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THEORETICAL ASPECTS OF TOTAL TIME ON TEST TRANSFORM OF WEIGHTED VARIABLES AND APPLICATIONS

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Although the total time on test (TTT) transform is not a newly discovered concept, it has many applications in various fields. On the other hand, weighted distributions are extensively developed by the statisticians to tackle the insufficiency of the standard statistical distributions in modeling the arising data from real-world problems in the contexts like medicine, ecology, and reliability engineering. This paper develops the TTT transform for the weighted random variables and investigates the behavior of the failure rate function of such variables based on the TTT transform. In addition, the conditions for establishing the TTT transform ordering for weight variables and its relationship with some stochastic orders have been investigated, and the conditions for establishing the TTT transform order as well as the presentation of the new better than used in total time on test transform ($NBUT$) class of the weighted variables have also been studied. Finally, by analyzing the real data sets, applications of the transform introduced in the fit of a model is presented, and it is shown that weighted models have a significant advantage over the base models.

Keywords: total time on test transform, generalized failure rate, generalized reversed failure rate, new better than used in total time on test transform, weight function

Classification: 62N05, 62E15, 62P99

1. INTRODUCTION

The total time on test (TTT) transform has attracted the attention of researchers in several fields of applications such as reliability engineering (Klefsjo, [25]; [26]; Bergman and Klefsjo,[11]; Nair and Sankaran,[34]; Gamiz et al.[20]), economics (Pham and Turkkan, [41]), analysis of censored data (Sun and Kececloglu,[48]), maintenance schedule (Kumar and Westberg, [30]), model identification (Franco-Pereira and Shaked, [18]), stochastic orders (Kochar et al. [29]; Shaked and Shantikumar,[45] , and so on. In the domain of stochastic ordering, Kochar et al. [29] implemented the TTT transform order for nonnegative random variables, which is closely related to the position location independent riskier (LIR) and excess wealth (EW) orders. As a result of this expansion, one can derive some new intrinsic properties of the TTT order. They also developed an

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intriguing separation results on the relation between the TTT and EW orders. In the context of reliability analysis, Nair et al. [35] investigated the properties of the TTT transform order and its applications.

Furthermore, Nair and Sankaran [34] described some known characterizations of common aging notions based on the TTT function. Finally, in the research direction of model identification on the basis of TTT function and the observed TTT (when the random variable X is observed), Franco-Pereira and Shaked [18] derived two characterizations of the decreasing percentile residual life of order α (DPRL(α)) aging notion. In addition, for distributions with decreasing generalized failure rate, Bieniek [12] and Bieniek and Szpak [13] derived optimal bounds for the mean of TTT. Spiroiu [47] also calculated the average time spent on the test, mean of TTT, for a large sample. All of these show the pivotal role of TTT transforms in data analysis. Refer to Barlow and Campo [4], Barlow [3], Bergman and Klefsjo [10], Klefsjo [27] and Klefsjo and Westberg[28] for further study on the application of diagrams. Especially, TTT plots were used by Rao and Prasad [43] to estimate maintenance intervals for failure data with increasing failure rate. The TTT plot is a suitable criterion for analyzing nonnegative data. Later, Lai and Xie [31] provided additional results for implementations of the TTT plots. Recently, Gamiz et al. [20] also used the TTT plot as a graphical tool for aging trends recognition.

Weighted distributions have been widely developed by statisticians and data analysts to provide efficient statistical models for the arising data from various domains, including medicine, industry, ecology, reliability and many other fields. In fact, they are milestones for efficient modeling and prediction of data when the standard distributions are not adequate. Sometimes there is no suitable random sampling frame to observe events and apply classical sampling. In practice, sometimes it is not possible to record and view all the data of a random sample of the investigated population. It is even possible to select an observation with a certain probability that depends on the characteristics of the sample, but the selected sample is not a random sample of the population under investigation. Therefore, the sample collected in this way is a weighted sample of the community. For example, in situations where there is unequal probability sampling, such as actuarial sciences, ecology, biomedicine biostatistics, and survival data analysis, weighted distributions are appropriate. Up to now, a large number of investigations on developing weight distributions have been conducted. In particular, over the last 25 years, the concept of weighted distributions has been used to collect appropriate models for the observed data. According to the literature, Fisher [19] was the first to present the concept of weighted distributions. Then, Cox [14] introduced the concept of length-biased sampling, and Rao[42] developed a unifying approach that can be applied to a variety of sampling situations and visualized using weighted distributions. Recently, Patil [39] fully reviewed the weight distributions and investigated their features, as well as provided examples of applications where these models can be applied. Saghir et al. [44] presented the characterizations of these distributions based on a simple relationship between two truncated moments. Nguyen and Nguyen[36]) introduced a linear time partitioning algorithm for frequency weighted impurity functions.

As far as we know, obtaining the TTT transform for the weighted variables has not yet been investigated in the literature. Hence, given the importance of the weighted random variables in data modeling and the widespread application of the TTT transform, we are motivated to derive the TTT transform for the weighted variables as an efficient tool for reliability analysis. The obtained results can be extended easily to the other domains of applied sciences. We further investigated the properties of the TTT transform corresponding to the weighted variables as well as provided various examples for illustration purposes. In addition, the reversed failure rate (RFR) and generalized failure rate (GFR) metrics are also assessed. Subsequently, the conditions for establishing a TTT stochastic order for the weighted random variables are checked, and the new better than used in total time on test transform ($NBUT$) class for such variables are obtained.

In reliability theory and lifetime testing, incorrect choice of weighted or original distribution in lifetime data analysis may lead to different results for different reliability measures, such as, for example, failure rate function and mean residual lifetime function. For size-biased or length-biased distributions in which the weighted function is monotonically increasing, a reasonably good approximation of the reliability function and the mean residual life function will usually be obtained. Therefore, it is important to examine the invariance of weighted distributions in relation to the invariance of the main distributions.

In this paper, we continue this study by deriving some new results on the preservation of some properties of aging and stochastic orders by weighted distributions. Also, considering the important application of TTT transform in identifying the behavior of the failure rate function, the performance of this transform is very important to the failure rate function of weighted variables. Also, comparing the failure rate function of the weighted and the original variables according to the behavior of the TTT plot can lead to interesting results.

The rest of the article is organized as follows: In section 2, some essential preliminaries are reviewed. We provide the extention of weighted TTT transform and main results of this study in section 3. Section 4 deals with the application of TTT transform for weighted function with some illustrative examples. Finally, concluding remarks are given in the last section.

2. PRELIMINARY

Let X and Y be two random variables with the corresponding distribution functions F and G and probability density functions f and g, respectively. In addition, let $\bar{F} = 1 - F$ $\big(\bar{G} = 1 - G\big)$ be the survival (tail) function, let $F^{-1}(u) = \inf\{x : F(x) \geq u\}$, $u \in (0,1)$ $\big(G^{-1}(v) = \inf\{y : G(y) \geq v\}, v \in (0,1)\big)$, be the quantile function, and let $F^{-1}(0)(G^{-1}(0))$ and $F^{-1}(1)(G^{-1}(1))$ be the lower bound and upper bound of the support of the random variable $X(Y)$, say S_F (S_G).

Definition 2.1. For the nonnegative continuous random variable X, the TTT transform is defined as

$$T_X(u) = \int_0^{F^{-1}(u)} \bar{F}(x)\,\mathrm{d}x, \qquad \text{for all } u \in (0,1).$$

Note that $T(1) = E(X)$, where the expectation $E(X)$ can be finite or infinite. Given the fact that $0 < \mu = E(X) < \infty$, the scaled TTT transform $\varphi(u)$ of X is shown as follows:

$$\varphi_F(u) = \frac{T_X(u)}{\mu}, \qquad \text{for all } u \in (0, 1).$$

Barlow and Campo [4] were the first to introduce the scaled TTT transform, which is also a free scale. The scaled TTT transform has proved to be an extremely useful tool in a variety of reliability applications, including model recognition, characterizing different aging properties and analyzing various maintenance and burn-in problems. According to Theorem 6 of Bartoszewicz and Skolimowska [6], it can be concluded that if X is the increasing failure rate (the decreasing failure rate) $IFR(DFR)$, then $\varphi_X(pq) \leq (\geq)\varphi_X(p)\varphi_X(q)$, for all $p, q \in (0, 1]$.

