V}(k-i)\binom{deg(v)}{i}p^i(1-p)^{deg(v)-i}.$$

Finally, by the first moment method [2], it yields that

$$\gamma_{kt}(G) \leq np + \sum_{i=0}^{k-1}\sum_{v \in V}(k-i)\binom{deg(v)}{i}p^i(1-p)^{deg(v)-i}. \tag{6}$$

This completes the proof of Theorem 2.1. □

Consider a connected graph $G = (V, E)$ of order n, minimum degree $\delta \geq k$ and maximum degree Δ, for $w \in [0, 1]$ let denote

$$P_G(w) = n - nw + \sum_{j=\delta}^{\Delta} a_j\left(kw^j + \sum_{i=1}^{k-1}(k-i)\binom{j}{i}(1-w)^i w^{j-i}\right),$$

where $a_j = |\{v \in V : deg(v) = j\}|$ with $j \in \{\delta, \delta+1, \cdots, \Delta\}$. Next substituting $w = 1-p$ on the upper bound obtained at Theorem 2.1, it follows that $\gamma_{kt}(G) \le P_G(w)$ for all $w \in [0,1]$.

Since, to $P_G(w)$ is a continuous function in $[0,1]$, making use of Maximun and Minimum Value Theorem [1], we can affirm that $P_G(w)$ attains a minimum in $[0,1]$, indeed if $k \ge 2$, we claim that $w = 1$ is not a minimum because $P_G(1) = kn > n = P_G(0)$. Consequently, $P_G(w^*)$ is the best possible upper bound for $\gamma_{kt}(G)$, where w^* is a minimum of $P_G(w)$ with $w \in [0,1)$. In addition, for $k = 2$, the upper bound $P_G(w^*)$ is sharp, see for instance the graphs illustrated in Figure 1 and concerning the sharpness of this bound for the other values of k, we highlight that research in this direction is still in progress.

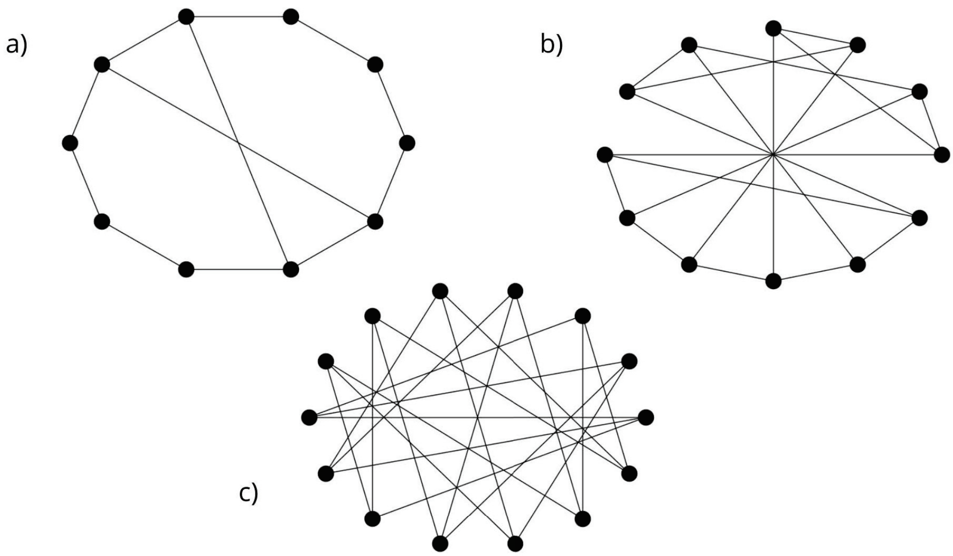

Fig. 1. Some examples of graphs where the bound is achieved: a)$P_G(w) = 10 + 2w + 12w^2 - 4w^3$, $\lfloor P_G(w^*) \rfloor = \gamma_{2t}(G) = 10$b)$P_G(w) = 12 - 12w + 36w^2 - 12w^3$, $\lfloor P_G(w^*) \rfloor = \gamma_{2t}(G) = 10$[7]c) The Heawood graph $P_G(w) = 14 - 14w + 42w^2 - 14w^3$, $\lfloor P_G(w^*) \rfloor = \gamma_{2t}(G) = 12$ [7].

