KYBERNETIKA — VOLUME 57 (2021), NUMBER 3, PAGES 426–445

A CONTINUOUS MAPPING THEOREM FOR THE ARGMIN-SET FUNCTIONAL WITH APPLICATIONS TO CONVEX STOCHASTIC PROCESSES

Dietmar Ferger

For lower-semicontinuous and convex stochastic processes Z_n and nonnegative random variables ϵ_n we investigate the pertaining random sets $A(Z_n, \epsilon_n)$ of all ϵ_n-approximating minimizers of Z_n. It is shown that, if the finite dimensional distributions of the Z_n converge to some Z and if the ϵ_n converge in probability to some constant c, then the $A(Z_n, \epsilon_n)$ converge in distribution to $A(Z, c)$ in the hyperspace of Vietoris. As a simple corollary we obtain an extension of several argmin-theorems in the literature. In particular, in contrast to these argmin-theorems we do not require that the limit process has a unique minimizing point. In the non-unique case the limit-distribution is replaced by a Choquet-capacity.

Keywords: convex stochastic processes, sets of approximating minimizers, weak convergence, Vietoris hyperspace topologies, Choquet-capacity

Classification: 60B05, 60B10, 60F99

1. INTRODUCTION AND MAIN RESULTS

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $Z : \Omega \times \mathbb{R}^d \to \overline{\mathbb{R}}$ be a bivariate function with values in the extended real line $\overline{\mathbb{R}}$ endowed with the Borel-σ algebra $\overline{\mathcal{B}}$. Such a function is called *stochastic process* or *integrand*, if $Z(\cdot, t) : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ is measurable for every $t \in \mathbb{R}^d$. It is convenient to identify a stochastic process with a function-valued map $Z : \Omega \to \overline{\mathbb{R}}^{\mathbb{R}^d}$. So, $Z(\omega) \equiv Z(\omega, \cdot)$ is a function from $\mathbb{R}^d$ into $\overline{\mathbb{R}}$, which is called the *trajectory* or *path* of Z pertaining to the sample point $\omega \in \Omega$. It takes the value $Z(\omega)(t) \equiv Z(\omega, t)$ at point $t \in \mathbb{R}^d$. Occasionally it is practical to write $Z(t)$ instead of $Z(\cdot, t)$ for this ambiguity in the notation explains in the context.

In this paper we focus on integrands Z which are lower-semicontinuous (lsc) and convex. For a lsc function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ and a real number $r \in \mathbb{R}_+ = [0, \infty)$ let

$$A(f, r) := \{t \in \mathbb{R}^d : f(t) \leq \inf_{s \in \mathbb{R}^d} f(s) + r\}$$

and

$$\text{Argmin}(f) := \{t \in \mathbb{R}^d : f(t) = \inf_{s \in \mathbb{R}^d} f(s)\}.$$

DOI: 10.14736/kyb-2021-3-0426

Thus $A(f,r)$ is the set of all *r-approximating minimizers* of f and $\text{Argmin}(f)$ consists of all minimizers of f. Obviously, $\text{Argmin}(f) = A(f,0)$. By lower-semicontinuity $A(f,r)$ is a closed subset of $\mathbb{R}^d$ (possibly empty), see Lemma 4.1 in the appendix. Consider the space S of all lower-semicontinuous functions from $\mathbb{R}^d$ into the extended real line $\overline{\mathbb{R}}$, i. e.

$$S := \{f : \mathbb{R}^d \to \overline{\mathbb{R}}; f \text{ lsc}\}.$$

Then the assignment $(f,r) \mapsto A(f,r)$ defines a map

$$A : S \times \mathbb{R}_+ \to \mathcal{F}_d,$$

where

$$\mathcal{F}_d := \mathcal{F}(\mathbb{R}^d) := \{F \subseteq \mathbb{R}^d : F \text{ is closed }\}$$

is the family of all closed subsets of $\mathbb{R}^d$. For a fixed lsc integrand Z and a nonnegative random real variable ϵ on $(\Omega, \mathcal{A})$ we have that $A(Z,\epsilon) := A \circ (Z,\epsilon)$ is a map from $(\Omega, \mathcal{A})$ into $\mathcal{F}_d$, or in other words a *$\mathcal{F}_d-$valued random element*.