Definition 2.2. Let X and Y be two random variables with distribution functions F and G that have finite means. Then X is said to be smaller than Y in the TTT transform order, denoted by $X \leq_{ttt} Y$, if

$$T_X(u) \leq T_Y(u), \qquad \text{for all } u \in (0, 1).$$

Let us proceed with the definition of the EW order. The EW order is defined in terms of the EW transform. Given the random variable X with distribution function F, the EW transform associated to X, denoted here by EW_X, is defined by

$$EW_X(u) = \int_{F^{-1}(u)}^{\infty} \bar{F}(x)\,\mathrm{d}x, \qquad \text{for all } u \in (0, 1).$$

Accordingly, given the random variables X and Y with finite means, X is said to be smaller than Y in the EW order, denoted by $X \leq_{ew} Y$, if

$$EW_X(u) \leq EW_Y(u), \quad u \in (0, 1).$$

When X and Y have an equal finite mean, that is, $E(X) = E(Y) < \infty$, it follows from the definitions that

$$X \leq_{ttt} Y \Longleftrightarrow X \geq_{ew} Y.$$

Jewitt [24] considered an order, called the LIR order, represented by $\leq_{lir}$. It is illustrated in Fagiuoli et al. [16] that, $X \leq_{lir} Y$ if and only if $-X \leq_{ew} -Y$. In other words, any result that holds for the order $\leq_{ew}$ can be rewritten with the assistance of order $\leq_{lir}$.

Jewitt [24] proved that if X and Y have the distribution functions F and G, then

$$X \leq_{lir} Y \Longleftrightarrow LIR_X(p) \leq LIR_Y(p), \quad p \in (0, 1),$$

where $LIR_X(p) = \int_{-\infty}^{F^{-1}(p)} F(x)\,\mathrm{d}x$ and $LIR_Y(p) = \int_{-\infty}^{G^{-1}(p)} G(y)\,\mathrm{d}y$, are the LIR transformations of X and Y, respectively. In what follows, we will provide a useful proposition for theIFR and DFR random variables.

A brief description of the weight function, the motivation for its creation, and its characteristics, as well as some required definitions, are given below. Let X be a discrete random variable with probability mass function p and support $S_X = \{x_1, x_2, \ldots\}$. Consider an observation recording system in nature where each value in the sample space S_X is recorded as a sample observation with a chance that depends on the magnitude of that value. If R is the occurrence of the observation record, then the sample obtained in this way is not a random sample of p but a random sample of the conditional probability mass function p_R, which is related to the conditional random variable $(X|R)$. As a result, using the Bayes formula, we have

$$\begin{aligned} p_R &= P(X = x|R) \\ &= \frac{P(R|X = x)P(X = x)}{P(R)} \\ &= \frac{w(x)p(x)}{E[w(X)]}, \quad x \in S_X, \end{aligned}$$

where $w(x) = P(R|X = x)$ is a value in $[0, 1]$, which is actually the probability of recording an observation with the value of x in the sample.

So in the general case if $w : R \longrightarrow R^+$ be a function for which $0 \leq E[w(X)] \leq \infty$. Then

$$F_w(x) = \frac{1}{E[w(X)]} \int_{-\infty}^{x} w(u)\,\mathrm{d}F(u) = \frac{1}{E[w(X)]} \int_{-\infty}^{F(x)} wF^{-1}(z)\,\mathrm{d}z,$$

is a distribution function, called the weighted distribution associated with baseline distribution F. If the density f of F exists, then $f_w(x) = \frac{w(x)f(x)}{E[w(x)]}$ is the density of $F_w(x)$. If $F(0) = 0$ and $w(x) = x^k$, where k is a positive integer, then we call $F_w(x)$ the length-biased (or size-biased) distribution of order k and denote it by $F_{w_k}(x)$ and simply by $F_{w_1}(x)$ if $k = 1$. If $E(X) < \infty$, then $F_{w_1}(x) = \frac{1}{E(X)} \int_0^x u\,\mathrm{d}F(u)$ and $f_{w_1}(x) = \frac{xf(x)}{E(X)}, x > 0$. It is obvious that the function $\frac{F^{-1}(u)}{E(X)}$ is a probability density function on $(0, 1)$.

Definition 2.3. Let F be a lifetime distribution.

- F is said to be IFR (or DFR) if $r(x)$ is increasing (or decreasing) on S_F being an interval, where $r(x) = \frac{f(x)}{\bar{F}(x)}$ is the hazard rate function of F.
- F is said to be increasing reversed failure rate (IRFR) if $\check{r}(x)$ is increasing, and F is said to be decreasing reversed failure rate (DRFR) if $\check{r}(x)$ is decreasing, where $\check{r}(x) = \frac{f(x)}{F(x)}$ is the reversed hazard rate function of F.
- F is said to be increasing generalized failure rate ($IGFR$) if $h(x)$ is increasing, and F is said to be decreasing generalized hazard rate ($DGFR$) if $h(x)$ is decreasing, where $h(x) = xr(x)$ is the generalized hazard rate function of F.
- F is said to be increasing generalized reversed failure rate ($IGRFR$) if $\check{h}(x)$ is increasing, and F is said to be decreasing generalized reversed hazard rate ($DGRFR$) if $\check{h}(x)$ is decreasing, where $\check{h}(x) = x\check{r}(x)$ is the generalized hazard rate function of F.

Throughout this study, we have used notations similar to Shaked and Shanthikumar [45]. On the other hand, stochastic orders are relationships between probability distributions.

Definition 2.4. Let X and Y be two random variables with distribution functions F and G.

- X is stochastically smaller than Y ($X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all x values.
- X is smaller than Y in the likelihood ratio order ($X \leq_{lr} Y$) if $\frac{g(x)}{f(x)}$ be increasing.
- X is smaller than Y in the hazard rate order ($X \leq_{hr} Y$) if $\frac{\bar{G}(x)}{\bar{F}(x)}$ is increasing or $r_F(x) \geq r_G(x)$ for all x values if F and G are absolutely continuous.
- X is smaller than Y in the reversed hazard rate order ($X \leq_{rh} Y$) if $\frac{G(x)}{F(x)}$ is increasing, or $\check{r}_F(x) \leq \check{r}_G(x)$ for all x values if F and G are absolutely continuous.
- X is smaller than Y in the dispersive order ($X \leq_{disp} Y$) if $F^{-1}(b) - F^{-1}(a) \leq G^{-1}(b) - G^{-1}(a)$ whenever $0 < a \leq b < 1$.
- X is smaller than Y in the star order ($X \leq_{*} Y$) if $G^{-1}(F(x))$ is star-shaped in x (that is, if $\frac{G^{-1}(F(x))}{x}$ increases in $x \geq 0$).
- X is smaller than Y in the Lorenz order ($X \leq_{Lorenz} Y$) if and only if $L_X(p) \geq L_Y(p)$, in which $L_X(p) = \frac{\int_0^p F^{-1}(u)\,\mathrm{d}u}{\int_0^1 F^{-1}(u)\,\mathrm{d}u}$, for all $p \in [0, 1]$.

It is also well known that

- $X \leq_{lr} Y \Longrightarrow X \leq_{hr} Y \Longrightarrow X \leq_{st} Y$,
- $X \leq_{lr} Y \Longrightarrow X \leq_{rh} Y \Longrightarrow X \leq_{st} Y$,
- $X \leq_{st} Y \Longrightarrow X \leq_{ttt} Y$.