As the next step, we prove the following important result which represents a decisive property in future observations.

Theorem 2.2. Let $k \ge 1$ be a positive integer. For any connected graph G on n vertices with minimum degree $\delta \ge k$ and maximum degree Δ, the following inequality holds

$$P_G(w) \le h_\delta(w)n,$$

for all $w \in [0,1]$, where

$$h_\delta(w) = 1 - w + kw^\delta + \sum_{i=1}^{k-1}(k-i)\binom{\delta}{i}(1-w)^i w^{\delta-i}.$$

Proof. For $r \geq \delta$ let us define the next function sequence on $[0,1]$

$$f_r(w) = kw^r + \sum_{i=1}^{k-1}(k-i)\binom{r}{i}(1-w)^i w^{r-i}.$$

We claim that $\{f_r\}_{r\geq\delta}$ is decreasing, i.e., $\blacktriangle_r := f_r(w) - f_{r+1}(w) \geq 0$, to prove this fact note that using $\binom{r+1}{i} = \binom{r}{i} + \binom{r}{i-1}$, it follows that

$$\blacktriangle_r = kw^r(1-w) + \sum_{i=1}^{k-1}(k-i)\binom{r}{i}(1-w)^{i+1}w^{r-i} - \sum_{i=1}^{k-1}(k-i)\binom{r}{i-1}(1-w)^i w^{r+1-i}.$$

Next, if we replace: $\sum_{i=1}^{k-2}(k-i)\binom{r}{i}(1-w)^{i+1}w^{r-i} + \binom{r}{k-1}(1-w)^k w^{r-(k-1)}$ and $\sum_{i=1}^{k-2}(k-(i+1))\binom{r}{i}(1-w)^{i+1}w^{r-i} + (k-1)(1-w)w^r$ with $\sum_{i=1}^{k-1}(k-i)\binom{r}{i}(1-w)^{i+1}w^{r-i}$ and $\sum_{i=1}^{k-1}(k-i)\binom{r}{i-1}(1-w)^i w^{r+1-i}$ respectively, we obtain that

$$\blacktriangle_r = w^r(1-w) + \binom{r}{k-1}(1-w)^k w^{r-(k-1)} + \sum_{i=1}^{k-2}\binom{r}{i}(1-w)^{i+1}w^{r-i} \geq 0.$$

Finally, to conclude the proof, it is only necessary to consider that $\sum_{j=\delta}^{\Delta} a_j = n$, $P_G(w) = n - nw + \sum_{j=\delta}^{\Delta} a_j f_j(w)$ and $h_\delta(w) = 1 - w + f_\delta(w)$. □

To present the rest of the manuscript, we must cite some related work. In 2019, Alipour and Jafari [1] proved Theorem 2.3 and Theorem 2.4, by using Turán's theorem.

Theorem 2.3. If G is a graph with minimum degree $\delta \geq k+1+d$ for $0 \leq d \leq k-1$ then

$$\gamma_{kt}(G) \leq \frac{2d+(k-d)(k-d+1)}{2d+(k-d)(k-d+1)+1}n. \tag{7}$$

Theorem 2.4. If G is a graph with n vertices and minimum degree $\delta \geq 3$ such that at least half of the vertices have degree at least 4, then $\gamma_{2t}(G) \leq \frac{5}{6}n$.