Now, let (Z_n) be a sequence of lsc and convex stochastic processes accompanied by a sequence (ϵ_n) of nonnegative random variables. Assume that

$$(Z_n(t_1), \ldots, Z_n(t_k)) \xrightarrow{\mathcal{D}} (Z(t_1), \ldots, Z(t_k)) \quad \text{in } \overline{\mathbb{R}}^k \text{ as } n \to \infty, \tag{1}$$

for all $t_1, \ldots, t_k \in D$, where D is any countable and dense subset of $\mathbb{R}^d$ (*convergence of the finite-dimensional distributions on* D). This is denoted by $Z_n \xrightarrow{fd}_D Z$. Further, assume that the sequence (ϵ_n) converges in probability:

$$\epsilon_n \xrightarrow{\mathbb{P}} c, \tag{2}$$

where $c \geq 0$ is a real constant.

We now state our main results. For that purpose let $\mathcal{F}_d = \mathcal{F}(\mathbb{R}^d)$ be endowed with either the *Vietoris topology* $\tau_V = \tau_V(\mathcal{F}_d)$ or the *upper Vietoris topology* $\tau_{uV} = \tau_{uV}(\mathcal{F}_d)$. Here, the Vietoris topology $\tau_V(\mathcal{F}_d)$ is generated through the system $\mathcal{S}_V := \{\mathcal{M}(F) : F \in \mathcal{F}_d\} \cup \{\mathcal{H}(G) : G \in \mathcal{G}_d\}$, where $\mathcal{G}_d$ denotes the class of all open subsets in $\mathbb{R}^d$, $\mathcal{M}(E) := \{F \in \mathcal{F}_d : F \cap E = \emptyset\}$ is the collection of all *missing sets* of a set $E \subseteq \mathbb{R}^d$ and $\mathcal{H}(E) := \{F \in \mathcal{F}_d : F \cap E \neq \emptyset\}$ is the collection of all *hitting sets* of E. The upper Vietoris topology τ_{uV} is generated by the sub-system $\mathcal{S}_{uV} := \{\mathcal{M}(F) : F \in \mathcal{F}_d\}$, whence it is coarser than the Vietoris topology.

The issue is to give minimal conditions such that our *basic assumptions* (1) and (2) ensure distributional convergence of $A(Z_n, \epsilon_n)$ to $A(Z,c)$ in the topological space $(\mathcal{F}_d, \tau_{uV})$ or $(\mathcal{F}_d, \tau_V)$, respectively. These conditions concern the path properties of Z and Z_n. For their description recall that a function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is called *proper* if $f(t) > -\infty$ for all $t \in \mathbb{R}^d$ and $f(t) < \infty$ for at least one $t \in \mathbb{R}^d$. The set $\text{dom } f := \{t \in \mathbb{R}^d : f(t) < \infty\}$ is called the *effective domain* of f. Further, f is *level-bounded*, if for every $\alpha \in \mathbb{R}$ the *level-set* $\{t \in \mathbb{R}^d : f(t) \leq \alpha\}$ is bounded (possibly empty). This is the same as having $f(t) \to \infty$ as $|t| \to \infty$, where $|\cdot|$ is the euclidian norm on $\mathbb{R}^d$. Henceforth, we can introduce the subspaces $S_0 := \{f \in S : f \text{ convex, proper and } \text{int}(\text{dom f}) \neq \emptyset\}$, where $\text{int}(E)$ denotes the interior of $E \subseteq \mathbb{R}^d$ and $S_1 := \{f \in S_0 : f \text{ level-bounded}\}$. It is easy to see that $S_0 = \{f \in S : f \text{ convex and finite on some non-empty open subset}\}$, see Lemma 4.4 in the appendix.

