# DELAY-DEPENDENT STABILITY OF HIGH-ORDER NEUTRAL SYSTEMS

Yanbin Zhao and Guang-Da Hu

In this note, we are concerned with delay-dependent stability of high-order delay systems of neutral type. A bound of unstable eigenvalues of the systems is derived by the spectral radius of a nonnegative matrix. The nonnegative matrix is related to the coefficient matrices. A stability criterion is presented which is a necessary and sufficient condition for the delay-dependent stability of the systems. Based on the criterion, a numerical algorithm is provided which avoids the computation of the coefficients of the characteristic function. Under some conditions, the presented results are less conservative than those reported. A numerical example is given to illustrate the main results.

*Keywords:* delay-dependent stability, high-order neutral delay systems, bound of unstable eigenvalues, argument principle, nonnegative matrix

*Classification:* 15A18, 34K06, 34K20

## 1. INTRODUCTION

Time-delay is a common phenomenon in real control systems. The time delay can greatly deteriorate the performance of the systems, and even drive the systems to be unstable. Thus it is necessary to analyze the effects of time-delays on dynamic systems so as to solve practical problems and avoid their adverse consequences. In [3, 4, 5, 6, 8, 9, 10, 11, 17], determinate delay systems are investigated. Stochastic delay systems have been discussed in [1, 14, 15, 16, 18, 19].

In this paper, we are concerned with the high-order delay differential system of neutral type is given by

$$x^{(n)}(t) + \sum_{l=1}^{n} \left[ A_l x^{(n-l)}(t) + \sum_{j=1}^{m} B_{lj} x^{(n-l)}(t-\tau_j) \right] + \sum_{j=1}^{m} C_j x^{(n)}(t-\tau_j) = 0, \qquad (1)$$

where, matrices $A_l, B_{lj}, C_j \in \mathbb{R}^{d\times d}$, for $l = 1, \ldots, n$ and $j = 1, \ldots, m$, $\tau_j > 0$, and the indices on $x$ denote derivatives with respect to the independent variable $t$. The neutral

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terms satisfy the condition

$$\rho(F) < 1, \quad \text{and} \quad F = \sum_{j=1}^{m} |C_j|. \tag{2}$$

Throughout the present note, we assume that condition (2) holds.

The high-order delay differential system appear in the bilateral control of tele-operation systems [6] and the active control of the dynamics of vibrating structures [17]. For system (1), when $A_l, B_{lj}, C_j$ are scalars, the stability of the high-order scalar neutral delay equations is discussed in [9]. For the case of $n = 1$, i. e., delay-dependent stability of the first-order neutral delay systems have been reported in the literature (e. g. [3, 5, 9]). The present note is a generalization, in one way or another, of the high-order scalar neutral delay equations and of the first-order neutral delay systems.

A scalar delay-independent stability test (e. g. [9]) need information of all the coefficients of the characteristic function. However, when $d$ or $n$ or $m$ are large, it is difficult to obtain the coefficients of the characteristic function, even if it is a polynomial. In [10], the delay-dependent stability is investigated for the second-order scalar neutral delay differential equations. It is difficult to extend the technique in [10] to the case of system (1). Recently a direct stability test has been reported in [4] for system (1) with the following condition

$$\sum_{j=1}^{m} ||C_j|| < 1. \tag{3}$$

The stability test in [4] does not involve the computation of the coefficients of the characteristic function. This note is a continuation of [4]. We emphasize that all the computations in this note involve only the matrices of size $d \times d$.

The main contributions of this note are summarised as follows.

1. A stability criterion is presented which is a necessary and sufficient condition for the delay-dependent stability of system (1) with (2).
2. Based on the criterion, a numerical algorithm is provided.

Throughout this note, the $j^{\text{th}}$ eigenvalue of $W$ is denoted by $\lambda_j(W)$, $\rho(W)$ represents the spectral radius. Let $W \in \mathbb{C}^{n\times n}$ with elements $w_{jk}$ and $|W|$ denotes the nonnegative matrix in $\mathbb{R}^{n\times n}$ with elements $|w_{jk}|$. Let $W = \{w_{jk}\}$ and $V = \{v_{jk}\}$ be matrices in $\mathbb{R}^{n\times n}$. We write $W \geq V$ if and only if $w_{jk} \geq v_{jk}$.