In addition, by applying Lovász's local lemma, Alipour and Jafari [1] were able to improve the bound expressed in (7) for $k=2$ in some special cases. In table 1, for given values of δ and Δ, the corresponding upper bound for the total number of 2 dominations is given. This number has been studied by different authors under different names, e.g., the double total dominance number. Focusing only on $k \geq 2$, we are now in a position to start a comparison process between the upper bound $P_G(w^*)$ and the aforementioned results, considering networks with $P_G(w^*) \neq n$.

Theorem 2.5. Let $k \geq 1$ be a positive integer. For any connected graph G on n vertices, minimum degree $\delta = k+1$ and maximum degree $\Delta \geq k+2$. Then $\gamma_{kt}(G) \leq P_G(w^*) \leq \epsilon_0^k$ if and only if $P_G(w) - \epsilon_0^k$ has a root in $(0, 1)$, where $\epsilon_0^k = \frac{5}{6}n$ if $k = 2$ and otherwise $\epsilon_0^k = \frac{k(k+1)}{k(k+1)+1}n$.

Proof. Suppose that there exists $w_r \in (0, 1)$ such that $P_G(w_r) = \epsilon_0^k$, consequently $P_G(w^*) \leq P_G(w_r) = \epsilon_0^k$. Conversely, assume that $P_G(w^*) - \epsilon_0^k \leq 0$, the case when $P_G(w^*) - \epsilon_0^k = 0$ is clear, in other matters, taking into account that $P_G(0) - \epsilon_0^k > 0$, applying Bolzano's Theorem [3] we can conclude that $P_G(w) - \epsilon_0^k$ has a root in $(0, 1)$. □

δ	Δ	**Upper bound**
7	7	$\frac{3}{4}n$
7	8	$\frac{3}{4}n$
9	9	$\frac{2}{3}n$
9	10	$\frac{2}{3}n$
9	11	$\frac{2}{3}n$
14	14	$\frac{1}{2}n$

Tab. 1. Upper bounds on the double total domination number for some values of δ and Δ [1].

For numerical purposes, it is important to specify that the problem of guaranteeing the existence of a root in $(0, 1)$ of the polynomial $P_G(w) - \epsilon_0^k$, mentioned in Theorem 2.5 can be restricted to values of w in the interval $(1 - \frac{\epsilon_0^k}{n}, 1)$.

In the following remark, we present some key points of the analytical comparative study.

Remark 2.6. a) Observe that there exist some graphs that satisfy Theorem 2.5, emphasizing that in the case $k = 2$ these graphs verify $a_3 \leq \frac{n}{2}$. Thus the upper bounds stated in Theorem 2.3 and Theorem 2.4 are improved, see for instance the graphs reported in Figure 2.

b) We are also interested in the relation between $\lfloor P_G(w^*) \rfloor$ and $\lfloor \epsilon_0^k \rfloor$, note that in most examples shown in Figure 2 the strict inequality $\lfloor P_G(w^*) \rfloor < \lfloor \epsilon_0^k \rfloor$ is confirmed. Furthermore, as n becomes larger this difference increases.

