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THE CONVERGENCE OF THE CORE OF A FUZZY EXCHANGE ECONOMY

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This paper focuses on a new model called fuzzy exchange economy (FXE), which integrates fuzzy consumption, fuzzy initial endowment and the agent's fuzzy preference (vague attitude) in the fuzzy consumption set. Also, the existence of the fuzzy competitive equilibrium for the FXE is verified through a related pure exchange economy. We define a core-like concept (called weak fuzzy core) of the FXE and prove that any fuzzy competitive allocation belongs to the weak fuzzy core. The fuzzy replica economy, which is the r-fold repetition of the FXE, is considered. Finally, we show that the weak fuzzy core of the r-fold fuzzy replica economy, i. e., the set of all fuzzy allocations which cannot be blocked by any coalition of agents, converges to the set of fuzzy competitive allocations of the FXE as r becomes large.

Keywords: pure exchange economy, fuzzy competitive equilibrium, fuzzy replica economy, weak fuzzy core, fuzzy Edgeworth equilibrium

Classification: 91A12, 91B08, 91B50

1. INTRODUCTION

A pure exchange economy is a market model with l different goods, m agents and without production. Each agent has an initial endowment of goods for trading and a definite order of preference on the set of all commodity bundles. The object is to determine a price and a consumption equilibrium which maximizes the preferences of agents described by real utility functions under their budget constraint. The concept of competitive equilibrium is a state of the market meeting "the law of supply and demand". It consists of a price structure where the total supply of each good balances the total demand, and an allocation that results from trading at these prices. The full recognition of the theory of competitive equilibrium is attributed to Walras [24]. He established a system of simultaneous equations that described an economy and derived the solutions to this system at equilibrium prices and quantities of commodities. However, the first rigorous result on the existence of equilibrium owes to Wald [23]. With these foundations, and the influence of the rapid development of linear programming, nonlinear analysis, and game theory, some discoveries about the existence of equilibrium were made by other researchers—such as Gale [12], Nikaido [18] and McKenzie [14]. In especial, Arrow and Debreu [1] applied the fixed point theory to equilibrium problems, generalizing Nash's

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theorem on the existence of equilibrium points for non-cooperative games [17], and then built the existence of an equilibrium in an abstract economy which is a variation on the notion of a non-cooperative game. Also, an alternative approach for the study of equilibrium was given by a suitable equivalent variational inequality (see Refs. [3, 4, 8, 9, 16]).

Moreover, some researchers studied the relationship between the core and the set of competitive allocations. For example, Aubin [2] obtained the existence of the coincidence of the fuzzy core and the set of competitive allocations by considering the pure exchange economy with fuzzy coalition of agent. From another aspect, Edgeworth [11] provided a prominent study of the pure exchange economy composed of two goods and two types of agents each of whom has a definite preference and an initial endowment. The exchange results in redistribution of the total amounts of the two goods. It may be characterized geometrically by a point in the Edgeworth box corresponding to that economy. Also, the concept of the contract curve was introduced in [11] to describe the possible allocations which are Pareto optimal and at least as desired by both agents as the allocation prevailing before exchanging. As Edgeworth remarks, a competitive allocation is on the contract curve. To select out the competitive allocations, Edgeworth [11] introduced an expanding economy which consists of $2r$ agents divided into two types. Everyone of the same type has an identical preference and identical initial endowment before trading takes place. The aim is to illustrate that more and more allocations are ruled out, and eventually only the competitive allocations remain as r becomes large. This statement can be explained as the contract curve shrinking to the set of competitive allocations as the number of agents becomes infinite.

Nevertheless, it does not seem to apply to the general case involving more than two goods and more than two types of agents. For this purpose, Debreu and Scarf [6] generalized Edgeworth's thought and took into account a pure exchange economy with an arbitrary number of types of agents and an arbitrary finite number of goods. Based on the game theory including a definition of the core, the r-fold replica economy of the pure exchange economy which is composed of r subeconomies identical to the pure exchange economy, was studied. Also, Debreu and Scarf defined the core of an economy as the set of allocations which cannot be blocked by any coalition of agents. In other words, an allocation is said to be in the core of an economy if no coalition of agents can force an outcome that is better for them than their current allocation. Furthermore, they proved that any competitive allocation belongs to the core of the pure exchange economy, and presented that no other allocation is in the core for all r besides the competitive allocations for the larger economy. These indicate that the competitive allocations are Pareto optimal. Furthermore, Urai and Murakami [22] extended the Debreu-Scarf limit theorem to double infinity monetary economies.

It is worth noting that a model called cooperative game with fuzzy payoffs presented by Mallozzi et al. [15] in cooperative game theory has been developed when the worth of any coalition is given by means of a fuzzy number. Also, Zhang et al. [26] proposed an agent's attitude is vague when facing a variety of alternative consumption vectors, and got that there exists a fuzzy utility function being in accordance with agent's fuzzy preference. In addition, Shapley and Shubik [20] acquired that a market can be used to generate a game by defining the characteristic function of coalition S as maximizing the total utility of coalition S. Motivated by these results, a new model called the fuzzy exchange economy (FXE) is presented in this paper, which integrates with fuzzy consumption, fuzzy initial endowment and fuzzy preference of any agent. Unarguably, the initial endowments of different agents in the pure exchange economy are practical measurements of such imprecision that they are given by exact figures with difficulty. So the initial endowments can be expressed in the form of fuzzy numbers. When we consider an FXE, two aspects should be confirmed. The first aspect necessary to be confirmed is the existence of the fuzzy competitive equilibrium for the FXE. Specifically, the weak fuzzy core of the FXE is the set of all fuzzy allocations which distribute the total fuzzy initial endowments, under the condition that the fuzzy allocations cannot be blocked by any coalition of agents. One immediate consequence of the weak fuzzy core is that a fuzzy allocation in it is Pareto optimal. Hence, the second aspect is to establish the relationship between the weak fuzzy core of the FXE and the fuzzy competitive allocation of that based on a generalized FXE consisting of l goods and m agents.

However, it is not enough to know that there may be some allocations in the weak fuzzy core. One can easily construct examples in game theory in which every indifference fuzzy imputation is blocked by some coalition such that the weak fuzzy core is empty. Meanwhile, the FXE with an empty weak fuzzy core may also be found in case the usual assumptions on fuzzy preferences are relaxed. Therefore, addressing these necessary aspects are as follows.

Firstly, we define an expected function of a fuzzy mapping which maps a fuzzy consumption set to the set of all fuzzy numbers to prove the existence of the fuzzy competitive equilibrium. Thus, the fuzzy utility function about l-dimensional fuzzy consumption vector can be described as a crisp utility function about $4l$-dimensional crisp consumption vector. The fuzzy competitive equilibrium consists of a fuzzy competitive price structure where the total fuzzy supply of each good exactly balances the total fuzzy demand, and a fuzzy competitive allocation that results from trading at these prices.

Secondly, we make some assumptions, in which case it can be shown that the weak fuzzy core is not empty. One way to do this is to illustrate that any fuzzy competitive allocation belongs to the weak fuzzy core. Moreover, we follow the procedure used in [6] for enlarging the market and imagine the fuzzy economy to be composed of m types of agents and r agents of any type. For any two agents with the same type, we require them to have precisely the same fuzzy preferences and the same fuzzy initial endowments. Hence, this forms r kinds of new fuzzy economies, each of which is called the r-fold fuzzy replica economy. We verify that if a fuzzy allocation in the weak fuzzy core of the r-fold fuzzy replica economy, all the agents with the same type are assigned the same consumption. A fuzzy Edgeworth equilibrium of the FXE is a fuzzy allocation such that the r-fold repetition of the fuzzy allocation belongs to the weak fuzzy core of the r-fold fuzzy replica economy for any r. Tracing back to the fuzzy allocation of the FXE, it is found that as the number of agents becomes infinite, the weak fuzzy core of the r-fold fuzzy replica economy converges to the set of fuzzy competitive allocations, i. e., a subset of the set of fuzzy Edgeworth equilibria.

The main contribution of this paper is to propose a new model of FXE, and show the relationship between the weak fuzzy core and the fuzzy competitive allocations of FXE. The rest of this paper is presented as follows. In Section 2, we recall some basic concepts of fuzzy numbers, fuzzy mappings, pure exchange economies and fuzzy preferences. Section 3 introduces an FXE and confirms the existence of a fuzzy competitive equilibrium for FXE. The weak fuzzy core of the FXE is defined and next the connection between the fuzzy competitive equilibrium and the weak fuzzy core of FXE is obtained in Section 4. Section 5 illustrates what the weak fuzzy core of the r-fold fuzzy replica economy becomes as the number of agents becomes infinite. The last section concludes this paper.

2. PRELIMINARIES

To define and characterize the fuzzy exchange economy (FXE), we first recall some basic definitions about fuzzy numbers, fuzzy mappings, pure exchange economies and fuzzy preferences [7, 8, 21, 26] in this section.