Theorem 1.1. Assume that Z and every Z_n have trajectories in S_0 and that $Z \in S_1$ $\mathbb{P}$-almost surely (a.s.) . Then $Z_n \xrightarrow{fd}_D Z$ and $\epsilon_n \xrightarrow{\mathbb{P}} c$ yield

$$A(Z_n, \epsilon_n) \twoheadrightarrow^{\sim} A(Z, c) \text{ in } (\mathcal{F}_d, \tau_{uV}), \tag{3}$$

where $\twoheadrightarrow^{\sim}$ denotes *convergence in Borel law.* Moreover, $A(Z, c)$ is a.s. non-empty and compact.

Convergence in Borel law is introduced and investigated by Hoffmann-Jørgensen [10]. Now, $\mathcal{S}_{uV}$ is actually a base of τ_{uV}, because $\cap_{i=1}^n \mathcal{M}(F_i) = \mathcal{M}(\cup_{i=1}^n F_i)$ whenever $F_1, \ldots, F_n$ are closed (or even arbitrary) subsets of $\mathbb{R}^d$. Therefore it follows from the Borel Law Portmanteau Theorem of [10] that (3) is equivalent to

$$\limsup_{n\to\infty} \mathbb{P}^*\Big(\bigcap_{F\in\mathcal{F}'} \{A(Z_n, \epsilon_n) \cap F \neq \emptyset\}\Big) \leq \mathbb{P}_*\Big(\bigcap_{F\in\mathcal{F}'} \{A(Z, c) \cap F \neq \emptyset\}\Big) \tag{4}$$

for every sub-collection $\mathcal{F}' \subseteq \mathcal{F}_d$ of closed sets in $\mathbb{R}^d$. Here, $\mathbb{P}^*$ and $\mathbb{P}_*$ denote the outer and inner probability of $\mathbb{P}$. An essential feature of convergence in Borel law is that the involved random elements are simply maps from Ω into $\mathcal{F}_d$ without any measurability requirement. In fact, there is no $\sigma-$algebra on $\mathcal{F}_d$ so far. So, let us endow $\mathcal{F}_d$ with the Borel-σ-algebra $\mathcal{B}_{uV} := \mathcal{B}_{uV}(\mathcal{F}_d) := \sigma(\tau_{uV}(\mathcal{F}_d))$ pertaining to the upper Vietoris topology. The following result sharpens the Borel law convergence (3) to classical weak convergence under the additional assumption that the Z_n in Theorem 1.1 are level bounded as well.

Theorem 1.2. Assume that Z and every Z_n have trajectories in S_1. Then $Z_n \xrightarrow{fd}_D Z$ and $\epsilon_n \xrightarrow{\mathbb{P}} c$ entail

$$A(Z_n, \epsilon_n) \xrightarrow{\mathcal{D}} A(Z, c) \text{ in } (\mathcal{F}_d, \tau_{uV}) \text{ as } n \to \infty. \tag{5}$$

Furthermore, $A(Z, c)$ and all $A(Z_n, \epsilon_n)$ are non-empty and compact.

Note that $(\mathcal{F}_d, \tau_{uV})$ is a topological space, which is not metrizable. Therefore, we need to say a few words about the meaning of (5). Firstly it means that the $A(Z_n, \epsilon_n)$ and $A(Z, c)$ are $\mathcal{A} - \mathcal{B}_{uV}$ measurable maps from Ω into $\mathcal{F}_d$ and secondly that the induced distributions $\mathbb{P} \circ A(Z_n, \epsilon_n)^{-1}$ converge in the weak topology to $\mathbb{P} \circ A(Z, c)^{-1}$. The classical Portmanteau Theorem, see Gänssler and Stute [6], Proposition 8.4.9, or Topsøe [21], Theorem 8.1, then gives that (5) is equivalent to

$$\limsup_{n\to\infty} \mathbb{P}\Big(\bigcap_{F\in\mathcal{F}'} \{A(Z_n, \epsilon_n,) \cap F \neq \emptyset\}\Big) \leq \mathbb{P}\Big(\bigcap_{F\in\mathcal{F}'} \{A(Z, c) \cap F \neq \emptyset\}\Big) \text{ for all } \mathcal{F}' \subseteq \mathcal{F}_d. \tag{6}$$