The work of Shaked and Shanthikumar [45] is a complete reference for studying the conditions for establishing the opposite of the above relations.

3. MAIN RESULTS

The concept of TTT transform is well known for its applications in different fields such as reliability analysis, econometrics, stochastic modeling, tail orderings, and ordering distributions. Given the importance of the weight distributions, the TTT transform for weight variables is interesting. The behavior of the failure rate function and the $NBUT$ class for the TTT transform of weighted variables will also be examined.

3.1. *TTT* transform of weighted variables

Let $Y = w(X)$ be a random variable, where $w(\cdot)$ is a derivative weight function with probability density function $f_w(x)$.

Proposition 3.1. For the nonnegative random variable Y, the TTT transform and scaled TTT transform are

$$T_Y(p) = \int_0^{F^{-1}(p)} \bar{F}(s)w^{'}(s)\,\mathrm{d}s, \tag{1}$$

$$\varphi_Y(p) = \frac{T_Y(p)}{E(w(X))}, \qquad \text{for all } p \in (0,1)$$

respectively. To obtain equation (1) is sufficient to write the TTT transform for the weighted variable Y, therefore

$$T_Y(p) = \int_0^{H_Y^{-1}(p)} \bar{H}(y)\,\mathrm{d}y = \int_0^{w(F^{-1}(p))} \bar{F}(w^{-1}(y))\,\mathrm{d}y,$$

where $\bar{H}(y)$ and $H_Y^{-1}(p)$ are survival and quantile functions of Y respectively. Now by changing the variable $w^{-1}(y) = x$ the desired result is obtained.

If $w(x) = K(F(x))$, then $w^{'}(x) = f(x)K'(F(x))$, and thus

$$T_Y(p) = \int_0^{F^{-1}(p)} f(s)\bar{F}(s)K^{'}(F(s))\,\mathrm{d}s = \int_0^{p} (1-z)K^{'}(z)\,\mathrm{d}z. \tag{2}$$

When the weight function is a function of the distribution function, Equation (2) is more suitable for obtaining the TTT transform of weighed variables.

Proposition 3.2. Let X be a nonnegative random variable with distribution function F and let $Y = w(X)$ be a weighted random variable. Then $T_Y(p)$ is convex (concave) if and only if $\frac{w^{'}(x)}{r(x)}$ be increasing (decreasing). Also, for X, a nonnegative random variable with distribution function F and $Y = w(X)$, a weighted random variable, $F_Y \in IFR(DFR)$ if

$$T_Y^{'}(p) = \frac{w^{'}(F^{-1}(p))(1-p)}{f(F^{-1}(p))} \quad \searrow (\nearrow) in \quad p, \qquad \text{for all } p \in (0,1).$$

Remark 3.3. Let X be a nonnegative random variable with a distribution function F and let $Y = w(X)$ be a weighted random variable. Then EW transform and LIR transform of Y are

- $EW_Y(p) = \int_{F^{-1}(p)}^{\infty} \bar{F}(x)w^{'}(x)\,\mathrm{d}x, \quad$ for all $p \in (0,1)$,
- $LIR_Y(p) = \int_{-\infty}^{F^{-1}(p)} F(x)w^{'}(x)\,\mathrm{d}x, \quad$ for all $p \in (0,1)$,

respectively, and for $w(x) = K(F(x))$, we have

- $EW_Y(p) = \int_p^1 (1-z)K^{'}(z)\,\mathrm{d}z, \quad$ for all $p \in (0,1)$,

- $LIR_Y(p) = \int_0^p zK^{'}(z)\,\mathrm{d}z, \qquad$ for all $p \in (0,1)$.

Corollary 3.4. Let X be a nonnegative random variable with distribution function F and let $Y = w(X)$ be a weighted random variable. With a simple calculation, we can show that

$$T_Y(p) + EW_Y(p) = E(w(X)) - w(0),$$

and in the special case for $w(X) = X$, we have $T_X(p) + EW_X(p) = E(X)$.

Proposition 3.5. Let X and Y be two nonnegative random variables with distribution functions F and G, respectively, and let $w(x)$ be a weighted random variable. Then,

a) $X <_{ttt(ew,lir)} X_w$ if and only if $w(x) \geq x + c$ where in c is constant.

b) If $X \leq_{st} Y$, then $X_w \leq_{ttt} Y_w$.

c) For two weight functions w_1 and w_2, if $X \leq_{st} Y$ and $\frac{\bar{F}(x)}{\bar{G}(x)} \leq \frac{w_2^{'}(x)}{w_1^{'}(x)}$, then $X_{w_1} \leq_{ttt} Y_{w_2}$.

Proof. To prove, it is enough to use the definition of stochastic orders. Here, we prove the problem for the TTT transform order. For weight function $w(\cdot)$, we have

$$X <_{ttt} X_w \Longleftrightarrow T_X(p) \leq T_{X_w}(p) \Longleftrightarrow \int_0^{F^{-1}(p)} \bar{F}(s)\,\mathrm{d}s \leq \int_0^{F^{-1}(p)} \bar{F}(s)w^{'}(s)\,\mathrm{d}s,$$

$$\Longleftrightarrow \bar{F}(s) \leq \bar{F}(s)w^{'}(s) \Longleftrightarrow w^{'}(s) \geq 1 \Longleftrightarrow w(s) \geq s + c.$$

Substituting x with s results in the desired result. Cases b and c are clear. □

Corollary 3.6. Let w_1 and w_2 be two derivative weight functions. In this case, it holds that

$$w_1^{'}(x) \leq w_2^{'}(x) \Longleftrightarrow X_{w_1} \leq_{ttt(ew,lir)} X_{w_2}.$$

Let us consider the the weight function $w_c(x) = x^c$ for some fixed $c \in (0,1]$. In this case, the distribution function F_{w_c} is usually called "size-biased"(see Patil and Ord [40]). If $c > (<)1$ and $F \in IGFR(DGFR)$, then $F_w \in IGFR(DGFR)$ and as a result $T_w(p)$ is concave (convex). Transforms regarding $w_c(x)$ are as follows:

- $T_{w_c}(p) = (1-p)(F^{-1}(p))^c + E[I\{x \leq F^{-1}(p)\}x^c]$,

- $EW_{w_c}(p) = E[I\{x \geq F^{-1}(p)\}x^c] - (1-p)(F^{-1}(p))^c$,

- $LIR_{w_c}(p) = p(F^{-1}(p))^c - E[I\{x \leq F^{-1}(p)\}x^c]$.

According to Theorem 2.2 in Franco-Pereira and Shaked [18], this transform in weight mode is also related to $DPRL(\alpha)$ as follows:

$$F_w \in DPRL(\alpha) \Longleftrightarrow \frac{(1-\alpha)q(\alpha+(1-\alpha)u)}{q(u)} \leq \frac{w^{'}(Q(u))}{w'(Q(\alpha+(1-\alpha)u))},$$

where $q(u) = \frac{d}{du}Q(u)$ and $t(u) = \frac{d}{du}T_w(u)$. It is useful to see Feranco-Pereira et al. [17] and Haines and Singpurwalla [21].

Three typical examples of such weighted function are $y_1 = F^\theta(x)$, $y_2 = 1 - \bar{F}^\theta(x)$, and $y_3 = F^\alpha(x)\bar{F}^\beta(x)$ that in the order named, respectively, the proportional reversed hazard rate family of distributions, proportional hazard rate family of distributions, and Jones model, where α, β, and θ are positive real parameters. In the following example, we present the introduced transforms for these three weighted functions.