2.1. Fuzzy numbers

Denote the set of all real numbers by R. A *fuzzy number* we treat in this paper is a fuzzy set $\tilde{A} : R \to [0, 1]$ which is *normal*, *has bounded support*, and is *upper semicontinuous* and *quasiconcave* as a function. That is, $\tilde{A}$ is a mapping with the following properties:

(i) $\tilde{A}$ is upper semi-continuous,

(ii) $\tilde{A}$ is convex fuzzy set, that is, function $\tilde{A}$ is quasiconcave, $\tilde{A}(\lambda x + (1-\lambda)y) \geq \min\{\tilde{A}(x), \tilde{A}(y)\}$ for all $x, y \in R, \lambda \in [0, 1]$,

(iii) $\tilde{A}$ is normal, i. e., $\exists x_0 \in R$ for which $\tilde{A}(x_0) = 1$,

(iv) supp $\tilde{A} = \{x \in R \mid \tilde{A}(x) > 0\}$ is a support of $\tilde{A}$ and its closure $cl(\text{supp } \tilde{A})$ is compact.

Let $\mathcal{F}R$ be the set of all fuzzy numbers in R.

For any $\tilde{A} \in \mathcal{F}R$, there exist $a, b, c, d \in R$, $\mathbb{L} : [a, b] \to [0, 1]$ non-decreasing and $\mathbb{R} : [c, d] \to [0, 1]$ non-increasing such that the membership function $\tilde{A}(x)$ is given as follows:

$$\tilde{A}(x) = \begin{cases} \mathbb{L}(x), & \text{if } a \leq x < b, \\ 1, & \text{if } b \leq x \leq c, \\ \mathbb{R}(x), & \text{if } c < x \leq d, \\ 0, & \text{ otherwise.} \end{cases}$$

A fuzzy number denoted by $\lfloor a, b, c, d \rfloor$ is *trapezoidal* if the functions $\mathbb{L}$ and $\mathbb{R}$ are linear. We denote the set of all trapezoidal fuzzy numbers by $\mathcal{T}R \subseteq \mathcal{F}R$. If $b - a = d - c$, then the trapezoidal fuzzy number is symmetrical.

The *α-level set* of a fuzzy number $\tilde{A} \in \mathcal{F}R, 0 \leq \alpha \leq 1$, denoted by $\tilde{A}[\alpha]$, is defined as

$$\tilde{A}[\alpha] = \begin{cases} \{x \in R | \tilde{A}(x) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1, \\ cl(\text{supp } \tilde{A}), & \text{if } \alpha = 0. \end{cases}$$

It is clear that the α-level set of a fuzzy number is a closed bounded interval $[A_*(\alpha), A^*(\alpha)]$, where $A_*(\alpha)$ and $A^*(\alpha)$ denote the left-hand and right-hand endpoints of $\tilde{A}[\alpha]$, respectively. Let $\tilde{A}$, $\tilde{B}$ be two fuzzy numbers and λ be a real number. The fuzzy addition

$\tilde{A}\tilde{+}\tilde{B}$, the multiplication $\tilde{A}\circ\tilde{B}$ and the scalar multiplication $\lambda\tilde{B}$ are fuzzy numbers that have the membership functions $(\tilde{A}\tilde{+}\tilde{B})(z)$, $(\tilde{A}\circ\tilde{B})(z)$ and $(\lambda\tilde{A})(z)$ defined as: for any $z\in R$,

$$(\tilde{A}\tilde{+}\tilde{B})(z)=\sup_{y\in R}\{\min(\tilde{A}(y),\tilde{B}(z-y))\},$$

$$(\tilde{A}\circ\tilde{B})(z)=\sup_{xy=z}\{\min(\tilde{A}(x),\tilde{B}(y))\},$$

$$(\lambda\tilde{A})(z)=\begin{cases}\tilde{A}(\frac{z}{\lambda}), & \text{if } \lambda\neq 0,\\ 0, & \text{if } \lambda=0.\end{cases}$$

Let $\tilde{A}=\lfloor a^1,a^2,a^3,a^4\rfloor$, $\tilde{B}=\lfloor b^1,b^2,b^3,b^4\rfloor$ be two trapezoidal fuzzy numbers and λ a real number. The fuzzy addition $\tilde{A}\tilde{+}\tilde{B}$ and the scalar multiplication $\lambda\tilde{B}$ are trapezoidal fuzzy numbers, as follows:

$$(\tilde{A}\tilde{+}\tilde{B})=\lfloor a^1+b^1,a^2+b^2,a^3+b^3,a^4+b^4\rfloor,$$

$$(\lambda\tilde{A})=\lfloor \lambda a^1,\lambda a^2,\lambda a^3,\lambda a^4\rfloor,\ \text{if } \lambda>0,$$

$$(\lambda\tilde{A})=\lfloor \lambda a^4,\lambda a^3,\lambda a^2,\lambda a^1\rfloor,\ \text{if } \lambda<0.$$

Moreover, a fuzzy number $\tilde{A}$ is non-negative iff $A_*(\alpha)\geq 0$ for each $\alpha\in(0,1]$. Supposing that $\tilde{A},\tilde{B}$ are non-negative trapezoidal fuzzy numbers, then

$$\tilde{A}\circ\tilde{B}=\lfloor a^1b^1,a^2b^2,a^3b^3,a^4b^4\rfloor.$$

Also, for any $\tilde{A},\tilde{B}\in\mathcal{F}R$, let the membership functions be $\tilde{A}(x)$ and $\tilde{B}(x)$ respectively. If $\tilde{A}(x)\leq\tilde{B}(x)$ for any $x\in R$, then we say $\tilde{B}$ contains $\tilde{A}$, denoted by $\tilde{A}\subseteq\tilde{B}$. Moreover, for any $\tilde{A},\tilde{B},\tilde{C}\in\mathcal{F}R$, the fuzzy addition and multiplication satisfy $\tilde{A}\circ(\tilde{B}\tilde{+}\tilde{C})\subseteq\tilde{A}\circ\tilde{B}\tilde{+}\tilde{A}\circ\tilde{C}$.

The expected value $E(\tilde{A})$ of a fuzzy number $\tilde{A}$ was defined by Heilpern [13] as follows:

$$E(\tilde{A})=\tfrac{1}{2}\int_0^1(A_*(\alpha)+A^*(\alpha))\,\mathrm{d}\alpha.$$

For any $\tilde{A},\tilde{B}\in\mathcal{F}R$, the expected values of fuzzy numbers satisfy the following properties:

$$E(\tilde{A}\tilde{+}\tilde{B})=E(\tilde{A})+E(\tilde{B}),E(\tilde{A}\tilde{-}\tilde{B})=E(\tilde{A})-E(\tilde{B}).$$

Notice that the expected value of a fuzzy number is the center of the expected value of an interval random set generated by this fuzzy number, where the interval random set is a measurable mapping from the probability space to the family of all closed intervals of real line. Based on the expected values of fuzzy numbers, Chanas and Kasperski [5] defined the partial order relation of fuzzy numbers, and ignored the relation of two fuzzy numbers with equal expected values. Also, a total order relation of fuzzy numbers was defined by Zhang et al. [25] using the expected values, that is, for any $\tilde{A},\tilde{B}\in\mathcal{F}R$, we say that $\tilde{A}$ is weakly superior to $\tilde{B}$, denoted by $\tilde{A}\succcurlyeq\tilde{B}$, iff $E(\tilde{A})\geq E(\tilde{B})$; $\tilde{A}$ and $\tilde{B}$ are an indifference relationship, denoted by $\tilde{A}\approx\tilde{B}$, iff $E(\tilde{A})=E(\tilde{B})$; $\tilde{A}$ is superior to $\tilde{B}$, denoted by $\tilde{A}\succ\tilde{B}$, iff $E(\tilde{A})>E(\tilde{B})$.

Observe that in view of the total order relation of fuzzy numbers, for the symmetrical trapezoidal fuzzy numbers, if $\tilde{A}\subseteq\tilde{B}$, then $\tilde{A}\approx\tilde{B}$, and, therefore, $\tilde{A}\circ(\tilde{B}\tilde{+}\tilde{C})\approx\tilde{A}\circ\tilde{B}\tilde{+}\tilde{A}\circ\tilde{C}$ if the fuzzy numbers $\tilde{A}\circ(\tilde{B}\tilde{+}\tilde{C})$ and $\tilde{A}\circ\tilde{B}\tilde{+}\tilde{A}\circ\tilde{C}$ are symmetrical.

2.2. Fuzzy mappings

In what follows, we recall some related concepts of fuzzy mappings introduced by Zhang et al. [27].

For any $\tilde{\mathbf{x}} \in \mathcal{F}R^l$ which is the set of all l-dimensional fuzzy vectors and $\tilde{\delta} \succ \tilde{0}$, let $\tilde{B}_{\tilde{\delta}}(\tilde{\mathbf{x}}) = \{\tilde{\mathbf{y}} \in \mathcal{F}R^l \mid \|E(\tilde{\mathbf{y}}) - E(\tilde{\mathbf{x}})\| < E(\tilde{\delta})\}$, where the expected value of a fuzzy vector $E(\tilde{\mathbf{x}})$ is the vector consisting of the expected value of each component of $\tilde{\mathbf{x}}$.