## 2. PRELIMINARIES

In this section, several definitions and lemmas are provided.

**Lemma 2.1.** (Lancaster [12]) Let $W \in \mathcal{C}^{d\times d}$. If $W < 1$, then $(I - W)^{-1}$ exits and

$$(I - W)^{-1} = I + W + W^2 + \cdots .$$

**Lemma 2.2.** (e. g. Lancaster [12]) For $W \in \mathbb{C}^{n\times n}$, $\rho(W) \leq \rho(|W|)$ holds.

The characteristic function of the neutral delay system (1) is as follows.

$$g(s) = \det P(s), \tag{4}$$

where $P(s)$ is defined by the following

$$P(s) = Is^n + \sum_{l=1}^{n} A_l s^{n-l} + \sum_{l=1}^{n}\sum_{j=1}^{m} B_{l,j} s^{n-l}\exp(-\tau_j s) + \sum_{j=1}^{m} C_j s^n \exp(-\tau_j s). \tag{5}$$

The root of the characteristic function $g(s)$ is called an eigenvalue of the neutral delay system (1). For $g(\xi) = 0$, if $\Re\xi < 0$, then the eigenvalue $\xi$ is called stable, otherwise, if $\Re\xi \geq 0$, then the eigenvalue $\xi$ is called unstable.

Now we consider the following first-order neutral delay differential system [4].

$$\dot{Y}(t) = \mathcal{A}Y(t) + \sum_{j=1}^{m} \mathcal{B}_j Y(t-\tau_j) + \sum_{j=1}^{m} \mathcal{C}_j \dot{Y}(t-\tau_j), \tag{6}$$

where $Y(t) \in \mathbb{R}^{dn}$,

$$\mathcal{A} = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & & \vdots \\ \vdots & \vdots & \vdots & & I \\ -A_n & -A_{n-1} & -A_{n-2} & \dots & -A_1 \end{bmatrix},$$

$$\mathcal{B}_j = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ -B_{n,j} & -B_{n-1,j} & -B_{n-2,j} & \dots & -B_{1,j} \end{bmatrix}$$

and

$$\mathcal{C}_j = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & -C_j \end{bmatrix}.$$

We have

$$\det\{sI - [\mathcal{A} + \sum_{j=1}^{m} \mathcal{B}_j \exp(-\tau_j s) + \sum_{j=1}^{m} \mathcal{C}_j s \exp(-\tau_j s)]\} = \det P(s), \tag{7}$$

see [4].

A definition of stability of the high-order neutral delay system (1) is as follows.

**Definition 2.3.** (Hu [4]) The neutral delay system (1) is called asymptotically stable if the first-order neutral delay system (6) is asymptotically stable.

It is obvious for us to obtain the following result.

**Lemma 2.4.** The neutral delay system (1) with (2) is asymptotically stable if and only if all the eigenvalues of it lie in the open left complex half-plane.

Proof. According to Definition 2.3, the neutral delay system (1) is asymptotically stable $\Leftrightarrow$ the first-order neutral delay system (6) is asymptotically stable. According to the definition of $\mathcal{C}_j$, we obtain

$$\rho(\sum_{j=1}^{m} \mathcal{C}_j \exp(i\omega_j)) = \rho\left(\sum_{j=1}^{m} -C_j \exp(i\omega_j)\right) \leq \rho(F) < 1, \tag{8}$$

where $F = \sum_{j=1}^{m} |C_j|$, $i^2 = -1$ and $\omega_j \in [0, 2\pi]$ for $j = 1, 2, \ldots, m$. This means that the difference operator in the first-order neutral delay system (6) is strongly stable under the condition (2)[3]. The first-order neutral delay system (6) with (2) is asymptotically stable if and only if all the roots of the characteristic function

$$\det\left\{ sI - [\mathcal{A} + \sum_{j=1}^{m} \mathcal{B}_j \exp(-\tau_j s) + \sum_{j=1}^{m} \mathcal{C}_j s \exp(-\tau_j s)] \right\} = 0$$

lie in the open left complex half-plane (e.g. [3]). According to (7), this means that all the roots of $\det P(s) = 0$ lie in the open left complex half-plane. The proof is completed. □