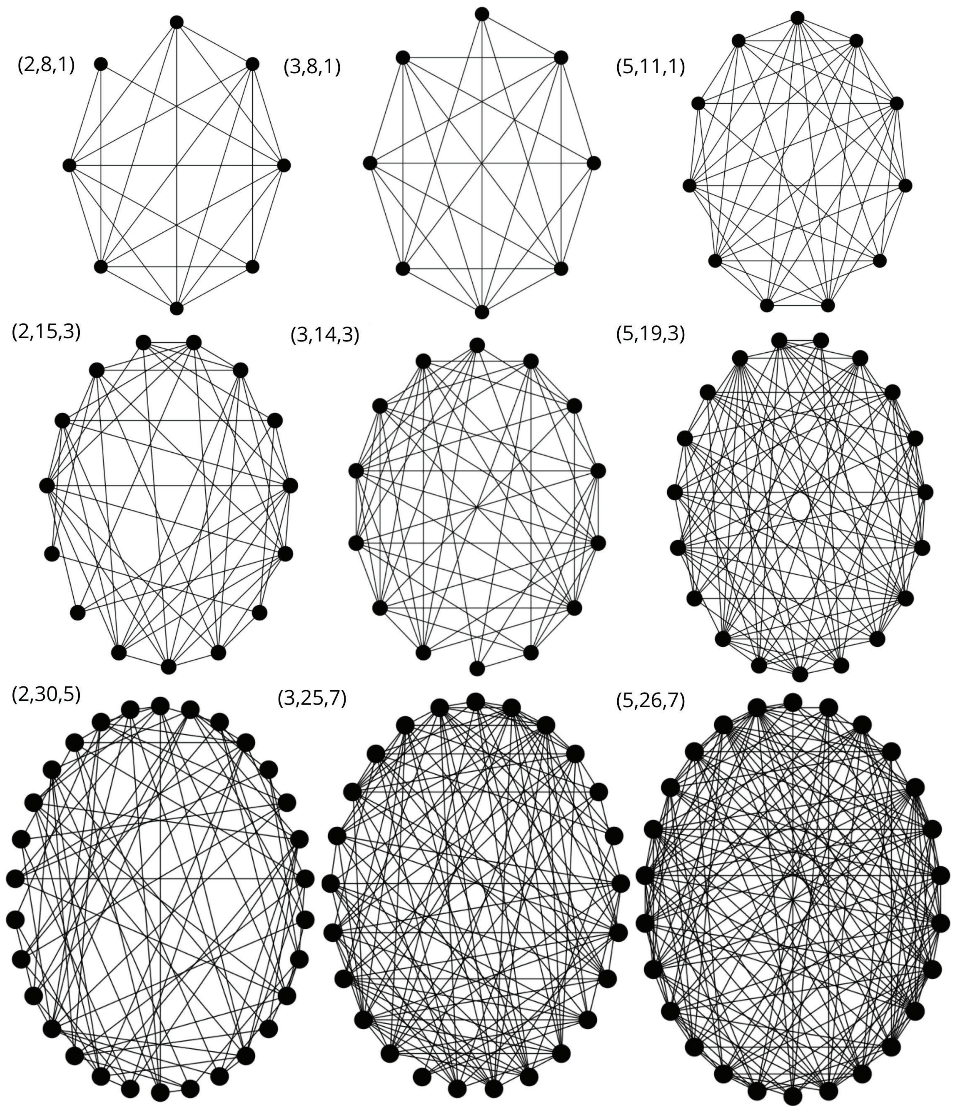

Fig. 2. The information of interest associated with each network is exposed in a triad with the structure $(k, |V|, \lfloor P_G(w^*) \rfloor - \lfloor \epsilon_0^k \rfloor)$.