Let $\tilde{X}$ be the non-empty subset of $\mathcal{F}R^l$. A *fuzzy mapping* $\tilde{f} : \tilde{X} \to \mathcal{F}R$ is said to be

(i) *upper semicontinuous* at $\tilde{\mathbf{x}}_0 \in \tilde{X}$ if for any $\tilde{\varepsilon} \succ \tilde{0}$ there exists a $\tilde{\delta} = \tilde{\delta}(\tilde{\mathbf{x}}_0, \tilde{\varepsilon}) \succ \tilde{0}$, such that

$$\tilde{f}(\tilde{\mathbf{x}}) \preccurlyeq \tilde{f}(\tilde{\mathbf{x}}_0) \tilde{+} \tilde{\varepsilon}, \tag{1}$$

for all $\tilde{\mathbf{x}} \in \tilde{X} \cap \tilde{B}_{\tilde{\delta}}(\tilde{\mathbf{x}}_0)$, and $\tilde{f} : \tilde{X} \to \mathcal{F}R$ is upper semicontinuous if it is upper semicontinuous at any $\tilde{\mathbf{x}} \in \tilde{X}$;

(ii) *lower semicontinuous* at $\tilde{\mathbf{x}}_0 \in \tilde{X}$ if for any $\tilde{\varepsilon} \succ \tilde{0}$ there exists a $\tilde{\delta} = \tilde{\delta}(\tilde{\mathbf{x}}_0, \tilde{\varepsilon}) \succ \tilde{0}$, such that

$$\tilde{f}(\tilde{\mathbf{x}}_0) \preccurlyeq \tilde{f}(\tilde{\mathbf{x}}) \tilde{+} \tilde{\varepsilon}, \tag{2}$$

for all $\tilde{\mathbf{x}} \in \tilde{X} \cap \tilde{B}_{\tilde{\delta}}(\tilde{\mathbf{x}}_0)$, and $\tilde{f} : \tilde{X} \to \mathcal{F}R$ is lower semicontinuous if it is lower semicontinuous at each $\tilde{\mathbf{x}} \in \tilde{X}$; and

(iii) *continuous* at $\tilde{\mathbf{x}}_0 \in \tilde{X}$ if it is upper semicontinuous and lower semicontinuous at $\tilde{\mathbf{x}}_0 \in \tilde{X}$.

Let $\tilde{X}$ be a subset of $\mathcal{F}R^l$ and $\tilde{f} : \tilde{X} \to \mathcal{F}R$ a fuzzy mapping parameterized by $\tilde{f}(\tilde{\mathbf{x}}) = \{(f(\tilde{\mathbf{x}})_*(\alpha), f(\tilde{\mathbf{x}})^*(\alpha), \alpha) : \alpha \in [0,1]\}$ for each $\tilde{\mathbf{x}} \in \tilde{X}$. The expected mapping $f_E(\tilde{\mathbf{x}})$ for any $\tilde{\mathbf{x}} \in \tilde{X}$ defined as $f_E(\tilde{\mathbf{x}}) = \frac{1}{2}\int_0^1 [f(\tilde{\mathbf{x}})_*(\alpha) + f(\tilde{\mathbf{x}})^*(\alpha)]\, \mathrm{d}\alpha$.

For the trapezoidal fuzzy numbers, Zhang et al. [27] got the following result.

Let $\tilde{\mathbf{x}} \in \tilde{X} \subseteq \mathcal{T}R^l_+$ which is the set of all l-dimensional non-negative trapezoidal fuzzy vectors and $\tilde{f} : \tilde{X} \to \mathcal{T}R$ a fuzzy mapping parameterized by $\tilde{f}(\tilde{\mathbf{x}}) = \{(f(\tilde{\mathbf{x}})_*(\alpha), f(\tilde{\mathbf{x}})^*(\alpha), \alpha) : \alpha \in [0,1]\}$ for each $\tilde{\mathbf{x}} \in \tilde{X}$. Then $\tilde{f}$ is continuous iff f_E is continuous.

This shows that the continuity of a fuzzy function about l-dimensional fuzzy vector can be translated into the continuity of a real function about $4l$-dimensional real vector.

Let $\tilde{X}$ be a non-empty subset of $\mathcal{F}R^l$. $\tilde{X}$ is said to be convex if for any $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \tilde{X}$ and $\lambda \in [0,1]$, $\lambda\tilde{\mathbf{x}} \tilde{+} (1-\lambda)\tilde{\mathbf{y}} \in \tilde{X}$.

2.3. Pure exchange economies

For the pure exchange economy, we take into account a marketplace consisting of l different goods indexed by $h = 1, \ldots, l$ and m agents denoted by $i = 1, \ldots, m$. Every agent i has an initial endowment vector: $\mathbf{w}_i = (w_{i1}, \ldots, w_{il}) \in R^l_+$. The consumption vector relative to the agent i is $\mathbf{x}_i = (x_{i1}, \ldots, x_{il}) \in X_i \subseteq R^l_+$, where x_{ih} is consumption relative to the commodity h, X_i is interpreted as the consumption set of agent i, and $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_m) \in \prod_{i=1}^m X_i$ represents the consumption of the market. We denote the price vector by $\mathbf{p} = (p_1, \ldots, p_l) \in R^l_+$, where $p_h (h = 1, \ldots, l)$ is the price of commodity h. As is standard in economic theory, the choice by the agent from a given set of alternative consumption vectors is assumed to be made in accordance with their preference $\succsim^i$. The reason is that there exists a utility function: $u_i : R^l_+ \to R$ such that $u_i(\mathbf{x}_i) \geq u_i(\mathbf{x}'_i)$ if and only if $\mathbf{x}_i$ is preferred or indifferent to $\mathbf{x}'_i$ for agent i. Thus, the pure exchange economy can be denoted by $\mathcal{E} = (R^l_+, X_i, \succsim^i, \mathbf{w}_i)$.

The goal of any agent becomes to find their optimal consumption vector which maximizes their utility by accomplishing the exchange of the goods in their budget set. It is required that the prices of different goods should be non-negative and not all zero. Moreover, one can normalize the price vector by restricting the sum of its coordinates to be 1. The market for any goods is usually considered to be in equilibrium if the supply of the good equals the demand of it. Thus, a pair $(\bar{\mathbf{p}}, \bar{\mathbf{x}})$ is said to be a *competitive equilibrium* of $\mathcal{E}$ if it satisfies the following conditions (1)-(3):

(1) $\bar{\mathbf{x}}_i$ is the optimum solution of $\max_{\mathbf{x}_i \in B_i(\bar{\mathbf{p}})} u_i(\mathbf{x}_i)$, where $B_i(\bar{\mathbf{p}}) = \{\mathbf{x}_i \mid \mathbf{x}_i \in X_i, \langle \bar{\mathbf{p}}, \mathbf{x}_i \rangle \leq \langle \bar{\mathbf{p}}, \mathbf{w}_i \rangle\}$, for all $i = 1, \ldots, m$.

(2) $\bar{\mathbf{p}} \in P = \{\mathbf{p} \mid \mathbf{p} \in R^l, \mathbf{p} \geqq \mathbf{0}, \sum_{h=1}^{l} p_h = 1\}$.

(3) $\bar{\mathbf{z}} \leqq \mathbf{0}, \langle \bar{\mathbf{p}}, \bar{\mathbf{z}} \rangle = 0$.

What follows are certain assumptions concerning the consumption units in a pure exchange economy.

(i) The set of consumption vectors X_i available to an individual $i = 1, 2, \ldots, m$ is a closed convex subset of R^l_+, i. e., $\mathbf{x}_i \geqq \mathbf{0}$, for all $\mathbf{x}_i \in X_i$.

(ii) For all $\mathbf{x}'_i \in X_i$, the sets $\{\mathbf{x}_i \in X_i \mid \mathbf{x}_i \precsim^i \mathbf{x}'_i\}$ and $\{\mathbf{x}_i \in X_i \mid \mathbf{x}'_i \precsim^i \mathbf{x}_i\}$ are closed.

(iii) For any $\mathbf{x}_i \in X_i$, there is $\mathbf{x}'_i \in X_i$ such that $u_i(\mathbf{x}'_i) > u_i(\mathbf{x}_i)$.

(iv) If $u_i(\mathbf{x}_i) > u_i(\mathbf{x}'_i)$ and $0 < \lambda < 1$, then $u_i[\lambda \mathbf{x}_i + (1 - \lambda)\mathbf{x}'_i] > u_i(\mathbf{x}'_i)$.

(v) For some $\mathbf{x}_i \in X_i$, there exists a $\mathbf{w}_i \in R^l_+$ such that $\mathbf{x}_i < \mathbf{w}_i$.