**Remark 2.5.** Introducing new state variables, we can rewrite the $n-$order neutral delay system (1) with parameter matrices of size $d \times d$ in the first-order neutral delay system (6) with parameter matrices of size $nd \times nd$. In the theoretical sense, the stability criteria for the first-order neutral delay system (6) can be directly applied to the $n-$order neutral delay system (1). However, since the parameter matrices of the first-order neutral delay system (6) are of size $nd \times nd$, much computational effort is needed to directly apply to the large problems. We emphasize that all the computations in this note involve only the matrices of size $d \times d$.

## 3. A BOUND FOR UNSTABLE EIGENVALUES

By means of Lemma 2.1, we obtain the bound for unstable eigenvalues of the neutral delay system (1) with (2) as follows.

**Theorem 3.1.** Every unstable eigenvalue $\xi$ of the neutral delay system (1) with (2) satisfies

$$|\xi| \leq \max\{1, h\}, \tag{9}$$

where the scalar $h$ is defined by

$$h = \rho(H), \quad \text{and} \quad H = [I - F]^{-1} \left[ \sum_{l=1}^{n} |A_l| + \sum_{l=1}^{n} \sum_{j=1}^{m} |B_{lj}| \right]. \tag{10}$$

P r o o f. Let $\xi$ be an eigenvalue of system (1), i. e., $g(\xi) = 0$. Since $\xi$ is an unstable root, $\Re\xi \geq 0$.

First, we consider the case of $|\xi| \geq 1$. According to (4), we have that

$$g(\xi) = \det P(\xi) = 0,$$

which implies that

$$\det \left[ I\xi^n + \sum_{l=1}^{n} A_l \xi^{n-l} + \sum_{l=1}^{n}\sum_{j=1}^{m} B_{lj}\xi^{n-l}\exp(-\tau_j\xi) + \sum_{j=1}^{m} C_j \xi^n \exp(-\tau_j\xi) \right] = 0. \quad (11)$$

Since $\Re\xi \geq 0$,

$$|\exp(-\tau_j\xi)| \leq 1 \quad (12)$$

holds. From (12), we obtain that

$$\left| \sum_{j=1}^{m} -C_j \exp(-\tau_j\xi) \right| \leq \sum_{j=1}^{m} |C_j| = F. \quad (13)$$

According to condition (2), we know that [12]

$$\left( I - \sum_{j=1}^{m} -C_j \exp(-\tau_j\xi) \right)^{-1} \quad \text{exists.}$$

$$\begin{aligned} &\left| \left( I + \sum_{j=1}^{m} C_j \exp(-\tau_j\xi) \right)^{-1} \right| \\ &= \left| I - \sum_{j=1}^{m} -C_j \exp(-\tau_j\xi) + (\sum_{j=1}^{m} -C_j \exp(-\tau_j\xi))^2 + \ldots \right| \\ &\leq I + F + F^2 + \ldots = (I - F)^{-1}. \end{aligned} \quad (14)$$

For $\Re\xi \geq 0$, we introduce the matrix

$$W(\xi) = -\left( I + \sum_{j=1}^{m} C_j \exp(-\tau_j\xi) \right)^{-1} \left( \sum_{l=1}^{n} A_l \xi^{1-l} + \sum_{l=1}^{n}\sum_{j=1}^{m} B_{lj}\xi^{1-l}\exp(-\tau_j\xi) \right). \quad (15)$$

Since $|\xi| \geq 1$, we can rewrite (11) as

$$\det P(\xi) = \det \left[ \xi^{n-1} \left( I + \sum_{j=1}^{m} C_j \exp(-\tau_j\xi) \right) \right] \det[I\xi - W(\xi)] = 0$$

which means

$$\det [\xi I - W(\xi)] = 0. \quad (16)$$

This implies that $\xi$ is an eigenvalue of the matrix $W(\xi)$ and there exists an integer $j (1 \le j \le d)$ such that