Our Theorems 1.1 and 1.2 can be viewed as *Continuous Mapping Theorems* for the functional A. They can easily be extended to *asymptotic subsets* C_n of $A(Z_n, \epsilon_n)$. By this we mean a sequence (C_n) of $\mathcal{F}_d-$ valued random elements such that

$$\liminf_{n\to\infty} \mathbb{P}_*(C_n \subseteq A(Z_n, \epsilon_n)) = 1, \tag{7}$$

which in fact is the same as

$$\lim_{n\to\infty} \mathbb{P}_*(C_n \subseteq A(Z_n, \epsilon_n)) = 1. \tag{8}$$

For example, if $C_n \subseteq A(Z_n, \epsilon_n)$ a.s. for eventually all $n \in \mathbb{N}$, then the sequence (C_n) consists of asymptotic subsets.

Corollary 1.3. Let the assumptions of Theorem 1.1 be fulfilled. If $C_n, n \in \mathbb{N}$, are asymptotic subsets of $A(Z_n, \epsilon_n)$, then

$$C_n \twoheadrightarrow^{\sim} A(Z, c) \text{ in } (\mathcal{F}_d, \tau_{uV}). \tag{9}$$

If additionally the C_n are $\mathcal{A} - \mathcal{B}_{uV}$ measurable and $Z \in S_1$, then actually

$$C_n \xrightarrow{\mathcal{D}} A(Z, c) \quad \text{in } (\mathcal{F}_d, \tau_{uV}). \tag{10}$$

Measurability of C_n is guaranteed for instance, if C_n is a convex and bounded *random closed set*, see Proposition 2.11 below. (The notion of *random closed set* will be explained later on). Also, we show that C_n is $\mathcal{A} - \mathcal{B}_{uV}$ measurable, if it consists of finitely many random variables in $\mathbb{R}^d$, see Lemma 4.6 in the appendix.

Next, we ask for convergence in Borel law if $\mathcal{F}_d$ is equipped with the Vietoris-topology τ_V. Since τ_V is finer (stronger) than the upper Vietoris-topology τ_{uV} it does not surprise that here additional assumptions are necessary. In short we need that $c = 0$ and that Z has at most one minimizing point with probability one.

Theorem 1.4. Let the assumptions of Theorem 1.1 be fulfilled with $c = 0$. Further, assume that

$$\text{Argmin}(Z) \subseteq \{\xi\} \text{ a.s. for some random variable } \xi. \tag{11}$$

Then actually

$$\text{Argmin}(Z) = \{\xi\} \ \text{ a.s.} \tag{12}$$

and

$$A(Z_n, \epsilon_n) \twoheadrightarrow^{\sim} \text{Argmin}(Z) \text{ in } (\mathcal{F}_d, \tau_V). \tag{13}$$

If the Z_n in Theorem 1.4 are level-bounded one might expect that (13) could be sharpened to classical weak convergence. However, for this we needed that the underlying random sets are $\mathcal{A} - \mathcal{B}_V$ measurable, which is not self-evident and in fact questionable. Therefore we consider the *Fell-topology* $\tau_F = \tau_F(\mathcal{F}_d)$ on $\mathcal{F}_d$, which is generated by the system $\mathcal{S}_F := \{\mathcal{M}(K), K \in \mathcal{K}_d\} \cup \{\mathcal{H}(G), G \in \mathcal{G}_d\}$, where $\mathcal{K}_d$ is the family of all compact sets in $\mathbb{R}^d$. Since $\mathcal{S}_F \subseteq \mathcal{S}_V$ the Fell-topology is coarser than the Vietoris-topology. The hyperspace $(\mathcal{F}_d, \tau_F)$ is known to be compact, second-countable and Hausdorff and hence it is metrizable. In fact one can specify a metrization δ, e. g., the *Painlevé–Kuratowski-metric*, see Pflug [17].

Theorem 1.5. Let the assumptions of Theorem 1.2 be fulfilled with $c = 0$ and assume that (11) holds. Then (12) is true and

$$A(Z_n, \epsilon_n) \xrightarrow{\mathcal{D}} \text{Argmin}(Z) \text{ in } (\mathcal{F}_d, \tau_F). \tag{14}$$

In applications, e. g., in statistics or in stochastic optimization, one considers *measurable selections* ξ_n of $A(Z_n, \epsilon_n)$, that means $\xi_n : (\Omega, \mathcal{A}) \to (\mathbb{R}^d, \mathcal{B}_d)$ is measurable with $\xi_n \in A(Z_n, \epsilon_n)$ a.s. Here, $\mathcal{B}_d = \mathcal{B}(\mathbb{R}^d)$ is the Borel-σ algebra on $\mathbb{R}^d$.