Under Assumptions (i)$-$(v), for all $i = 1, \ldots, m$, Arrow and Debreu [1] verified that there exists a competitive equilibrium of a pure exchange economy using fixed point theorem.

2.4. Fuzzy preferences

Here, we recall the definitions of a fuzzy binary relation and fuzzy preferences defined by Zhang et al. [26].

Let X be a reference set. A *fuzzy binary relation* $\mathcal{G}$ of X is characterized by a membership function $\mu_{\mathcal{G}} : X \times X \to \mathcal{F}R$.

Based on the fuzzy binary relation $\mathcal{G}$, Zhang et al. [26] defined a fuzzy preference relation $\succsim_{\mathcal{G}}$ on a reference set X as follows.

For any $\mathbf{x}, \mathbf{y} \in X$, if $\mu_{\mathcal{G}}(\mathbf{x}, \mathbf{y}) \succcurlyeq \mu_{\mathcal{G}}(\mathbf{y}, \mathbf{x})$, we say $\mathbf{x}$ is *fuzzily weakly preferred* to $\mathbf{y}$, denoted by $\mathbf{x} \succsim_{\mathcal{G}} \mathbf{y}$; if $\mu_{\mathcal{G}}(\mathbf{x}, \mathbf{y}) \approx \mu_{\mathcal{G}}(\mathbf{y}, \mathbf{x})$, we say $\mathbf{x}$ is *fuzzily indifferent* to $\mathbf{y}$, denoted by $\mathbf{x} \sim_{\mathcal{G}} \mathbf{y}$; if $\mu_{\mathcal{G}}(\mathbf{x}, \mathbf{y}) \succ \mu_{\mathcal{G}}(\mathbf{y}, \mathbf{x})$, we say $\mathbf{x}$ is *fuzzily preferred* to $\mathbf{y}$, denoted by $\mathbf{x} \succ_{\mathcal{G}} \mathbf{y}$.

Notice that the fuzzy preference relation $\mu_{\mathcal{G}}$ of a reference set X satisfies reflexive, antisymmetric, complete, and transitive. For a given fuzzy preference relation $\mu_{\mathcal{G}}$, there exists a fuzzy utility function $\tilde{u}$ that maps reference set X into the set of all fuzzy numbers.

Let us illustrate the practical significance of fuzzy binary relation of X with the following example similar to Example 1 in [27].

Example 2.1. For two given commodity bundles of mask and non-necessity good (e. g., apple) $\mathbf{x}$ and $\mathbf{y}$, where the amount of mask of $\mathbf{x}$ is greater than that of $\mathbf{y}$, and the amount of apple of $\mathbf{x}$ is less than that of $\mathbf{y}$, let p be the pollution index. We assume that $p \in [201, 300]$ indicates severe pollution, and the maximum pollution index accepted by humans outdoors is $p = 300$. An agent's satisfaction degree of this commodity bundle can be given as follows:

$$\mu_{\mathcal{G}}(\mathbf{x},\mathbf{y})(p) = \begin{cases} \frac{1}{201}p, & \text{if } 0 \leq p < 201, \\ 1, & \text{if } 201 \leq p \leq 300, \\ 0, & \text{otherwise,} \end{cases}$$

which means that the agent's satisfaction degree of this commodity bundle varies continuously in pace with the changeable air pollution; that is, $\mu_{\mathcal{G}}(\mathbf{x},\mathbf{y}) \in \mathcal{F}R$.

3. THE EXISTENCE OF FUZZY COMPETITIVE EQUILIBRIUM FOR FXE

In our paper, for any real vectors $\mathbf{x}_i, \mathbf{y}_i \in R^l$, $\mathbf{x}_i > \mathbf{y}_i$ means $x_{ih} > y_{ih}$ for all h; $\mathbf{x}_i \geqq \mathbf{y}_i$ means $x_{ih} \geq y_{ih}$ for all h; and $\mathbf{x}_i \geqslant \mathbf{y}_i$ means $\mathbf{x}_i \geqq \mathbf{y}_i$ but not $\mathbf{x}_i = \mathbf{y}_i$. The scalar product $\sum_{h=1}^{l} x_{ih}y_{ih}$ of two members $\mathbf{x}_i$ and $\mathbf{y}_i$ of R^l is denoted by $\langle \mathbf{x}_i, \mathbf{y}_i \rangle$.

For any fuzzy vectors $\tilde{\mathbf{x}}_i, \tilde{\mathbf{y}}_i \in \mathcal{F}R^l$, $\tilde{\mathbf{x}}_i \succ \tilde{\mathbf{y}}_i$ means $\tilde{x}_{ih} \succ \tilde{y}_{ih}$ for all h; $\tilde{\mathbf{x}}_i \gtrapprox \tilde{\mathbf{y}}_i$ means $\tilde{x}_{ih} \succcurlyeq \tilde{y}_{ih}$ for all h; and $\tilde{\mathbf{x}}_i \gtrsim \tilde{\mathbf{y}}_i$ means $\tilde{\mathbf{x}}_i \gtrapprox \tilde{\mathbf{y}}_i$ but not $\tilde{\mathbf{x}}_i \approx \tilde{\mathbf{y}}_i$. The scalar product $\tilde{\sum}_{h=1}^{l} \tilde{x}_{ih} \circ \tilde{y}_{ih}$ of two members $\tilde{\mathbf{x}}_i$ and $\tilde{\mathbf{y}}_i$ of $\mathcal{F}R^l$ is denoted by $\langle \tilde{\mathbf{x}}_i, \tilde{\mathbf{y}}_i \rangle$.

In the pure exchange economy, the initial endowments of agents are practical measurements of such inaccuracy that they are given by real numbers with difficulty. Thus, the initial endowments can be represented by fuzzy numbers. Besides, the agent's attitude is vague when facing all sorts of alternative fuzzy consumption vectors. Moreover, Zhang et al. [26] defined a fuzzy binary relation $\mathcal{G}$ of a reference set X characterized by this membership function $\mu_{\mathcal{G}} : X \times X \to \mathcal{F}R$.

To propose a new model which we study in this paper, here given that the reference set in [26] is a fuzzy set $\tilde{X} \subseteq \mathcal{F}R^l$, we can obtian a fuzzy binary relation on $\tilde{X}$.

An agent i's fuzzy preference relation on his fuzzy consumption set is denoted by $\succsim_{\mathcal{G}}^{i}$. Moreover, the crisp consumption set is the special case of the fuzzy consumption set. Therefore, the fuzzy preference relation $\succsim_{\mathcal{G}}^{i}$ on agent's fuzzy consumption set becomes the preference relation $\succsim^{i}$ on his consumption set.

Definition 3.1. An *FXE* is

$$\tilde{\mathcal{E}} = (\mathcal{F}R_{+}^{l}, \tilde{X}_i, \succsim_{\mathcal{G}}^{i}, \tilde{\mathbf{w}}_i)$$

consisting of m agents indexed by $i \in I = \{1, \ldots, m\}$, each of which has a fuzzy preference $\succsim^i_{\mathcal{G}}$, a fuzzy initial endowment vector $\tilde{\mathbf{w}}_i = (\tilde{w}_{i1}, \ldots, \tilde{w}_{il}) \in \mathcal{F}R^l_+$, and trades l different goods denoted by $h \in H = \{1, \ldots, l\}$, where $\tilde{X}_i \subseteq \mathcal{F}R^l_+$ is the consumption set of agent i and its element $\tilde{\mathbf{x}}_i = (\tilde{x}_{i1}, \ldots, \tilde{x}_{il})$ is a consumption vector of the agent.

Remark 3.2. The pure exchange economy is the special case of the FXE when the agent's satisfaction degree for any fuzzy consumption vector is only 0 or 1 and the fuzzy initial endowments are real numbers.