$$\xi = \lambda_j(W(\xi)). \tag{17}$$

According to $|\xi| \ge 1$, for $k \ge 0$,

$$|\xi^{-k}| \le 1 \tag{18}$$

holds. By means of (17) and Lemma 2.1, we have that

$$|\xi| = |\lambda_j(W(\xi))| \le \rho(|W(\xi)|). \tag{19}$$

According to (15), (12), (14) and (18), we have

$$\begin{aligned}
|W(\xi)| &= \left| -(I + \sum_{j=1}^{m} C_j \exp(-\tau_j \xi))^{-1} \left( \sum_{l=1}^{n} A_l \xi^{1-l} + \sum_{l=1}^{n}\sum_{j=1}^{m} B_{lj} \xi^{1-l} \exp(-\tau_j \xi) \right) \right| \\
&\le \left\{ \left| (I + \sum_{j=1}^{m} C_j \exp(-\tau_j \xi))^{-1} \right| \right\} \left\{ \left| \sum_{l=1}^{n} A_l \xi^{1-l} \right| + \left| \sum_{l=1}^{n}\sum_{j=1}^{m} B_{lj} \xi^{1-l} \exp(-\tau_j \xi) \right| \right\} \\
&\le (I - F)^{-1} \left\{ \sum_{l=1}^{n} |A_l| + \sum_{l=1}^{n}\sum_{j=1}^{m} |B_{lj}| \right\} \\
&= H.
\end{aligned}$$

By means of (19), we have that for $|\xi| \ge 1$ and $\Re\xi \ge 0$, $|\xi| \le h$ holds. Thus for any eigenvalue $\xi$ with $\Re\xi \ge 0$,

$$|\xi| \le \max\{1, h\}$$

holds. Thus the proof is completed. □

## 4. STABILITY CRITERION

We now investigate the delay-dependent stability of the neutral delay system (1) with (2).

**Definition 4.1.** For the neutral delay system (1) with (2), the region $D$ is defined by

$$D = \{s : \Re s \ge 0 \quad \text{and} \quad |s| \le \beta\},$$

and its boundary is denoted by $C$. Here $\beta$ is given by Theorem 3.1, i. e.

$$\beta = \max\{1, h\}, \tag{20}$$

where

$$h = \rho(H), \quad \text{and} \quad H = [I - F]^{-1} \left[ \sum_{l=1}^{n} |A_l| + \sum_{l=1}^{n}\sum_{j=1}^{m} |B_{lj}| \right]. \tag{21}$$

From the definition of $D$, it is obvious that $D \subset \mathbb{C}^+$. Using the argument principle, the following two theorems can be derived in the same way as those in [4].

**Theorem 4.2.** The neutral delay system (1) with (2) is asymptotically stable if and only if

$$g(s) \neq 0 \quad \text{for} \quad s \in C \tag{22}$$

and

$$\triangle_C \arg g(s) = 0 \tag{23}$$

hold. Here $g(s) = \det P(s)$, $\arg g(s)$ stands for the argument of $g(s)$ and $\triangle_C \arg g(s)$ change of the argument of $g(s)$ along the curve $C$.

**Theorem 4.3.** If

$$g(s) \neq 0 \quad \text{for} \quad s \in C \tag{24}$$

and

$$\frac{1}{2\pi} \triangle_C \arg g(s) = z \tag{25}$$

hold, then the number of the unstable eigenvalues of the neutral delay system (1) with (2) is $z$. Here, $g(s) = \det P(s)$

Now we describe an algorithm to check the delay-dependent stability of the neutral delay system (1) with (2) due to Theorem 4.2.

**Algorithm 1**

**Step 0.** We calculate $\beta$ according to (20). Then as the boundary of $D$, we have the closed contour $C$. The closed contour $C$ consists of two parts, i. e., the segment $\{s = it;\ -\beta \leq t \leq \beta\}$ and the half-circle $\{s;\ |s| = \beta \text{ and } -\pi/2 \leq \arg s \leq \pi/2\}$. Notice that the closed contour $C$ is positively oriented when it is in the counterclockwise direction.

**Step 1.** Take a sufficiently large integer $N \in \mathbb{N}$ and distribute $N$ node points $\{s_j\}$ ($j = 1, 2, \ldots, N$) on $C$ as uniformly as possible. For each $s_j$, we evaluate $g(s_j)$ by computing the determinant as

$$g(s_j) = \det P(s_j).$$

Also we decompose $g(s_j)$ into its real and imaginary parts for the computation of the argument.