Example 3.7. Let X be a nonnegative random variable with distribution function F and let y_1, y_2, y_3 be weighted random variables with TTT transform of the form (2). Then

- $T_{Y_1}(p) = p^\theta(\frac{1+\theta-\theta p}{1+\theta})$, $\quad T_{Y_2}(p) = \frac{\theta}{1+\theta}(1-(1-p)^{\theta+1})$,
  $T_{Y_3}(p) = \alpha\frac{\Gamma(\alpha)\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)}Beta(p,\alpha,\beta+2) - \beta\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}Beta(p,\alpha+1,\beta+1)$,
- $EW_{Y_1}(p) = \frac{1}{\theta+1} - p^\theta(1-\frac{\theta}{\theta+1}p)$, $\quad EW_{Y_2}(p) = \frac{\theta}{\theta+1}(1-p)^{\theta+1}$,

$$\begin{aligned} EW_{Y_3}(p) &= \alpha\frac{\Gamma(\alpha)\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)}(1-Beta(p,\alpha,\beta+2)) \\ &\quad -\beta\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}(1-Beta(p,\alpha+1,\beta+1)), \end{aligned}$$

- $LIR_{Y_1}(p) = \frac{\theta}{\theta+1}p^{\theta+1}$, $\quad LIR_{Y_2}(p) = \frac{1}{\theta+1}(1-(1-p)^{\theta+1}) - p(1-p)^\theta$,

$$\begin{aligned} LIR_{Y_3}(p) &= \alpha\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}Beta(p,\alpha+1,\beta+1) \\ &\quad -\beta\frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)}Beta(p,\alpha+2,\beta). \end{aligned}$$

Note that $Beta(p,\alpha,\beta)$ is Beta distribution function with parameters α and β in p.

For some weight functions, the weighted TTT transform has an explicit form, which is presented in Table 1. Also, the EW transformation and LIR transformation for the weights in Table 1 are shown in Table 2.

In addition, the behavior of TTT transform can be investigated by TTT plot. Figures 1 to 4 show the TTT plots for some of the weight functions in Table 1. It can be easily concluded that the TTT plot of w_3 and w_7 is first convex and then concave, which means that the distributions of w_3 and w_7 have a bathtub-shaped failure rate. Also, the TTT plot of w_4 and w_8 is concave, which means distributions of w_4 and w_8 have increasing failure rates, and for w_4, as the parameter increases, the failure rate decreases. For w_8, as the parameter increases, the failure rate also increases.

	Weight function	TTT Transform
$w_1(x)$	$Constant$	0
$w_2(x)$	x	$T_X(p)$
$w_3(x)$	$F(x)+\bar{F}(x)\ln\bar{F}(x)$	$\frac{1}{2}(1-p)^2\ln(1-p)-\frac{1}{4}(1-p)^2+\frac{1}{4}$
$w_4(x)$	$1-(1-\alpha)\bar{F}(x)$	$(1-\alpha)p(1-\frac{p}{2})$
$w_5(x)$	$1-(1-\alpha)\bar{F}^{\beta}(x)$	$\frac{\beta(1-\alpha)}{\beta+1}(1-(1-p)^{\beta+1})$
$w_6(x)$	$\alpha F^{\alpha-1}(x)$	$Beta(p,\alpha-1,2)$
$w_7(x)$	$\frac{a}{(1-\bar{a}\bar{F}(x))^2}$	$\frac{a}{\bar{a}(1-\bar{a}(1-p))^2}-\frac{2a}{\bar{a}(1-\bar{a}(1-p))}-\frac{1}{a\bar{a}}+\frac{2}{\bar{a}}$
$w_8(x)$	$1-\bar{F}^{a}(x)$	$\frac{a}{a+1}(1-(1-p)^{a+1})$
$w_9(x)$	$F(x)(1+a\bar{F}(x))$	$\frac{1}{2}-\frac{\alpha}{3}-\frac{(1-p)^2}{2}+\frac{\alpha(1-p)^3}{3}+\frac{\alpha p^2}{2}-\frac{\alpha p^3}{3}$
$w_{10}(x)$	$\frac{aF(x)}{1-(1-a)F(x)}$	$a^2-\frac{a^2}{1-(1-a)p}+a\ln(1-(1-a)p)$

Tab. 1. TTT transforms regarding some well-known weight functions.

$w(x)$	EW transform	LIR transform
$w_1(x)$	0	0
$w_2(x)$	$EW_X(p)$	$LIR_X(p)$
$w_3(x)$	$\frac{1}{4}(1-p)^2-\frac{1}{2}(1-p)^2\ln(1-p)$	$\frac{1}{2}[(1-p)^2\ln(1-p)-(1-p^2)-\frac{(1-p)^2}{2}+\frac{3}{2}]$
$w_4(x)$	$\frac{(1-\alpha)}{2}(1-p)^2$	$\frac{(1-\alpha)}{2}p^2$
$w_5(x)$	$\frac{\beta(1-\alpha)}{\beta+1}((1-p)^{\beta+1})$	$\frac{1-\alpha}{\beta+1}B(p,2,\beta)$
$w_6(x)$	$1-Beta(p,\alpha-1,2)$	$(\alpha-1)p^{\alpha}$
$w_7(x)$	$\frac{2a}{\bar{a}}[\frac{1}{1-\bar{a}(1-p)}-\frac{1}{2(1-\bar{a}(1-p))^2}+\frac{1}{2}]$	$\frac{2a}{\bar{a}(1-\bar{a}(1-p))}-\frac{a^2}{\bar{a}(1-\bar{a}(1-p))^2}+\frac{1}{a\bar{a}}-\frac{2}{\bar{a}}-\frac{1}{a}$
$w_8(x)$	$\frac{a}{a+1}(1-p)^{a+1}$	$\frac{1}{\alpha+1}B(p,2,\alpha)$
$w_9(x)$	$\frac{(1+\alpha)}{2}(1-p)^2+2\alpha(\frac{p^2}{2}-\frac{p^3}{3})$	$\frac{(1+\alpha)^2}{2}p-\frac{2\alpha p^3}{3}$
$w_{10}(x)$	$\frac{\alpha}{(1-\alpha)^2}[\ln(1-(1-\alpha)p)+\frac{\alpha}{1-(1-\alpha)p}-\ln\alpha-1]$	$\frac{\alpha}{(1-\alpha)^2}[\ln(1-(1-\alpha)p)+\frac{1}{1-(1-\alpha)p}-1]$

Tab. 2. EW and LIR transformations for some weights.

3.2. Weighted distributions with monotone failure rates properties

In this section, we examine the relationship between the weighted TTT transform and failure rate concepts. For this purpose, considering the different modes, failure rate, RFR, GFR, and reversed generalized failure rates are rewritten in terms of the TTT transform of the weighted random variable, and these relationships are explained by providing examples. Also, using a special weight function, a characterization of distribution F based on the GFR function is presented. Denote $Y = w(X)$, and let φ_Y be the scaled TTT transform of Y. Considering the relationship between TTT transform and Lorenz curve as well as Theorem 1 of Bartoszewicz and Skolimowska [6], we have the following statement for weighted variables. Accordingly, it can be easily shown that $F_w(x) = \varphi_w(F(x))-\frac{w(x)\bar{F}(x)}{E(w(X))}$, if w is increasing and that $F_w(x) = 1-\varphi_w(\bar{F}(x))+\frac{w(F^{-1}(\bar{F}(x)))F(x)}{E(w(X))}$, if w is decreasing. Table 3 shows the relationship between the types of failure rate functions and TTT transform of weighted variables.

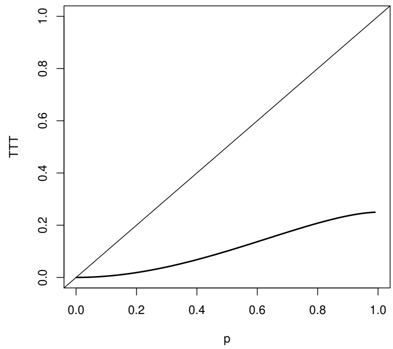

Fig. 1. TTT plot of w_3.