Theorem 1.6. Assume that $Z_n \in S_0$ for every $n \in \mathbb{N}$ and $Z \in S_1$. For every $n \in \mathbb{N}$ let ξ_n be a measurable selection of $A(Z_n, \epsilon_n)$. Then $Z_n \xrightarrow{fd}_D Z$ and $\epsilon_n \xrightarrow{\mathbb{P}} c$ ensure that

$$\{\xi_n\} \xrightarrow{\mathcal{D}} A(Z, c) \text{ in } (\mathcal{F}_d, \tau_{uV}). \tag{15}$$

Moreover, it follows that

$$\limsup_{n\to\infty} \mathbb{P}(\xi_n \in F) \le T(F) \text{ for all closed subsets } F \subseteq \mathbb{R}^d, \tag{16}$$

where T is the *Choquet-capacity functional* of the *random closed* set $A(Z, c)$, that is

$$T(F) = \mathbb{P}(A(Z, c) \cap F \neq \emptyset),\ F \in \mathcal{F}_d. \tag{17}$$

Here, a *random closed set (in $\mathbb{R}^d$)* is a measurable map $C : (\Omega, \mathcal{A}) \to (\mathcal{F}_d, \mathcal{B}_{uF})$, where $\mathcal{B}_{uF} = \mathcal{B}_{uF}(\mathcal{F}_d)$ is the Borel$-\sigma$ algebra induced by the *upper Fell-topology* $\tau_{uF} = \tau_{uF}(\mathcal{F}_d)$. This is the topology on $\mathcal{F}_d$ generated by $\{\mathcal{M}(K) : K \in \mathcal{K}_d\} \subseteq \mathcal{S}_{uV}$. Thus τ_{uF} is coarser than τ_{uV}, whence $\mathcal{B}_{uF} \subseteq \mathcal{B}_{uV}$. Therefore $A(Z, c)$ is a random closed set in $\mathbb{R}^d$, since it is even $\mathcal{A} - \mathcal{B}_{uV}$ measurable by Theorem 1.2. As any capacity functional our T can be extended to the Borel-σ algebra $\mathcal{B}_d$ such that (17) holds for all Borel-sets $F \in \mathcal{B}_d$, see, e. g., Molchanov [16]. So, formally (16) looks exactly like the characterization of weak convergence given in the Portmanteau-Theorem. However, $T : \mathcal{B}_d \to [0, 1]$ in general is not a probability measure, since it lacks additivity. Consequently, we can not deduce weak convergence for the random points ξ_n at least as long as T is not a probability measure. On the other hand, if $c = 0$ and $A(Z, 0) =$Argmin(Z) consists of a single random variable ξ, which means that Z has a **unique minimizer**, then T is equal to the distribution of ξ and (16) is the same as $\xi_n \xrightarrow{\mathcal{D}} \xi$. To sum up, in the unique case we obtain classical weak convergence, whereas in the **non-unique case** the ξ_n *converge weakly to a Choquet-capacity* under which we exactly mean (16), see Ferger [5] for a detailed characterization of this generalized concept of weak convergence. A distinction between the two cases is no longer necessary when considering the **sets** $\{\xi_n\}$ instead of the single points ξ_n. In either case we have weak convergence of the **singletons** $\{\xi_n\}$ in the hyperspace $\mathcal{F}_d$ endowed with the upper Vietoris topology. Thus this topology matches perfectly in our framework.