**Step 2.** We examine whether $g(s_j) = 0$ holds for each $s_j$ ($j = 1, \ldots, N$) by checking its magnitude satisfies $|g(s_j)| \leq \delta_1$ with the preassigned tolerance $\delta_1$. If it holds, i. e., $s_j \in C$ is a root of $g(s)$, then the neutral delay system (1) with (2) is not asymptotically stable and stop the algorithm. Otherwise, to go to the next step.

**Step 3.** We examine whether $\triangle_C \arg g(s) = 0$ holds along the sequence $\{g(s_j)\}$ by checking $|\triangle_C \arg g(s)| \leq \delta_2$ with the preassigned tolerance $\delta_2$. If it holds, this means that the change of the argument is 0 along $C$, then the neutral delay system (1) with (2) is asymptotically stable, otherwise not asymptotically stable.

**Remark 4.4.** Algorithm 1 avoids the computation of the coefficients of the characteristic function $g(s) = \det P(s)$. Instead it evaluates the determinant of numerical matrix $P(s_j)$ through the elementary row (or column) operations which are relatively efficient ways (e. g. [7]). A scalar stability test (e. g. [9]) needs information of all the coefficients of the characteristic function $g(s)$. It is an ill-posed problem to compute the coefficients of the characteristic function $g(s)$ for large problems (when $d$ or $n$ or $m$ are large), even if $g(s)$ is a polynomial (e. g. [13]). Although we may obtain the coefficients of the characteristic function $g(s)$ from $P(s)$ in theoretical sense, it can not work well in practice for large problems (when $d$ or $n$ or $m$ are large).

**Remark 4.5.** If there are $z_0$ eigenvalues on the boundary $C$, we can construct a modified curve [2] which replaces the boundary $C$. Theorem 4.3 can be extended and the number of the unstable eigenvalues are $z_0 + z$. Only modifying Step 3 in Algorithm 1, we may obtain a numerical algorithm to check Theorem 4.3.

We discuss the difference between Theorems 3.1, 4.2, 4.3 and those in [4]. In the results in [4] require condition (3) holds. However, condition (2) is demanded in Theorems 3.1, 4.2 and 4.3. In general, the two conditions complement each other. Now we provide sufficient conditions that (2) is less conservative than (3).

**Theorem 4.6.** Let $F = \sum_{j=1}^{m} |C_j|$, we have

$$\rho(F) \leq \sum_{j=1}^{m} ||C_j|| \tag{26}$$

if one of the following two conditions holds:

**(i)** $\quad C_j \geq 0 \quad \text{for} \quad j = 1, \ldots, m.$

**(ii)** $\quad C_j \leq 0 \quad \text{for} \quad j = 1, \ldots, m.$

Proof. When $C_j \geq 0$, we have $|C_j| = C_j$ for $j = 1, \ldots, m$. We obtain

$$\sum_{j=1}^{m} |C_j| = \sum_{j=1}^{m} C_j. \tag{27}$$

According to (27),

$$\rho(F) = \rho\left[\sum_{j=1}^{m} C_j\right] \leq \left\|\sum_{j=1}^{m} C_j\right\| \leq \sum_{j=1}^{m} \|C_j\|. \tag{28}$$

Similar to the above. When $C_j \leq 0$, $|C_j| = -C_j$ for $j = 1, \ldots, m$. We have

$$\sum_{j=1}^{m} |C_j| = -\sum_{j=1}^{m} C_j. \tag{29}$$

According to (29),

$$\rho(F) = \rho\left[\sum_{j=1}^{m} -C_j\right] \le \left\|\sum_{j=1}^{m} -C_j\right\| \le \sum_{j=1}^{m} \|-C_j\| = \sum_{j=1}^{m} \|C_j\|. \tag{30}$$

By means of (28) and (30), the proof is completed. □

**Remark 4.7.** Theorem 4.6 shows that under some conditions, the presented results, Theorems 4.2 and 4.3 are less conservative than those reported [4].