In an FXE, the aim of any agent is to choose the best fuzzy consumption vector by performing the exchange of the goods in his fuzzy budget set, admissible fuzzy consumption vectors of which are affordable for the agent at fuzzy price vector $\tilde{\mathbf{p}} = (\tilde{p}_1, \ldots, \tilde{p}_l) \in \mathcal{F}R^l_+$ with the value of his fuzzy initial endowment vector $\tilde{\mathbf{w}}_i$. Following from Theorem 2 confirmed by Zhang et al. [26], we can acquire a fuzzy utility function $\tilde{u}_i(\tilde{\mathbf{x}}_i)$ which maps a fuzzy consumption set into the set of all fuzzy numbers. Likewise, the fuzzy utility function on the agent i's consumption set is continuous under some conditions. Accordingly, an agent's motivation in the choice of a fuzzy consumption vector is to maximize his fuzzy utility among all fuzzy consumption vectors that belong to his fuzzy budget set. In turn, the agent's income can be regarded as the receipts from sales of the fuzzy initial endowments. This leads to the following optimization problem: for all $i \in I$ and $\tilde{\mathbf{p}} \in \tilde{P}$,

$$\tilde{u}_i(\tilde{\mathbf{x}}_i^*) \approx \max_{\tilde{\mathbf{x}}_i \in \tilde{B}_i(\tilde{\mathbf{p}})} \tilde{u}_i(\tilde{\mathbf{x}}_i), \tag{3}$$

where

$$\tilde{B}_i(\tilde{\mathbf{p}}) = \{\tilde{\mathbf{x}}_i \mid \tilde{\mathbf{x}}_i \in \tilde{X}_i, \langle \tilde{\mathbf{p}}, \tilde{\mathbf{x}}_i \rangle \preccurlyeq \langle \tilde{\mathbf{p}}, \tilde{\mathbf{w}}_i \rangle\} \tag{4}$$

is i's fuzzy budget set. Moreover, we can assume that

$$\tilde{\mathbf{p}} \in \tilde{P} = \{\tilde{\mathbf{p}} \mid \tilde{\mathbf{p}} \in \mathcal{F}R^l, \tilde{\mathbf{p}} \gtrapprox \tilde{\mathbf{0}}, \tilde{\sum}_{h \in H} \tilde{p}_h \approx \tilde{1}\}, \tag{5}$$

where $\tilde{\mathbf{p}} \gtrapprox \tilde{\mathbf{0}}$ means that $E(\tilde{\mathbf{p}}) \geqq \mathbf{0}$ and $\tilde{\sum}_{h \in H} \tilde{p}_h \approx \tilde{1}$ iff $\sum_{h \in H} E(\tilde{p}_h) = 1$, because for all $\lambda > 0$, $\tilde{B}_i(\tilde{\mathbf{p}}) = \tilde{B}_i(\lambda \tilde{\mathbf{p}})$.

From now on to the end of this section, we suppose that for any $i \in I$, $h \in H$, $\tilde{x}_{ih} = \lfloor x^1_{ih}, x^2_{ih}, x^3_{ih}, x^4_{ih} \rfloor$, $\tilde{w}_{ih} = \lfloor w^1_{ih}, w^2_{ih}, w^3_{ih}, w^4_{ih} \rfloor$, $\tilde{p}_h = \lfloor p^1_h, p^2_h, p^3_h, p^4_h \rfloor$ are non-negative trapezoidal fuzzy numbers.

Lemma 3.3. For any fuzzy optimization problem (3) with constraint conditions (4) and (5), there exists a crisp optimization problem with a solution $\mathbf{x}_i^* = (x^*_{i1}, x^*_{i2}, \ldots, x^*_{i,4l}) \in X_i \subseteq R^{4l}_+$ such that the corresponding $\tilde{\mathbf{x}}_i^* \in \tilde{X}_i$ with $\tilde{\mathbf{x}}^*_{ih} = \lfloor \mathbf{x}^*_{i,4h-3}, \mathbf{x}^*_{i,4h-2}, \mathbf{x}^*_{i,4h-1}, \mathbf{x}^*_{i,4h} \rfloor$ for any $h = 1, \ldots, l$ is a solution of (3).

P r o o f. From the total order relation of fuzzy numbers and the expected function $u^i_E(\tilde{\mathbf{x}}_i)$ of the fuzzy utility function $\tilde{u}_i(\tilde{\mathbf{x}}_i)$, it yields that (3) is equivalent to saying that

$$u^i_E(\tilde{\mathbf{x}}_i^*) = \max_{\tilde{\mathbf{x}}_i \in \tilde{B}_i(\tilde{\mathbf{p}})} u^i_E(\tilde{\mathbf{x}}_i) \text{ for all } i \in I \text{ and } \tilde{\mathbf{p}} \in \tilde{P}. \tag{6}$$

The optimization problem (6) is the same as

$$u_E^i(\mathbf{x}_i^*) = \max_{\mathbf{x}_i \in B_i(\mathbf{p})} u_E^i(\mathbf{x}_i), \text{ for all } i \in I \text{ and } \mathbf{p} \in P, \tag{7}$$

where $\mathbf{x}_i = (x_{i1}^1, x_{i1}^2, x_{i1}^3, x_{i1}^4, \ldots, x_{il}^1, x_{il}^2, x_{il}^3, x_{il}^4) \equiv (x_{i1}, x_{i2}, \ldots, x_{i,4l}) \in X_i \subseteq R_+^{4l}$, $\mathbf{w}_i = (w_{i1}^1, w_{i1}^2, w_{i1}^3, w_{i1}^4, \ldots, w_{il}^1, w_{il}^2, w_{il}^3, w_{il}^4) \equiv (w_{i1}, w_{i2}, \ldots, w_{i,4l}) \in R_+^{4l}$, $\mathbf{p} = (p_1^1, p_1^2, p_1^3, p_1^4, \ldots, p_l^1, p_l^2, p_l^3, p_l^4) \equiv (p_1, p_2, \ldots, p_{4l}) \in P \subseteq R_+^{4l}$,

$$\begin{aligned} B_i(\mathbf{p}) &= \{\mathbf{x}_i \mid \mathbf{x}_i \in X_i, \sum_{h \in H} \sum_{j=1}^{4} p_h^j x_{ih}^j \leq \sum_{h \in H} \sum_{j=1}^{4} p_h^j w_{ih}^j\} \\ &= \{\mathbf{x}_i \mid \mathbf{x}_i \in X_i, \sum_{h'=1}^{4l} p_h x_{ih} \leq \sum_{h'=1}^{4l} p_h w_{ih}\} \\ &= \{\mathbf{x}_i \mid \mathbf{x}_i \in X_i, \langle \mathbf{p}, \mathbf{x}_i \rangle \leq \langle \mathbf{p}, \mathbf{w}_i \rangle\}, \text{ and} \end{aligned} \tag{8}$$

$$\begin{aligned} P &= \{\mathbf{p} \mid \mathbf{p} \in R^{4l}, \mathbf{p} \geqq \mathbf{0}, \sum_{h \in H} \sum_{j=1}^{4} p_h^j = 1\} \\ &= \{\mathbf{p} \mid \mathbf{p} \in R^{4l}, \mathbf{p} \geqq \mathbf{0}, \sum_{h'=1}^{4l} p_{h'} = 1\}. \end{aligned} \tag{9}$$

Furthermore, the set of fuzzy price vector $\tilde{P}$, i. e., (5) is equal to

$$\begin{aligned} \tilde{P}' = \{\tilde{\mathbf{p}} \mid \tilde{\mathbf{p}} \in \mathcal{T}R^l, (\frac{p_1^1 + p_1^2 + p_1^3 + p_1^4}{4}, \ldots, \frac{p_l^1 + p_l^2 + p_l^3 + p_l^4}{4}) \geqq \mathbf{0}, \\ \frac{p_1^1 + \ldots + p_l^1 + p_1^2 + \ldots + p_l^2 + p_1^3 + \ldots + p_l^3 + p_1^4 + \ldots + p_l^4}{4} = 1\}. \end{aligned}$$

The corresponding real price vector set is

$$P' = \{\mathbf{p}' \mid \mathbf{p}' \in R^{4l}, \mathbf{p}' \geqq \mathbf{0}, \sum_{h \in H} \sum_{j=1}^{4} p_h'^j = 1\} = \{\mathbf{p}' \mid \mathbf{p}' \in R^{4l}, \mathbf{p}' \geqq \mathbf{0}, \sum_{h'=1}^{4l} p'_{h'} = 1\}, \tag{10}$$

where $\mathbf{p}' = (\frac{p_1^1}{4}, \frac{p_1^2}{4}, \frac{p_1^3}{4}, \frac{p_1^4}{4}, \ldots, \frac{p_l^1}{4}, \frac{p_l^2}{4}, \frac{p_l^3}{4}, \frac{p_l^4}{4}) \equiv (\frac{p_1}{4}, \frac{p_2}{4}, \ldots, \frac{p_{4l}}{4}) \in P' \subseteq R_+^{4l}$, $p_h'^j = \frac{p_h^j}{4}$, i. e., $p'_{h'} = \frac{p_{h'}}{4}, h' = 1, \ldots, 4l$.