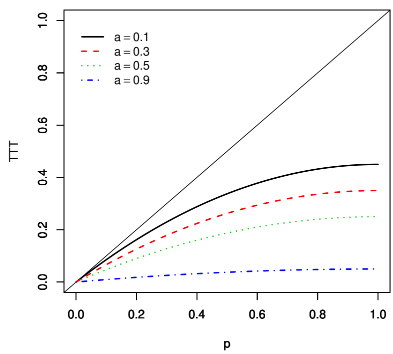

Fig. 2. TTT plot of w_4 with different values of parameter a.

In that follows, the relationship between the weighted TTT transform and the GFR expression will be discussed.

Proposition 3.8. Let F be a life distribution and let $w(\cdot)$ be a weight function. Given $w(x) = \ln x$, for the weighted distribution F_w, we have

(a) F_w is $IGFR\,(DGFR)$ if and only if $T_w(p)$ is concave (convex) for $0 < p < 1$.

(b) F_w is IGFR and, as a result, $\varphi_w(p)$ is concave, if $w(x)h_F(x)$ or $w(x)\check{h}_F(x)$ is increasing.

Proof. (a) By taking the first derivation of $T_w(p)$ with respect to p, it can be derived that $T_w^{'}(p) = \frac{(1-p)w^{'}(F^{-1}(p))}{f(F^{-1}(p))}$ and $T_w^{'}(p) = \frac{w^{'}(x)}{r(x)}$ when $p = F(x)$. Now, considering

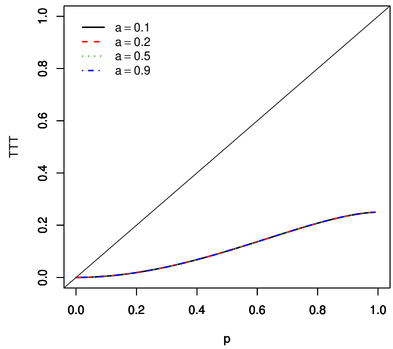

Fig. 3. TTT plot of w_7 with different values of parameter a.

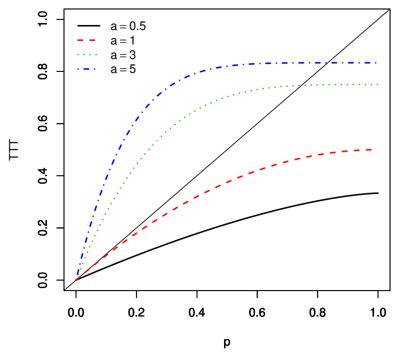

Fig. 4. TTT plot of w_9 with different values of parameter a.

$w(x) = \ln x$ with $w^{'}(x) = \frac{1}{x}$, it is easy to see that

$$T_w^{'}(p) = \frac{1}{xr(x)} = \frac{1}{h(x)}.$$

To verify the implications in the table for the $IGFR(DGFR)$ case, first, let us assume that F_w is absolutely continuous with the *GFR* function $h(\cdot)$. If F_w is $IGFR(DGFR)$, then $T_w^{'}(p)|_{p=F(x)} = \frac{1}{h(x)}$ is decreasing (increasing) in x, which implies that T_w is concave (convex). Conversely, if T_w is concave (convex), then the *GFR* function would be increasing (decreasing).

(b) Considering (a) and equations (1) and (4) along with Theorem 3.13 of Behdani et al. [8], the proof would be completed. □

FR function	w is increasing	w is decreasing
$r_{F_w}(x)$	$\frac{w(x)r_F(x)}{E(w(X))} \times \frac{\bar{F}(x)}{1-\varphi_w(F(x))+\frac{w(x)\bar{F}}{E(w(X))}}$	$\frac{w(x)r_F(x)}{E(w(X))} \times \frac{\bar{F}(x)}{\varphi_w(\bar{F}(x))-\frac{w(F^{-1}(\bar{F}(x)))F(x)}{E(w(X))}}$
$\check{r}_{F_w}(x)$	$\frac{w(x)\check{r}_F(x)}{E(w(X))} \times \frac{F(x)}{\varphi_w(F(x))-\frac{w(x)\bar{F}}{E(w(X))}}$	$\frac{w(x)\check{r}_F(x)}{E(w(X))} \times \frac{F(x)}{1-\varphi_w(\bar{F}(x))+\frac{F(x)w(F^{-1}(\bar{F}(x))}{E(w(X))}}$
$h_{F_w}(x)$	$h(x) \times \frac{w(x)}{E(w(X))} \times \frac{\bar{F}(x)}{1-\varphi_w(F(x))+\frac{\bar{F}(x)w(x)}{E(w(X))}}$	$h(x) \times \frac{w(x)}{E(w(X))} \times \frac{\bar{F}(x)}{\varphi_w(\bar{F}(x))-\frac{F(x)w(F^{-1}(\bar{F}(x))}{E(w(X))}}$
$\check{h}_{F_w}(x)$	$\check{h}(x) \times \frac{w(x)}{E(w(X))} \times \frac{F(x)}{\varphi_w(F(x))-\frac{\bar{F}(x)w(x)}{E(w(X))}}$	$\check{h}(x) \times \frac{w(x)}{E(w(X))} \times \frac{F(x)}{1-\varphi_w(\bar{F}(x))+\frac{F(x)w(F^{-1}(\bar{F}(x))}{E(w(X))}}$

Tab. 3. Failure rate functions and TTT transform of weighted variables.

Also, the weighted TTT transform can be written according to the GFR and the generalized reversed failure rate. Let F be a life distribution and let $w(\cdot)$ be a weight function. Then

$$T_w(p) = (1-p)w(F^{-1}(p)) + \int_0^{F^{-1}(p)} (\frac{h(x)\check{h}(x)}{h(x)+\check{h}(x)}) \frac{w(x)}{x} \, \mathrm{d}x. \tag{3}$$

In the equation (3), the expression in brackets can be supplanted with $h(x)\bar{F}(x)$ and $\check{h}(x)F(x)$. Next, equation (3) is obtained for some specific weight functions.

Example 3.9. Let F be a lifetime distribution and let $w(\cdot)$ be a weight function.

(a) For the length (size)-biased weight function, we have

$$T_w(p) = (1-p)F^{-1}(p) + \int_0^{F^{-1}(p)} (\frac{h(x)\check{h}(x)}{h(x)+\check{h}(x)}) \, \mathrm{d}x.$$

(b) For the length (size)-biased of order α, $\alpha = 1, 2, \ldots,$ the weight function is

$$T_w(p) = (1-p)(F^{-1}(p))^{\alpha} + \int_0^{F^{-1}(p)} (\frac{h(x)\check{h}(x)}{h(x)+\check{h}(x)}) x^{\alpha-1} \, \mathrm{d}x.$$

(c) For an inverse failure rate weight function with $\alpha = -1$, we have

$$T_w(p) = \frac{1-p}{p} + \int_0^{F^{-1}(p)} \check{h}(x) \, \mathrm{d}x.$$

(d) For a failure rate weight function with $\beta = -1$, it is

$$T_w(p) = 1 + \int_0^{F^{-1}(p)} r(x) \, \mathrm{d}x.$$

reliability measures	length-biased	equilibrium
$F_w(x)$	$\varphi_X(F(x)) - \frac{x\bar{F}(x)}{\mu}$	$\varphi_X(F(x))$
$r_{F_w}(x)$	$\frac{\bar{F}(x)+\frac{xf(x)-x}{\mu}}{f(x)(1-\varphi_X(F(x))+\frac{x\bar{F}(x))}{\mu}}$	$\frac{\bar{F}(x)}{E(X)(1-\varphi_X(F(x))+\frac{x\bar{F}(x)}{\mu})-x\bar{F}(x)}$
$\check{r}_{F_w}(x)$	$\frac{\bar{F}(x)+\frac{xf(x)-x}{\mu}}{f(x)(\varphi_X(F(x))-\frac{x\bar{F}(x))}{\mu}}$	$\frac{\bar{F}(x)}{x\bar{F}(x)+E(X)(\varphi_X(F(x))-\frac{x\bar{F}(x)}{\mu})}$
$m_{F_w}(x)$	$\frac{\int_x^\infty 1-\varphi_X(F(t))+\frac{t\bar{F}(t)}{\mu}\,\mathrm{d}t}{1-\varphi_X(F(x))+\frac{x\bar{F}(x)}{\mu}}$	$\frac{\int_x^\infty 1-\varphi_X(F(t))+\frac{t\bar{F}(t)}{\mu}\,\mathrm{d}t}{1-\varphi_X(F(x))+\frac{x\bar{F}(x)}{\mu}}$

Tab. 4. Reliability measures and TTT transform of weighted variables.