## 5. NUMERICAL EXAMPLE

Consider system (1) with the parameter matrices

$$A_1 = \begin{bmatrix} 5 & -3 & -5 \\ 0 & 4 & -1 \\ 0 & 0 & 8 \end{bmatrix}, \ A_2 = \begin{bmatrix} 4 & -12 & -20 \\ 0 & 4 & -2 \\ 0 & 0 & 15 \end{bmatrix}, \ B_{11} = \begin{bmatrix} 0.9 & 0.1 & -0.4 \\ 1 & 0.8 & -0.9 \\ -1 & 0 & -1.6 \end{bmatrix},$$

$$B_{12} = \begin{bmatrix} -1.1 & -0.7 & 0.5 \\ -0.2 & -1.6 & 0.1 \\ -1.1 & 1.3 & -0.8 \end{bmatrix}, \ B_{13} = \begin{bmatrix} -0.1 & 0.3 & 1.2 \\ -1 & -0.6 & 1.3 \\ -0.1 & 1.3 & -1.8 \end{bmatrix},$$

$$B_{21} = \begin{bmatrix} 1 & -3 & -8 \\ 1 & -1 & -9 \\ -1 & 3 & 2 \end{bmatrix}, \ B_{22} = \begin{bmatrix} -1 & 1 & 6 \\ -0 & -4 & -1 \\ -1 & 5 & 1 \end{bmatrix}, \ B_{23} = \begin{bmatrix} 0 & 0 & 3 \\ -1 & 3 & 8 \\ 0 & 2 & -7 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} -0.1 & -0.1 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.2 \end{bmatrix}, \ C_2 = \begin{bmatrix} -0.1 & 0 & 0 \\ -0.2 & -0.1 & 0 \\ -0.1 & 0 & -0.1 \end{bmatrix}, \ C_3 = \begin{bmatrix} -0.6 & 0 & -0.1 \\ 0 & -0.3 & 0 \\ -0.1 & 0 & -0.1 \end{bmatrix}.$$

It is not difficult to check that $\rho(F) = 0.8917 < 1$, the condition (2) holds. By direct calculation, we have $h = 413.9733$ and the radius of the semicircle $\beta = 413.9733$. The boundary of $C$ is shown in Figure 1. In Figs. 2 – 4, the horizontal axis $(j)$ and the vertical axis (Argument g(s)) denote the node points $s_j$ on the closed contour C and the argument $g(s_j)$ with $s_j \in C$, respectively.

The case of $\tau_1 = 0.7$, $\tau_2 = 0.8$, $\tau_3 = 0.9$. We can analyze the stability of the system by Theorem 4.2. By means of **Algorithm 1,** we know that $g(s) \neq 0$ for $s \in C$, the argument of $g(s)$ along the curve $C$ are shown in Figure 2, and $\triangle_C \arg g(s) = 0$ along the curve $C$, Theorem 4.2 tells that the system with the given parameter matrices is asymptotically stable.

The case of $\tau_1 = 0.9$, $\tau_2 = 2$, $\tau_3 = 3$. We can analyze the stability of the system by Theorem 4.3. By means of **Algorithm 1,** we know that $g(s) \neq 0$ for $s \in C$, the argument of $g(s)$ along the curve $C$ are shown in Figure 3, and $\triangle_C \arg g(s) = 2 \neq 0$ along the curve $C$. Theorem 4.3 shows that the number of the unstable eigenvalues of the neutral delay system is 2.

The case of $\tau_1 = 5$, $\tau_2 = 10$, $\tau_3 = 20$. We can analyze the stability of the system by Theorem 4.3. By means of **Algorithm 1,** we know that $g(s) \neq 0$ for $s \in C$, the argument of $g(s)$ along the curve $C$ are shown in Figure 4, and $\triangle_C \arg g(s) = 4 \neq 0$ along the curve $C$. Theorem 4.3 shows that the number of the unstable eigenvalues of the system is 4.

**Remark 5.1.** For the above example, we can check that $\sum_{j=1}^{m} ||C_j|| = 1.0811 > 1$, which means that the condition (3) does not hold, and the stability criterion in [4] can not work, but the stability of the system can be determined by Theorems 4.2 and 4.3 in this note. The above example also shows that system (1) is stable for $\tau_1 = 0.7$, $\tau_2 = 0.8$ , $\tau 3 = 0.9$ but unstable for $\tau_1 = 0.9$, $\tau_2 = 2$, $\tau_3 = 3$, and $\tau_1 = 5$, $\tau_2 = 10$, $\tau_3 = 20$, respectively. This means that the stability criterion, Theorem 4.2 is delay-dependent.