It is found that (9) becomes

$$P = \{\mathbf{p} \mid \mathbf{p} \in R^{4l}, \mathbf{p}' \in P', \mathbf{p} = 4\mathbf{p}'\}. \tag{11}$$

Hence, for the pure exchange economy $(R_+^{4l}, X_i, \succsim^i, \mathbf{w}_i)$, if the the maximization problem (7) with the constraint conditions (8) and (9) has a solution $\mathbf{x}_i^*(\mathbf{p})$ for each $\mathbf{p} \in P$, denoted by $\mathbf{x}_i^* = (x_{i1}^*, x_{i2}^*, \ldots, x_{i,4l}^*)$, then the maximization problem (7) with the constraint conditions (8) and (10) has a solution $\mathbf{x}_i^*$ as well, since for all $\lambda > 0$,

$\tilde{B}_i(\tilde{\mathbf{p}}) = \tilde{B}_i(\lambda\tilde{\mathbf{p}})$ and the link between P and P' is as (11). In addition, the maximization problem (3) with constraint conditions (4) (5) is equivalent to the maximization problem (7) with the constraint conditions (8) and (10). Finally, it is true that the maximization problem (3) has a solution $\tilde{\mathbf{x}}_i^*(\tilde{\mathbf{p}})$ for each $\tilde{\mathbf{p}} \in \tilde{P}$, denoted by $\tilde{\mathbf{x}}_i^*$ with $\tilde{\mathbf{x}}_{ih}^* = \lfloor \mathbf{x}_{i,4h-3}^*, \mathbf{x}_{i,4h-2}^*, \mathbf{x}_{i,4h-1}^*, \mathbf{x}_{i,4h}^* \rfloor$ for any $h = 1, \dots, l$. $\square$

Notice that we say that the pair $(\mathbf{p}^*, \mathbf{x}^*) \in P \times B(\mathbf{p}^*)$ is the expected pair of $(\tilde{\mathbf{p}}^*, \tilde{\mathbf{x}}^*) \in \tilde{P} \times \tilde{B}(\tilde{\mathbf{p}}^*)$.

Similarly, the fuzzy market for any good is usually considered to be in equilibrium if the fuzzy supply of the goods equals the fuzzy demand of those. However, the price of some good may be $\tilde{0}$, which means that fuzzy supply will exceed fuzzy demand. The fuzzy aggregate excess demand is $\tilde{z}_h = \sum_{i\in I}(\tilde{x}_{ih} \tilde{-} \tilde{w}_{ih})$ and $\tilde{x}_{ih} \tilde{-} \tilde{w}_{ih}$ is the individual fuzzy excess demand of agent i relative to good h. Thus, $\tilde{\mathbf{z}} = (\tilde{z}_1, \dots, \tilde{z}_l) \in \mathcal{T}R^l$.

Definition 3.4. For FXE $\tilde{\mathcal{E}}$, let $\tilde{\mathbf{p}}^* \in \tilde{P}$ and $\tilde{\mathbf{x}}^* \in \tilde{B}(\tilde{\mathbf{p}}^*) = \prod_{i\in I} \tilde{B}_i(\tilde{\mathbf{p}}^*)$. The pair $(\tilde{\mathbf{p}}^*, \tilde{\mathbf{x}}^*) \in \tilde{P} \times \tilde{B}(\tilde{\mathbf{p}}^*)$ is a *fuzzy competitive equilibrium* iff

$$\tilde{u}_i(\tilde{\mathbf{x}}_i^*) \approx \max_{\tilde{\mathbf{x}}_i \in \tilde{B}_i(\tilde{\mathbf{p}}^*)} \tilde{u}_i(\tilde{\mathbf{x}}_i), \text{ for all } i \in I, \text{ and} \tag{12}$$

$$\tilde{z}_h = \sum_{i\in I}(\tilde{x}_{ih}^* - \tilde{w}_{ih}) \preccurlyeq \tilde{0}, \text{ for any } h \in H.$$

According to the total order relation of fuzzy numbers, expected function of a fuzzy mapping and Lemma 3.3, we get the following theorem about the existence result of fuzzy competitive equilibria immediately by an associated pure exchange economy.

Theorem 3.5. The pair $(\tilde{\mathbf{p}}^*, \tilde{\mathbf{x}}^*) \in \tilde{P} \times \tilde{B}(\tilde{\mathbf{p}}^*)$ is a fuzzy competitive equilibrium of an FXE iff the expected pair $(\mathbf{p}^*, \mathbf{x}^*)$ is a competitive equilibrium of the corresponding pure exchange economy.

4. THE WEAK FUZZY CORE OF AN FXE

Depending on the following assumptions concerning the consumption units in an FXE, we get some conclusions about the relationship between the weak fuzzy core and the set of all fuzzy competitive allocations.

Assumption I Let $\tilde{\mathbf{x}}_i \gtrapprox \tilde{\mathbf{0}}$. We assume that there is a commodity bundle $\tilde{\mathbf{x}}_i' \in \tilde{X}_i$ such that $\tilde{u}_i(\tilde{\mathbf{x}}_i') \succ \tilde{u}_i(\tilde{\mathbf{x}}_i)$ for any $\tilde{\mathbf{x}}_i \in \tilde{X}_i$.

That means there is no fuzzy consumption vector which individual would fuzzily prefer to all others.

Assumption II If $\tilde{u}_i(\tilde{\mathbf{x}}_i) \succ \tilde{u}_i(\tilde{\mathbf{x}}_i')$ and $0 < \lambda < 1$, then $\tilde{u}_i[\lambda\tilde{\mathbf{x}}_i + (1-\lambda)\tilde{\mathbf{x}}_i'] \succ \tilde{u}_i(\tilde{\mathbf{x}}_i')$.

Assumption III For all $\tilde{\mathbf{x}}'_i \in \tilde{X}_i$, the sets $\{\tilde{\mathbf{x}}_i \in \tilde{X}_i \mid \tilde{\mathbf{x}}_i \precsim^i_{\mathcal{G}} \tilde{\mathbf{x}}'_i\}$ and $\{\tilde{\mathbf{x}}_i \in \tilde{X}_i \mid \tilde{\mathbf{x}}'_i \precsim^i_{\mathcal{G}} \tilde{\mathbf{x}}_i\}$ are closed.

This assumption ensures the continuity of $\tilde{u}_i(\tilde{\mathbf{x}}_i)$ demonstrated by Theorem 2 obtained by Zhang et al. [26].

In the FXE, the result of exchanging consists of a fuzzy allocation of the total fuzzy supply $\tilde{\sum}_{i\in I}\tilde{\mathbf{w}}_i$ described by a collection of m non-negative commodity bundles $(\tilde{\mathbf{x}}_1,\dots,\tilde{\mathbf{x}}_m)$ such that $\tilde{\sum}_{i\in I}\tilde{\mathbf{x}}_i \approx \tilde{\sum}_{i\in I}\tilde{\mathbf{w}}_i$. The set of all fuzzy allocations is denoted by $F(\tilde{\mathcal{E}})$.

A coalition of agents is the subset of I. A non-empty coalition S is said to block a fuzzy allocation $\tilde{\mathbf{x}}$ if there exist commodity bundles $\tilde{\mathbf{x}}'_i \in \tilde{X}_i$ for all $i \in S$ such that

$$\tilde{\sum}_{i\in S}\tilde{\mathbf{x}}'_i \approx \tilde{\sum}_{i\in S}\tilde{\mathbf{w}}_i,$$

$$\tilde{u}_i(\tilde{\mathbf{x}}'_i) \succ \tilde{u}_i(\tilde{\mathbf{x}}_i), \quad \text{for all } i \in S. \tag{13}$$

The formula (13) amounts to $\tilde{\mathbf{x}}'_i \succsim^i_{\mathcal{G}} \tilde{\mathbf{x}}_i$ for all $i \in S$, with a strict preference at least one member of S.

Definition 4.1. A *weak fuzzy core* $C(\tilde{\mathcal{E}})$ of an FXE $\tilde{\mathcal{E}}$ is defined as the set of all fuzzy allocations which are not blocked via any non-empty coalition.

Theorem 4.2. Any fuzzy competitive allocation belongs to the weak fuzzy core of an FXE.

Proof. Let $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_1,\dots,\tilde{\mathbf{x}}_m)$ be a fuzzy competitive allocation. If there exists a commodity bundle $\tilde{\mathbf{x}}'_i$ satisfying $\tilde{\mathbf{x}}'_i \succ^i_{\mathcal{G}} \tilde{\mathbf{x}}_i$, then $\langle\tilde{\mathbf{p}},\tilde{\mathbf{x}}'_i\rangle \succ \langle\tilde{\mathbf{p}},\tilde{\mathbf{w}}_i\rangle$. Otherwise, $\tilde{\mathbf{x}}_i$ does not satisfy the fuzzy preference of ith agent under his budget constraint. Also, note that $\tilde{\mathbf{x}}'_i \succsim^i_{\mathcal{G}} \tilde{\mathbf{x}}_i$ implies $\langle\tilde{\mathbf{p}},\tilde{\mathbf{x}}'_i\rangle \succcurlyeq \langle\tilde{\mathbf{p}},\tilde{\mathbf{w}}_i\rangle$. The reason is that if $\langle\tilde{\mathbf{p}},\tilde{\mathbf{x}}'_i\rangle \prec \langle\tilde{\mathbf{p}},\tilde{\mathbf{w}}_i\rangle$, then there is a consumption vector in the neighborhood of $\tilde{\mathbf{x}}'_i$ satisfying the budget constraint and being fuzzy preferred to $\tilde{\mathbf{x}}_i$ according to Assumptions I and II.