It is worth noting that the length-biased distribution and equilibrium distribution (as a weighted distribution) resemble the Lorenz curve quite closely. Now considering the relationship between the TTT transform and Lorenz curve as well as Propositions 5.9 and 5.14 of Behdani et al. [7], provides helpful information regarding these relationships. For the length-biased distribution, we have the following expressions shown in Table 4 for some reliability measures and their relationships with the TTT transform, where $m_F(x) = \frac{\int_x^\infty \bar{F}(t)\,\mathrm{d}t}{\bar{F}(x)}, x \geq 0$, is the mean residual life time corresponding to F.

3.3. Stochastic ordering of weighted distributions

In this section, we will conduct a comparison between the weighted and original distributions in terms of stochastic ordering. We also look at the topic of stochastic ordering preservation under weighting. Clearly, the definition leads to the following result. Let X and Y be two random variables, let F and G be their respective distribution functions, and let $w(\cdot)$ be a monotone left continuous weight function. According to the results of section 2.4 in Bartoszewicz and Skolimowska[6] and the relations between stochastic orders, it is simple to conclude that if w is increasing (decreasing), then $X \leq_{ttt} X_w (X_w \leq_{ttt} X)$. Also, if $X \leq_{hr(rh)} Y$, then $X_w \leq_{ttt} Y_w$ for all w increasing (decreasing). In addition, if $X \leq_{hr(rh)} Y$ and $\frac{r_G(x)}{r_F(x)} \left(\frac{\check{r}_G(x)}{\check{r}_F(x)}\right)$ is increasing, or F is the DFR and G is the IFR (F is the $DRFR$ and G is the $IRFR$), then $X_w \leq_{ttt} Y_w$. Moreover, according to Theorem 3 of Bartoszewicz [5], if $X_w \leq_{disp} Y_w$, then $X_w \leq_{ttt} Y_w$.

In the following, the storage conditions of the TTT transform order are expressed by some stochastic orders.

Remark 3.10. Let $X \leq_{st} Y$, let $w(X) \leq_{Lorenz} w(Y)$, and let w be monotone left continuous. If w is increasing (decreasing), then $X_w \leq_{ttt} Y_w (Y_w \leq_{ttt} X_w)$. Also if $X \leq_{st} Y$, $X \leq_* Y$, and $w(x) = x^p$, $p \neq 0$, then $X_w \leq_{ttt} Y_w (Y_w \leq_{ttt} X_w)$ for $p > (<) 0$.

Remark 3.11. Let F and G be absolutely continuous distribution functions, let $F(0) = G(0) = 0$, let F be the DFR, let G be the IFR, and let $X \leq_{disp} Y$. Then the following properties hold:

(a) If w is decreasing and convex, then $X_w \leq_{ttt} Y_w$.

(b) If w is increasing left continuous and $w(x)r_F(x)$ is decreasing, then $X_w \leq_{ttt} Y_w$.

Remark 3.12. (a) Let $X \leq_{disp}$, let $Y \leq_{st} X$, let F be $DRFR$, and let G be $IRFR$. If w is increasing and $w(x)\breve{r}_G(x)$ is increasing, then $X_w \leq_{ttt} Y_w$.

(b) Let $X \leq_{disp}$, let $Y \leq_{st} X$, let F be DFR, and let G be $IRFR$. If w is decreasing convex, then $Y_w \leq_{ttt} X_w$.

Let X and Y be two random variables, let F and G be their distribution functions, and let $w_i, i = 1, 2$ be two monotone left continuous functions. Misra et al. [33], Izadkhah et al. [22], Izadkhah et al. [23], and Behdani et al. [7] obtained the criteria for preserving some stochastic orders under various assumptions. All of these findings lead to the conclusion that the TTT order has been formed as $X_{w_1} \leq_{ttt} Y_{w_2}$.

Marshall and Olkin [32] suggested a way for increasing the flexibility of a family of distributions by introducing a parameter. They established the family of survival functions

$$g = \{\bar{G}_\alpha : \bar{G}_\alpha(x) = \frac{\alpha\bar{F}(x)}{1 - \bar{\alpha}\bar{F}(x)} = \frac{\alpha\bar{F}(x)}{F(x) + \alpha\bar{F}(x)}; x \in R, \alpha > 0, \bar{\alpha} = 1 - \alpha\},$$

where $\bar{F}$ is a survival function. According to Theorem 3 and Corollaries 1 and 2 of Benduch-Fraszczak [9], for the weight function $w(\cdot)$ and $0 < \alpha < \beta < \infty$, it is easily obtained $G_\alpha^w \leq_{ttt} G_\beta^w$. Also $G_\alpha^w \leq_{ttt} F_w (F_w \leq_{ttt} G_\alpha^w), 0 < \alpha < 1 (\alpha \geq 1)$.

Esfahani et al. [15] showed that if X is a random variable with a distribution function F and

$$w(x) = ae^{-a\bar{F}(x)} + e^{-a}, a > 0, \tag{4}$$

is an increasing weight function, then X_w is a weighted variable with distribution function of the form

$$F_w(x) = e^{-a[\bar{F}(x)]} - e^{-a}[\bar{F}(x)].$$

For this weight, we have

- $T_w(p) = (a(1-p)+1)e^{-a(1-p)} - (a+1)e^{-a}$,
- $EW_w(p) = 1 - (a(1-p)+1)e^{-a(1-p)}$,
- $LIR_w(p) = ape^{-a(1-p)} - e^{-a(1-p)} + e^{-a}$,

and $E[w(X)] = T_w(p) + W_w(p) + w(0)$. Figure 5 shows the TTT plot of $T_w(p)$ for different values of a.

It can be easily concluded that

(a) $T_w(p)$ is increasing for $p \in (0, 1)$,

(b) $T_w(p)$ is convex (concave) for $p \leq (\geq) \frac{a-1}{a}$.

From (b), it can be derived that $F_w \in DFR(IFR)$ if and only if $p \leq (\geq) \frac{a-1}{a}$. This can be clearly seen in Figure 5. Also Figure 6 shows the TTT plot of $T_w(p)$ for the breaking stresses of carbon fibers data set (Padgett and Spurrier [38]) and their weighted values for the weight function $w(x) = ae^{-a\bar{F}(x)} + e^{-a}, a > 0$. Esfahani et al.[15] derived another suitable and well-fitted density using the random conditional presenting of a lifetime distribution with using the Weibull distribution as a baseline density, and they showed the optimal value for $a = 8.1$. As can be seen, the weight function under study changes the shape of TTT plot, and by examining the graph, it can be concluded that the original data has a distribution with an increasing failure rate, but the weighted data have a distribution with a decreasing failure rate.