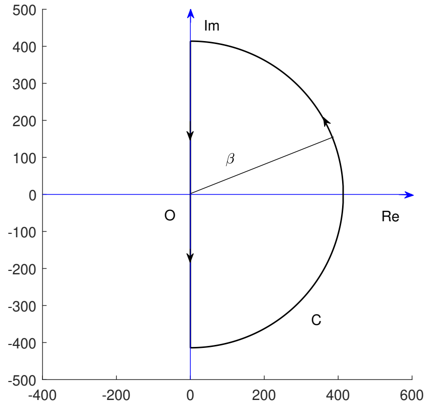


**Fig. 1.** The closed contour C.

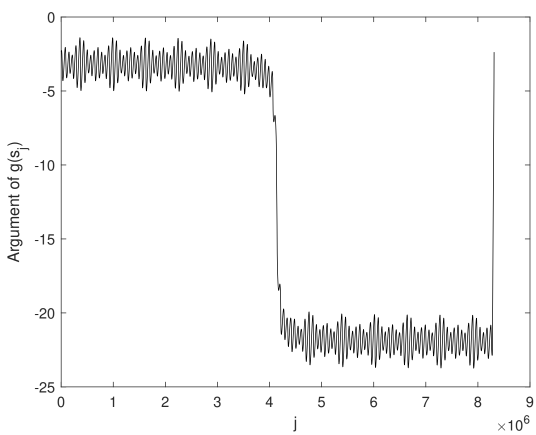


**Fig. 2.** Argument change of $g(s) = 0$ when $\tau_1 = 0.7, \tau_2 = 0.8, \tau_3 = 0.9$.

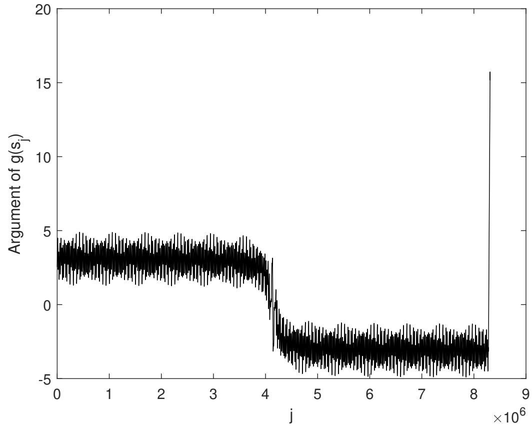


**Fig. 3.** Argument change of $g(s) = 2 \times 2\pi$ when $\tau_1 = 0.9, \tau_2 = 2, \tau_3 = 3$.

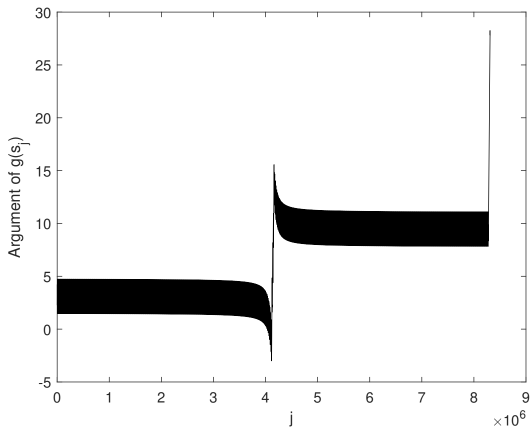


**Fig. 4.** Argument change of $g(s) = 4 \times 2\pi$ when $\tau_1 = 5, \tau_2 = 10, \tau_3 = 20$.

## 6. CONCLUSIONS

By means of the spectral radius of a nonnegative matrix, a bound is derived for the unstable eigenvalues of the high-order neutral delay systems. Based on the bound, a computable stability criterion is presented. The criterion is a necessary and sufficient condition for the delay-dependent stability of the systems. Under some conditions, the presented results are less conservative than those reported.

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