Let S be a possible blocking set. It is easily seen that $\tilde{\sum}_{i\in S}(\tilde{\mathbf{x}}'_i \tilde{-} \tilde{\mathbf{w}}_i) \approx \tilde{\mathbf{0}}$ and $\tilde{\mathbf{x}}'_i \succsim^i_{\mathcal{G}} \tilde{\mathbf{x}}_i$, for all $i \in S$ with a strict fuzzy preference for at least one $i \in S$. As mentioned in the above paragraph, $\langle\tilde{\mathbf{p}},\tilde{\mathbf{x}}'_i\rangle \succcurlyeq \langle\tilde{\mathbf{p}},\tilde{\mathbf{w}}_i\rangle$ for all $i \in S$ with a strict inequality for at least one $i \in S$. So then

$$\tilde{\sum}_{i\in S}\langle\tilde{\mathbf{p}},\tilde{\mathbf{x}}'_i\rangle \succ \tilde{\sum}_{i\in S}\langle\tilde{\mathbf{p}},\tilde{\mathbf{w}}_i\rangle.$$

Since the property of fuzzy arithmetic, under certain conditions, we get that

$$\tilde{0} \prec \tilde{\sum}_{i\in S}\langle\tilde{\mathbf{p}},\tilde{\mathbf{x}}'_i\rangle \tilde{-} \tilde{\sum}_{i\in S}\langle\tilde{\mathbf{p}},\tilde{\mathbf{w}}_i\rangle \approx \tilde{\sum}_{i\in S}\langle\tilde{\mathbf{p}},\tilde{\mathbf{x}}_i \tilde{-} \tilde{\mathbf{w}}_i\rangle,$$

which contradicts with $\tilde{\sum}_{i\in S}(\tilde{\mathbf{x}}'_i \tilde{-} \tilde{\mathbf{w}}_i) \approx \tilde{\mathbf{0}}$. □

5. THE WEAK FUZZY CORE AS THE NUMBER OF AGENTS BECOMES INFINITE

Owing to the replica economy presented by Debreu and Scarf [6], we consider a fuzzy economy composed of m types of agents, with r agents of each type. Notice that all agents of the same type have precisely the same fuzzy preferences and the same fuzzy initial endowment vectors. Actually, the r-fold fuzzy replica economy, which is the r-fold repetition of the FXE $\tilde{\mathcal{E}}$, denoted by $\tilde{\mathcal{E}}_r$ for each positive integer r. The r-fold fuzzy replica economy $\tilde{\mathcal{E}}_r$ consists of mr agents, whom we index by i^k with $i \in I$ and $k \in K = \{1, \ldots, r\}$, where i refers to the type of the agent and k distinguishes different agents of the same type. It is evident that the FXE $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}_1$.

A fuzzy allocation is characterized by a collection of mr non-negative commodity bundles $\tilde{\mathbf{x}}_{i^k}$ meeting $\tilde{\sum}_{i\in I}\tilde{\sum}_{k\in K}\tilde{\mathbf{x}}_{i^k} \approx r\tilde{\sum}_{i\in I}\tilde{\mathbf{w}}_i$.

A fuzzy allocation $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_{i^k})_{i\in I, k\in K}$ is blocked by coalition $S_r \subseteq N_r = \{i^k \mid i \in I \text{ and } k \in K\}$ if it is possible to find commodity bundles $\tilde{\mathbf{x}}'_{i^k}$ for all $i^k \in S_r$ such that

$$\tilde{\sum}_{i\in S'_r}\tilde{\sum}_{k\in S_r(i)}\tilde{\mathbf{x}}'_{i^k} \,\tilde{-}\, \tilde{\sum}_{i\in S'_r}|S_r(i)|\tilde{\mathbf{w}}_i \approx \tilde{\mathbf{0}},$$

$$\tilde{u}_{i^k}(\tilde{\mathbf{x}}'_{i^k}) \succ \tilde{u}_{i^k}(\tilde{\mathbf{x}}_{i^k}), \text{ for all } i^k \in S_r, \tag{14}$$

where (14) means $\tilde{\mathbf{x}}'_{i^k} \succsim^i_{\mathcal{G}} \tilde{\mathbf{x}}_{i^k}$, for all $i^k \in S_r$ with a strict preference for at least one member of S_r, $S'_r = \{i \in I \mid i^k \in S_r\}$, $S_r(i) = \{k \in K \mid i^k \in S_r\}$, and $|S_r(i)|$ denotes the number of agents of type i in $S_r(i)$.

Definition 5.1. The *weak fuzzy core* $C(\tilde{\mathcal{E}}_r)$ of the r-fold fuzzy replica economy is defined as the set of all fuzzy allocations which are not blocked by any non-empty coalition.

Theorem 5.2. For the r-fold fuzzy replica economy, if there exists a fuzzy allocation in the weak fuzzy core $C(\tilde{\mathcal{E}}_r)$, then the fuzzy allocation assigns the same consumption to all agents of the same type.

Proof. Assume that $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_{i^k})_{i\in I, k\in K} \in C(\tilde{\mathcal{E}}_r)$. For any particular type i, let $\tilde{\mathbf{x}}_i$ represent the least desired of the fuzzy consumption vectors $\tilde{\mathbf{x}}_{i^k}$ according to the fuzzy preferences for the agents of this type.

We suppose that two agents of a certain type i' have been assigned different commodity bundles. It holds that

$$\frac{1}{r}\tilde{\sum}_{k\in K}\tilde{\mathbf{x}}_{i^k} \succsim^i_{\mathcal{G}} \tilde{\mathbf{x}}_i, \text{ for all } i,$$

with strict fuzzy preference holding for i'. Nevertheless, $\tilde{\sum}_{i\in I}(\frac{1}{r}\tilde{\sum}_{k\in K}\tilde{\mathbf{x}}_{i^k} \,\tilde{-}\, \tilde{\mathbf{w}}_i) \approx \tilde{\mathbf{0}}$. Thus, the set consisting of one agent of each type, each of whom receives a least fuzzy preferred consumption, would block. Clearly, a fuzzy allocation in the weak fuzzy core $C(\tilde{\mathcal{E}}_r)$ of the r-fold fuzzy replica economy may be described by r-fold repetition of a fuzzy allocation for the FXE $\tilde{\mathcal{E}}$. □

Definition 5.3. A *fuzzy Edgeworth equilibrium* of an FXE $\tilde{\mathcal{E}}$ is a fuzzy allocation $\tilde{\mathbf{x}} \in F(\tilde{\mathcal{E}})$ such that for any given positive integer r, the r-fold repetition of $\tilde{\mathbf{x}}$ belongs to the weak fuzzy core of the r-fold fuzzy replica economy $\tilde{\mathcal{E}}_r$. Moreover, the set of all fuzzy Edgeworth equilibria is denoted by $C^E(\tilde{\mathcal{E}})$ as follows

$$C^E(\tilde{\mathcal{E}}) = \left\{\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_m) \in F(\tilde{\mathcal{E}}) \mid (\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_m, \ldots, \tilde{\mathbf{x}}_m) \in C(\tilde{\mathcal{E}}_r), \text{ for any } r\right\}.$$

Observe that for a given r, the set of all fuzzy allocations of the FXE arisen from the weak fuzzy core allocations in the r-fold fuzzy replica economy is defined as

$$C_r^E(\tilde{\mathcal{E}}) = \left\{\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_m) \in F(\tilde{\mathcal{E}}) \mid (\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_m, \ldots, \tilde{\mathbf{x}}_m) \in C(\tilde{\mathcal{E}}_r) \right\}.$$

Obviously, $C_{r+1}^E(\tilde{\mathcal{E}}) \subseteq C_r^E(\tilde{\mathcal{E}})$ for any $r > 1$, seeing that a coalition which blocks in the economy with r repetitions will certainly be available for blocking in the economy with $r + 1$ repetitions.

Provided that we consider a fuzzy competitive allocation in the economy consisting of one participant of each type and repeat the fuzzy allocation when the economy is enlarged to r participants of each type, the fuzzy allocation is competitive for the larger economy and therefore is in the weak fuzzy core $C(\tilde{\mathcal{E}}_r)$ of the r-fold fuzzy replica economy. Also, $C_r^E(\tilde{\mathcal{E}})$ forms a non-increasing sequence of sets, following from enlarging the number of r, each of which contains the collection of fuzzy competitive allocations of the FXE. Finally, we prove that $C_r^E(\tilde{\mathcal{E}})$, that is, the set of all fuzzy allocations of the FXE arising from the weak fuzzy core allocations in the r-fold fuzzy replica economy shrinks to the fuzzy competitive allocations of the FXE as the number r of agents of each type gradually increases.

Theorem 5.4. If $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_m) \in C_r^E(\tilde{\mathcal{E}})$ for any $r = 1, 2, \ldots$, then $\tilde{\mathbf{x}}$ is a fuzzy competitive allocation of an FXE $\tilde{\mathcal{E}}$.