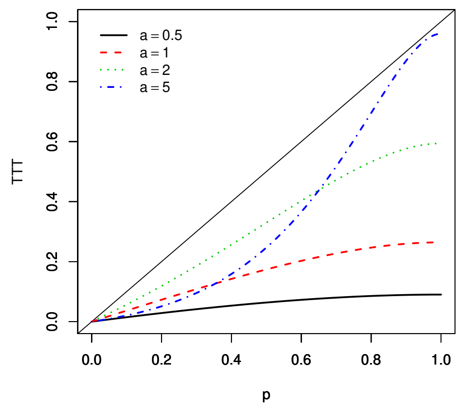

Fig. 5. TTT plot of $T_w(p)$ for different values of a.

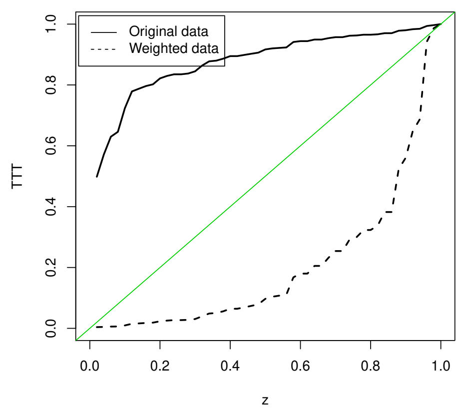

Fig. 6. TTT plot for 50 experimental data of the breaking stress of carbon fibers and their weighted data.

3.4. Weighted random variables in $NBUT$ class

In the reliability theory, the residual life and inactivity time variables are critical and the study of their properties has always been considered. For this reason, in this part, the conditions for putting the weighted variables in class $NBUT$ are inspected. First, the definition of class $NBUT$ is stated, and then this class is rewritten for weighted variables residual life. Next, conditions for establishing TTT transform order by inactivity time and double truncate variables are expressed.
A random variable X or F is said to be new better than used in TTT transform order denoted by $NBUT$ if

$$X_t \leq_{ttt} X,$$

or equivalently, $X \in NBUT$ if and only if

$$\int_0^{F_{X_t}^{-1}(p)} \bar{F}(x+t)\,\mathrm{d}x \leq \bar{F}(t) \int_0^{F^{-1}(p)} \bar{F}(x)\,\mathrm{d}x, \quad p \in (0,1),$$

where $X_t = [X - t|X > t], t \in \{x : F(x) < 1\}$, named residual life variable and denotes a random variable whose distribution is the same as the conditional distribution of X_t given that $X > t$. When X is the lifetime of a device, X_t can be regarded as the residual lifetime of the device at time t, given that the device has survived up to time t.
Now, let X^w be a weighted random variable. Then for $p \in (0,1)$,

$$X^w \in NBUT \Longleftrightarrow X_t^w \leq_{ttt} X^w$$

$$\Longleftrightarrow \int_{w^{-1}(t)}^{A(t))} \frac{\bar{F}(w^{-1}(x+t))w^{'}(x)}{\bar{F}(w^{-1}(t))}\,\mathrm{d}x \leq \int_0^{F^{-1}(p)} \bar{F}(x)w^{'}(x)\,\mathrm{d}x,$$

and also for t_1 and t_2, $X_{t_1}^w \leq_{ttt} X_{t_2}^w$ if and only if

$$\int_{w^{-1}(t_1)}^{A(t_1)} \frac{\bar{F}(w^{-1}(x+t_1))}{\bar{F}(w^{-1}(t_1))} w^{'}(x)\,\mathrm{d}x \leq \int_{w^{-1}(t_2)}^{A(t_2)} \frac{\bar{F}(w^{-1}(x+t_2))}{\bar{F}(w^{-1}(2_1))} w^{'}(x)\,\mathrm{d}x,$$

where $A(x) = F^{-1}(p + (1-p)F(w^{-1}(x)))$.
For any random variable X, let

$$X^t = [t - X|X < t], \quad t \in \{x : F(x) < 1\}.$$

When X is the lifetime of a device, X^t can be regarded as the inactivity time of the device at time t. If w is a weight function, then for $p \in (0,1)$, $X_w^t \leq_{ttt} X_w$ if and only if

$$\int_{B(t)}^{w^{-1}(t)} \frac{F(w^{-1}(t-x))}{F(w^{-1}(t))} w^{'}(x)\,\mathrm{d}x \leq \int_0^{F^{-1}(p)} \bar{F}(x)w^{'}(x)\,\mathrm{d}x.$$

For t_1 and t_2, $X_w^{t_1} \leq_{ttt} X_w^{t_2}$ if and only if

$$\int_{B(t_1)}^{w^{-1}(t_1)} \frac{F(w^{-1}(t_1-x))}{F(w^{-1}(t_1))} w^{'}(x)\,\mathrm{d}x \leq \int_{B(t_2)}^{w^{-1}(t_2)} \frac{F(w^{-1}(t_2-x))}{F(w^{-1}(t_2))} w^{'}(x)\,\mathrm{d}x,$$

where $B(x) = F^{-1}((1-p)F(w^{-1}(x)))$.

For any random variable X, the variable $X_{t_1,t_2} = [X|t_1 < X < t_2]$ is named double truncate variable. By performing simple calculations similar to the previous cases, we can write $X^w_{t_1,t_2} \leq_{ttt} X^w$ if and only if

$$\int_0^{C(t_1,t_2)} \frac{F(w^{-1}(t_2)) - F(x)}{F(w^{-1}(t_2)) - F(w^{-1}(t_1))} w^{'}(x)\,\mathrm{d}x \leq \int_0^{F^{-1}(p)} \bar{F}(x) w^{'}(x)\,\mathrm{d}x,$$

and for t_1, t_2 and $t_1^{'}, t_2^{'}$, $X_{t_1,t_2} \leq_{ttt} X_{t_1^{'},t_2^{'}}$ if and only if

$$\int_0^{C(t_1,t_2)} \frac{F(w^{-1}(t_2)) - F(x)}{F(w^{-1}(t_2)) - F(w^{-1}(t_1))} w^{'}(x)\,\mathrm{d}x \leq \int_0^{C(t_1^{'},t_2^{'})} \frac{F(w^{-1}(t_2^{'})) - F(x)}{F(w^{-1}(t_2^{'})) - F(w^{-1}(t_1^{'}))} w^{'}(x)\,\mathrm{d}x,$$

where $C(x,y) = F^{-1}(p(F(w^{-1}(y)) - F(w^{-1}(x))) + F(w^{-1}(x)))$.

4. APPLICATION

Here, we present two different examples aiming to provide excellent illustrations for a better understanding of our applications studies. The Weibull distribution is used to model a wide range of data types. In the following, by introducing some generalized models of Weibull distribution, we show that the new distributions give a better fit to the real data than the basic Weibull distribution, and we also examine these distributions from the point of view of weight variables.

Example 4.1. Let $X_1, X_2, \ldots, X_N$ be the lifetimes of the components of a parallel system, which N is a discrete random variable with the probability mass function

$$P(N = k) = \frac{\theta^{k-1} e^{-\theta}}{(k-1)!}, \quad k = 1, 2, ..., \quad \theta \geq 0.$$

Then $g_N(t) = E(t^N) = t\exp(-\theta(1-t)), 0 \leq t \leq 1$, is a distortion function and

$$F_g(x) = g_N(F(x)) = F(x)\exp(-\theta\bar{F}(x)) \tag{5}$$

is named the distorted distribution function of $F(x)$. It is clear that $F_g(x)$ is the lifetime of the parallel system introduced above. Also it is simply shown that $f_g(x) = w(x)f(x)$ is the density of distorted distribution function, where in $w(x) = (1+\theta F(x))\exp(-\theta\bar{F}(x))$ is an increasing function. Therefore if X_w has a distribution of $F_g(x)$, then X_w is a weighted random variable with density function $f_g(x)$. For this weight function, we have

- $T_w(p) = (1 + \theta p(1-p))\exp(-\theta(1-p)) - e^{-\theta}$,
- $EW_w(p) = 1 - (1 + \theta p(1-p))\exp(-\theta(1-p))$,
- $LIR_w(p) = \theta p^2 \exp(-\theta(1-p))$.