k	**d**	$h_{k+1+d}(\dot{w})$	$\frac{\epsilon_d^k}{n}$	**k**	**d**	$h_{k+1+d}(\dot{w})$	$\frac{\epsilon_d^k}{n}$	**k**	**d**	$h_{k+1+d}(\dot{w})$	$\frac{\epsilon_d^k}{n}$
2	1	0.78	0.80	20	11	0.80	0.98	60	40	0.71	0.99
3	1	0.86	0.88	20	12	0.78	0.98	60	48	0.68	0.99
3	2	0.78	0.85	20	16	0.72	0.97	60	59	0.65	0.99
4	1	0.90	0.93	20	18	0.69	0.97	**70**	1	0.99	0.99
4	2	0.83	0.90	20	19	0.67	0.97	70	3	0.99	0.99
4	3	0.77	0.88	**30**	2	0.98	0.99	70	9	0.95	0.99
5	1	0.92	0.95	30	4	0.95	0.99	70	10	0.94	0.99
5	2	0.86	0.94	30	8	0.90	0.99	70	13	0.92	0.99
5	3	0.81	0.92	30	9	0.88	0.99	70	16	0.90	0.99
5	4	0.75	0.90	30	11	0.85	0.99	70	18	0.88	0.99
6	1	0.94	0.96	30	15	0.80	0.99	70	21	0.86	0.99
6	2	0.89	0.96	30	18	0.76	0.99	70	23	0.85	0.99
6	3	0.83	0.94	30	20	0.74	0.99	70	24	0.84	0.99
6	4	0.79	0.93	30	24	0.70	0.98	70	28	0.81	0.99
6	5	0.74	0.92	30	26	0.68	0.98	70	37	0.76	0.99
7	1	0.95	0.97	30	28	0.66	0.98	70	43	0.68	0.99
7	2	0.90	0.97	30	29	0.65	0.98	70	58	0.57	0.99
7	3	0.86	0.96	**40**	2	0.98	0.99	**80**	2	0.99	0.99
7	4	0.81	0.95	40	4	0.97	0.99	80	4	0.98	0.99
7	5	0.77	0.94	40	5	0.95	0.99	80	8	0.96	0.99
7	6	0.74	0.93	40	10	0.90	0.99	80	13	0.93	0.99
8	1	0.96	0.98	40	14	0.85	0.99	80	18	0.89	0.99
8	2	0.92	0.97	40	20	0.79	0.99	80	20	0.88	0.99
8	3	0.87	0.97	40	23	0.76	0.99	80	32	0.81	0.99
8	4	0.83	0.96	40	26	0.74	0.99	80	44	0.75	0.99
8	5	0.80	0.95	40	28	0.72	0.99	80	49	0.74	0.99
8	6	0.76	0.94	40	32	0.69	0.99	80	57	0.73	0.99
8	7	0.73	0.94	40	35	0.66	0.99	80	62	0.72	0.99
9	1	0.96	0.98	40	38	0.64	0.99	80	69	0.71	0.99
9	2	0.93	0.98	**50**	4	0.97	0.99	80	74	0.69	0.99
9	3	0.89	0.97	50	6	0.96	0.99	**90**	3	0.99	0.99
9	4	0.85	0.97	50	9	0.93	0.99	90	20	0.89	0.99
9	5	0.81	0.96	50	12	0.90	0.99	90	25	0.86	0.99
9	6	0.78	0.96	50	16	0.86	0.99	90	37	0.80	0.99
9	7	0.75	0.95	50	18	0.84	0.99	90	47	0.77	0.99
9	8	0.72	0.94	50	23	0.80	0.99	90	52	0.76	0.99
10	1	0.97	0.98	50	28	0.76	0.99	90	61	0.75	0.99
10	2	0.93	0.98	50	35	0.71	0.99	90	64	0.74	0.99
10	3	0.90	0.98	50	40	0.68	0.99	90	74	0.72	0.99
10	4	0.86	0.98	50	43	0.66	0.99	90	76	0.71	0.99
10	5	0.83	0.97	50	49	0.60	0.99	**100**	2	0.99	0.99
10	6	0.80	0.96	**60**	3	0.98	0.99	100	14	0.94	0.99
10	7	0.77	0.96	60	7	0.95	0.99	100	27	0.87	0.99
10	8	0.74	0.95	60	9	0.94	0.99	100	31	0.85	0.99
10	9	0.72	0.95	60	11	0.92	0.99	100	33	0.84	0.99
20	1	0.98	0.99	60	12	0.91	0.99	100	35	0.83	0.99
20	3	0.95	0.99	60	14	0.90	0.99	100	59	0.77	0.99
20	5	0.91	0.99	60	19	0.86	0.99	100	62	0.76	0.99
20	6	0.89	0.99	60	27	0.80	0.99	100	77	0.74	0.99
20	7	0.87	0.99	60	33	0.76	0.99	100	91	0.71	0.99
20	8	0.85	0.99	60	34	0.75	0.99	100	95	0.70	0.99

Tab. 2. Comparison between $h_{k+1+d}(\dot{w})$ and $\frac{\epsilon_d^k}{n}$ for some values of k.