Proof. Let $\tilde{\Gamma}_i$ be the fuzzy set of all $\tilde{\mathbf{z}}$ in the commodity space such that $\tilde{\mathbf{z}} \tilde{+} \tilde{\mathbf{w}}_i \succ_{\mathcal{G}}^i \tilde{\mathbf{x}}_i$, and $\hat{\Gamma}_i$ be the set of all expected values of the fuzzy numbers in $\tilde{\Gamma}_i$. Suppose that $\hat{\Gamma}$ is the convex hull of the union of the sets $\hat{\Gamma}_i$. If this is the case, we say $\tilde{\Gamma}$ is the convex hull of the union of the fuzzy sets $\tilde{\Gamma}_i$. For every i, $\tilde{\Gamma}_i$ is convex and non-empty. Then $\hat{\Gamma}$ is convex and non-empty as a result of the definition of convexity for fuzzy set. $\hat{\Gamma}$ consists of the set of all vectors $E(\tilde{\mathbf{z}})$ which may be written as $\sum_{i\in I} \alpha_i E(\tilde{\mathbf{z}}_i)$, where $\alpha_i \geq 0$, $\sum_{i\in I} \alpha_i = 1$, and $\tilde{\mathbf{z}}_i \tilde{+} \tilde{\mathbf{w}}_i \succ_{\mathcal{G}}^i \tilde{\mathbf{x}}_i$. The key for proving this theorem is to verify that the origin, i. e., $\mathbf{0}$ does not belong to the set $\hat{\Gamma}$ on the basis of the general case of an arbitrary finite number of goods and an arbitrary number types of agents.

We assume that the origin belongs to $\hat{\Gamma}$. That is, $\tilde{\sum}_{i\in I} \alpha_i E(\tilde{\mathbf{z}}_i) = \mathbf{0}$ with $\alpha_i \geq 0$, $\sum_{i\in I} \alpha_i = 1$, and $\tilde{\mathbf{z}}_i \tilde{+} \tilde{\mathbf{w}}_i \succ_{\mathcal{G}}^i \tilde{\mathbf{x}}_i$. Choose an integer t, which will eventually tend to $+\infty$. Let b_i^t be the smallest integer greater than or equal to $t\alpha_i$ and I_+ the set of i for which $\alpha_i > 0$.

For any $i \in I_+$, define $\tilde{\mathbf{z}}_i^t = \frac{t\alpha_i}{b_i^t} \tilde{\mathbf{z}}_i$ and notice that $\tilde{\mathbf{w}}_i \precsim \tilde{\mathbf{z}}_i^t \tilde{+} \tilde{\mathbf{w}}_i \precsim \tilde{\mathbf{z}}_i \tilde{+} \tilde{\mathbf{w}}_i$. It is obvious that $\tilde{\mathbf{z}}_i^t \tilde{+} \tilde{\mathbf{w}}_i$ tends to $\tilde{\mathbf{z}}_i \tilde{+} \tilde{\mathbf{w}}_i$ when t tends to infinity. Assumption III indicates that $\tilde{\mathbf{z}}_i^t \tilde{+} \tilde{\mathbf{w}}_i \succ_{\mathcal{G}}^i \tilde{\mathbf{x}}_i$ for sufficiently large t. Additionally, $\tilde{\sum}_{i\in I_+} b_i^t \tilde{\mathbf{z}}_i^t = t \tilde{\sum}_{i\in I_+} \alpha_i \tilde{\mathbf{z}}_i \approx \tilde{\mathbf{0}}$.

Take into account the coalition composed of b_i^t members of type i to every one of whom we distribute $\tilde{\mathbf{w}}_i \tilde{+} \tilde{\mathbf{z}}_i^t$, where i is selected from I_+. Such a coalition blocks the fuzzy allocation $(\tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_m)$ with $\max_{i \in I_+} b_i^t$ repetitions, which contradict with $(\tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_m) \in C_r^E(\tilde{\mathcal{E}})$ for any r.

Subsequently, it is verified that the origin does not belong to the convex set $\hat{\Gamma}$. Therefore, there exists a hyperplane $\langle E(\tilde{\mathbf{p}}), E(\tilde{\mathbf{z}}) \rangle = 0$ through origin $\mathbf{0}$ with $E(\tilde{\mathbf{p}})$ such that $\langle E(\tilde{\mathbf{p}}), E(\tilde{\mathbf{z}}) \rangle \geq 0$ for all $E(\tilde{\mathbf{z}}) \in \hat{\Gamma}$ by using the separation theorem of convex sets, under the condition that the expected values of fuzzy numbers for the multiplication satisfy $E(\tilde{A})E(\tilde{B}) = E(\tilde{A} \circ \tilde{B})$ for any two fuzzy numbers $\tilde{A}, \tilde{B} \in \mathcal{F}R$. That is, $\langle \tilde{\mathbf{p}}, \tilde{\mathbf{z}} \rangle \succcurlyeq \tilde{0}$ for all $\tilde{\mathbf{z}}$ in $\tilde{\Gamma}$.

If there is an $\tilde{\mathbf{x}}_i'$ such that $\tilde{\mathbf{x}}_i' \succ_{\mathcal{G}}^i \tilde{\mathbf{x}}_i$, i.e., $\tilde{\mathbf{x}}_i' \tilde{-} \tilde{\mathbf{w}}_i \tilde{+} \tilde{\mathbf{w}}_i \succ_{\mathcal{G}}^i \tilde{\mathbf{x}}_i$, then $\tilde{\mathbf{x}}_i' \tilde{-} \tilde{\mathbf{w}}_i \in \tilde{\Gamma}_i$, so in $\tilde{\Gamma}$. Clearly, it holds that $\langle \tilde{\mathbf{p}}, \tilde{\mathbf{x}}_i' \rangle \succcurlyeq \langle \tilde{\mathbf{p}}, \tilde{\mathbf{w}}_i \rangle$. On account of Assumption II, in every neighbourhood of $\tilde{\mathbf{x}}_i$, there are consumption vectors strictly fuzzily preferred to $\tilde{\mathbf{x}}_i$. Therefore, it is true that $\langle \tilde{\mathbf{p}}, \tilde{\mathbf{x}}_i \rangle \succcurlyeq \langle \tilde{\mathbf{p}}, \tilde{\mathbf{w}}_i \rangle$. However, $\tilde{\sum}_{i \in I} (\tilde{\mathbf{x}}_i \tilde{-} \tilde{\mathbf{w}}_i) \approx \tilde{\mathbf{0}}$. So then $\langle \tilde{\mathbf{p}}, \tilde{\mathbf{x}}_i \rangle \approx \langle \tilde{\mathbf{p}}, \tilde{\mathbf{w}}_i \rangle$ for any i.

We have demonstrated the existence of fuzzy prices $\tilde{\mathbf{p}}$ such that for any i, (a) $\tilde{\mathbf{x}}_i' \succ_{\mathcal{G}}^i \tilde{\mathbf{x}}_i$ implies that $\langle \tilde{\mathbf{p}}, \tilde{\mathbf{x}}_i' \rangle \succcurlyeq \langle \tilde{\mathbf{p}}, \tilde{\mathbf{w}}_i \rangle$; (b) $\langle \tilde{\mathbf{p}}, \tilde{\mathbf{x}}_i \rangle \approx \langle \tilde{\mathbf{p}}, \tilde{\mathbf{w}}_i \rangle$. There remains to show that the inequality in (a) is strict. In fact, for $\tilde{\mathbf{w}}_i$ whose components are strictly positive, there is a non-negative $\tilde{\mathbf{x}}_i^0$ strictly below the budget hyperplane. Suppose that for some $\tilde{\mathbf{x}}_i''$, both $\tilde{\mathbf{x}}_i'' \succ_{\mathcal{G}}^i \tilde{\mathbf{x}}_i$ and $\langle \tilde{\mathbf{p}}, \tilde{\mathbf{x}}_i'' \rangle \approx \langle \tilde{\mathbf{p}}, \tilde{\mathbf{w}}_i \rangle$. The elements between $\tilde{\mathbf{x}}_i^0$ and $\tilde{\mathbf{x}}_i''$ close enough to $\tilde{\mathbf{x}}_i''$ would be strictly preferred to $\tilde{\mathbf{x}}_i$ and strictly below the budget hyperplane, a contradiction of (a). Namely, $\tilde{\mathbf{x}}$ is the fuzzy competitive allocation of the FXE. $\square$

6. CONCLUSIONS

The reality in a pure exchange economy is that there is vagueness in agents' initial endowments and preferences. This paper fills the gap in the literature by proposing a new model of FXE. Also, a weak fuzzy core of an FXE is defined as the set of all fuzzy allocations which cannot be improved by any coalition of agents. The relationship between the weak fuzzy core and the set of fuzzy competitive allocations of FXE is given as well by replicating the FXE in this paper. Actually, Remark 3.2 indicates that the FXE is a generalized pure exchange economy. In other words, we came to the same concepts and results in this paper as Debreu and Scarf [6] when the FXE degenerates into the pure exchange economy. Finally, future work can be carried out to determine if good results could be generalized by considering different economic models with fuzzy preferences, such as the restricted participation on financial markets proposed by Donato et al. [10] and the abstract economy studied by Rim and Kim [19].

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