If the base Weibull distribution is used in the distorted distribution function (5), then the newly obtained distribution function is called the distorted Weibull distribution and is denoted by $G_w(\alpha, \beta, \theta)$, that is,

$$G_w^1(x, \alpha, \beta, \theta) = (1 - \exp(-(\frac{x}{\beta})^\alpha)) \exp(-\theta \exp(-(\frac{x}{\beta})^\alpha)). \tag{6}$$

The newly introduced distribution can be a good supersede for the well-known Weibull distribution. For more illustrations, we provide an applicable analysis for the wind speed data collected in the Sweden country utilized in Abd-Elfattah [1], and the corresponding fitted criteria are given by distorted Weibull distribution. Fit test criteria shown in Tables 5 and 6 show the maximum likelihood estimator of unknown parameters. Also, the *TTT* plot of wind speed data and its weighted values based on the desired weight are shown in Figure 7. The results not only show the good performances of the new distribution but also provide a concave curve that is a suitable example of increasing failure rate. See [4] for more details.

Model	AIC	CAIC	BIC	HQIC	KS
Weibull	103.0283	103.4727	105.8307	103.9248	0.1723
G_w^1	99.5867	100.5098	103.7903	100.9315	0.0884

Tab. 5. The goodness-of-fit measures for wind speed data.

Model	α	β	θ
Weibull	$\hat{\alpha} = 3.1408$	$\hat{\beta} = 4.2363$	$---$
G_w^1	$\hat{\alpha} = 0.7054$	$\hat{\beta} = 0.3476$	$\hat{\theta} = 115.1782$

Tab. 6. MLE values of the parameters.

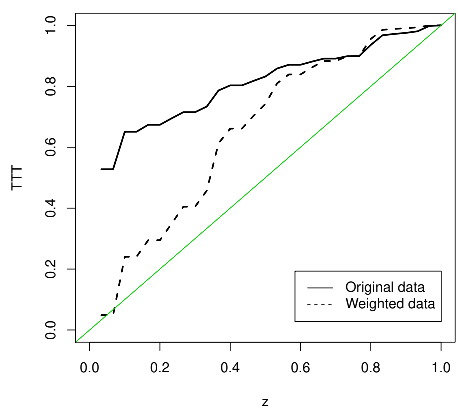

Fig. 7. *TTT* plot of Wind speed data and their weighted.

Example 4.2. Let $X_1, X_2, \ldots, X_N$ be the lifetimes of the components of a parallel system, in which N is a discrete random variable with the probability mass function,

$$P(N=k) = \begin{cases} (1+a)e^{-a}, & k=1, \\ \frac{a^k e^{-a}}{k!}, & k=2,3,\ldots \qquad a>0, \end{cases} \tag{7}$$

and X_i are independent and identically distributed random variables with distribution $F(x)$ that are independent from N. It can be simply said as in the previous cases that $Y = \max\{X_1, \ldots, X_N\}$ is the lifetime of this system, which has a distribution function of the form

$$F_Y(x) = e^{-a\bar{F}(x)} - e^{-a}\bar{F}(x) = g_N(F(x)), \tag{8}$$

where $g_N(t)$ is the probability generating function of N. Therefore $F_Y(x)$ is the same as the distorted distribution function of $F(x)$. Model (8) is a member of the distribution family introduced by Esfahani et al. [15]. They introduced a new distorted family of distributions using the distortion function. As mentioned it is clear that F is a weighted distribution with weight (4).

The following is a case study of the breaking stress of carbon fibers (GPa) in Nichols and Padgett [37]. The samples are 100 experimental data of the breaking stress of carbon fibers. It is simply shown that the Gompertz distribution with parameters shape 0.077 and the scale 0.791 is an appropriate distribution for this data. Also, by considering the Gompertz distribution as the base distribution in Equation (8), we will have a new distribution that can easily show the suitability of this distribution for the data under study, which is in the form below:

$$G_w^2(x,\alpha,\beta,a) = \exp\{-ae^{-\frac{\alpha}{\beta}(e^{\beta x}-1)}\} - e^{-a-\frac{\alpha}{\beta}(e^{\beta x}-1)}, \qquad \alpha,\beta,a>0. \tag{9}$$

The parameter $a = 6.6242$ is obtained in the new distribution. If the TTT plot examines the original data as well as the weighted data with the weight function (4), then it is found that the weight function studied improves the data failure rate and reduces it. Figure 8 shows the TTT plot of the fiber strength data and their weighted values. According to this figure, the original data has an increasing failure rate and the weighted data has a decreasing failure rate. The goodness-of-fit measures for breaking stress of carbon fibers data are shown in Tables 7 and 8.

Model	AIC	CAIC	BIC	HQIC	KS
Gompertz	302.25	302.37	307.46	304.56	0.0962
G_w^2	289.01	289.26	296.82	292.17	0.0684

Tab. 7. The goodness-of-fit measures for fiber strength data.

Model	α	β	θ
Gompertz	$\hat{\alpha} = 0.0769$	$\hat{\beta} = 0.7911$	$---$
G_w^2	$\hat{\alpha} = 0.6596$	$\hat{\beta} = 0.2184$	$\hat{a} = 6.6242$

Tab. 8. MLE values of the parameters.

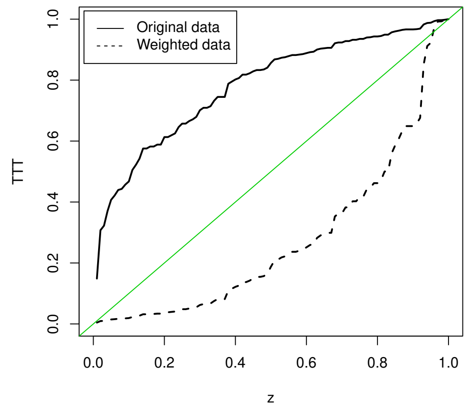

Fig. 8. TTT plot for 100 experimental data of the breaking stress of carbon fibers (GPa) and their weighted data.

Remark 4.3. Model (8) in the analysis of income data also leads to the production of more suitable distributions. For example, we study the Texas city income data presented in Arnold [2]. By choosing Fisk and Burr XII as the base distribution in model (8), we conclude that the distortion (weighted) models of these distributions lead to better results. A summary of the obtained results is presented in Table 9.

Model	AIC	CAIC	BIC	HQIC	KS
$Fisk$	1777.83	1777.91	1783.94	1780.31	0.1424
$BurrXII$	2083.58	2083.66	2089.69	2086.63	0.5119
$DistortionFisk$	1733.12	1733.28	1742.29	1736.84	0.0934
$DistortionBurrXII$	1733.01	1733.18	1742.18	1736.74	0.0933

Tab. 9. Model evaluation criteria.

The presented examples show that the extended distributions have a better fit to the applied data compared to the basic usage distribution in each case. Also, the increase in flexibility of the new distribution compared to the basic distributions can be seen in all three examples.

5. CONCLUSIONS

The main useful distorted form of random variables called weighted random variables have been investigated under TTT transform. To this end, many situations like decreasing and increasing failure rates for these variables were considered, and new transforms were widely discussed. In the following, not only the corresponding relationships were provided, but also we presented many relating conditions organizing the TTT transforms ordering for these variables, in addition to other types of truncated weighted ones that are obtained in a detailed form. As our comparison results, some figures were also given showing the behavior of TTT transform for such variables. Finally, three real data sets were analyzed, and the applications of TTT transform for these considered variables were provided including investigating weighted models, truncated forms, and different kinds of failure rates. Increasing the flexibility and improving the fit of the extended distributions compared to the base distribution used in each example showed the superiority of the introduced weighted models compared to the base model used.

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