Hereinafter, for convenience the upper bound given in (7) is denoted as ϵ_d^k for $1 \le d \le k-1$, i. e.

$$\epsilon_d^k := \frac{2d + (k-d)(k-d+1)}{2d + (k-d)(k-d+1) + 1} n.$$

Continuing the process of comparison, note that in Theorem 2.5 the results exposed in Theorem 2.3 were studied only for the case $d = 0$, so that only $1 \le d \le k-1$ remains to be treated. In this sense, for a fixed positive integer k considering Theorem 2.2, a sufficient condition to guarantee an improvement of the bound ϵ_d^k is the confirmation that

$$h_{k+1+d}(\dot{w}) \le \frac{\epsilon_d^k}{n},$$

for all $1 \le d \le k-1$, where $\dot{w}$ is a minimum of $h_{k+1+d}(w)$ with $w \in [0,1)$. This proof has been shown numerically for all $2 \le k \le 100$ and $1 \le d \le k-1$. As an example, let us consider some special cases shown in Table 2. In short, everything mentioned so far proves an improvement of Theorem 2.3 for all $2 \le k \le 100$ and $1 \le d \le k-1$. Aditionally, the behavior observed for $2 \le k \le 100$ conjectures an improvement of Theorem 2.3 for all $k \ge 101$ and $1 \le d \le k-1$.

Now, considering that for all $n \in \mathbb{N}$, $\lfloor P_G(w^*) \rfloor \le \lfloor h_{k+1+d}(\dot{w}) n \rfloor$, the former data lead to the conclusion that the strict inequality $\lfloor P_G(w^*) \rfloor < \lfloor \epsilon_d^k \rfloor$ is valid in most cases, as previously commented in Remark 2.6 b).

To complete the analysis of the proposed bound, it is important to compare it with the upper bounds given in Table 1, since in some special cases they are an improvement of Theorem 2.3 for $k = 2$. By applying Theorem 2.2 again, one can ensure that $P_G(w^*)$ provides a better upper bound than the corresponding upper bounds for the double total domination number given in Table 1. This fact is shown in detail in Table 3.

δ	Δ	$h_\delta(\dot{w})$
7	7	0.56 n
9	9	0.48 n
14	14	0.35 n

Tab. 3. Upper bound on the double total domination number obtained as a corollary of Theorem 2.1 for given values of δ and Δ.

Moreover, the graphical profile of the interactions between floor parts associated with the third columns of each table (considering the indicated colors) is shown in Figure 3, suggesting that $\lfloor \epsilon_d^k \rfloor$ is always greater than $\lfloor P_G(w^*) \rfloor$ for each n, confirming earlier remarks about the existing order relation between these integers.

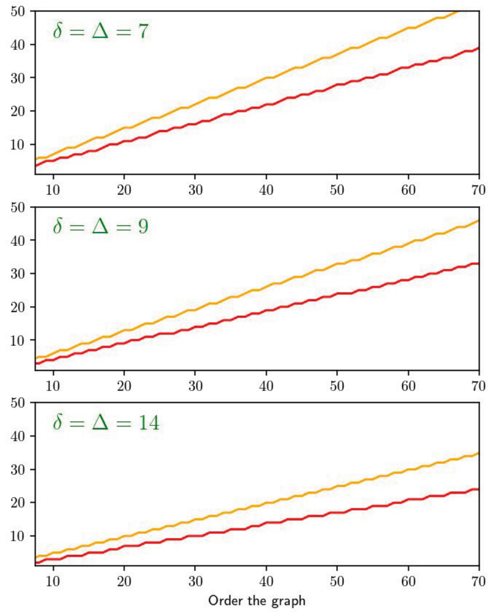

Fig. 3. Comparison between Table 1 and Table 3.

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