As our short discussion of Theorem 1.6 reveals the special case $c = 0$ plays a peculiar role. The uniqueness condition occurring there can be slightly weakened:

Theorem 1.7. Let Z and $Z_n, n \in \mathbb{N}$, be with trajectories in S_0. Further assume that $Z \in S_1$ a.s. and that

$$\text{Argmin}(Z) \subseteq \{\xi\} \text{ a.s. for some random variable } \xi. \tag{18}$$

If $Z_n \xrightarrow{fd}_D Z$ and $\epsilon_n \xrightarrow{\mathbb{P}} 0$, then for every sequence (ξ_n) of random variables with $\xi_n \in A(Z_n, \epsilon_n)$ a.s. we can infer that

$$\xi_n \xrightarrow{\mathcal{D}} \xi \quad \text{in } \mathbb{R}^d. \tag{19}$$

Notice: If $Z_n \in S_1$ then $A(Z_n, \epsilon_n)$ is non-empty and $\mathcal{A} - \mathcal{B}_{uV}$ measurable by Theorem 1.2. In particular, it is a non-empty random closed set. The Fundamental selection theorem, see Molchanov [16], then guarantees the existence of a measurable selection ξ_n.

Special cases of Theorem 1.7 include former results of the literature. We start with:

Corollary 1.8. (Geyer [7]) Let Z and $Z_n, n \in \mathbb{N}$, be with trajectories in S_0, where Z a.s. possesses the random variable ξ as its unique minimizing point. Consider non-negative constants c_n converging to zero and random variables ξ_n which are the c_n-approximating minimizers of Z_n. Then $Z_n \xrightarrow{fd}_D Z$ implies

$$\xi_n \xrightarrow{\mathcal{D}} \xi \quad \text{in } \mathbb{R}^d. \tag{20}$$

This result goes back to Geyer [7]. It is well-known in the statistical literature and has been cited in more than 100 contributions even though the paper of Geyer [7] is an unpublished manuscript. For the special choice $c_n = 0$ the ξ_n in (20) are the minimizers of the Z_n. The great utility of Corollary 1.8 has been demonstrated, e. g., by Chernozhukov [3], Geyer [7], Knight [12], [13], [14] or Wagener and Dette [22] to mention only a few. For example Knight [14] rediscovers Smirnov's [20] four types of all possible limiting distributions for quantile-estimators. Here, it is inevitable that the limit process Z may assume the value infinity. Indeed, stochastic processes taking the value infinity arise canonically in stochastic optimization problems with constraints, see Pflug [17], [18] and Knight [13]. In contrast, Davis, Knight and Liu [4] exclude this profitable case, since they only investigate *real-valued* stochastic processes $Z_n, Z : \Omega \times \mathbb{R}^d \to \mathbb{R}$ with convex trajectories.

Corollary 1.9. (Davis et al. [4]) Let Z and $Z_n, n \in \mathbb{N}$, be real-valued and convex stochastic processes and let ξ_n minimize Z_n and ξ minimize Z, where ξ is unique with probability 1. If

$$(Z_n(t_1), \ldots, Z_n(t_k)) \xrightarrow{\mathcal{D}} (Z(t_1), \ldots, Z(t_k)) \quad \text{in } \mathbb{R}^k \text{ as } n \to \infty, \tag{21}$$

for all $t_1, \ldots, t_k \in \mathbb{R}^d$, then

$$\xi_n \xrightarrow{\mathcal{D}} \xi \quad \text{in } \mathbb{R}^d. \tag{22}$$

Haberman [8] investigates a very broad class of M-estimators based on convex criterion functions. His proof of asymptotic normality (Theorem 6.1) is rather long, but using the above corollary can make it much less difficult.

A significant simplification of assumption (21) is possible, if the Z_n allow a second-order expansion.

Corollary 1.10. (Hjort and Pollard [9]) Let $Z_n, n \in \mathbb{N}$, be real-valued and convex stochastic processes and let ξ_n minimize Z_n. Assume there exists a sequence (U_n) of random vectors with $U_n \xrightarrow{\mathcal{D}} U$ in $\mathbb{R}^d$, and a sequence (V_n) of matrices with $V_n \xrightarrow{\mathbb{P}} V$, where V is positive definite. If Z_n has the representation

$$Z_n(t) = U_n' t + \frac{1}{2} t' V_n t + r_n(t),$$

where $r_n(t) \xrightarrow{\mathbb{P}} 0$ for every $t \in \mathbb{R}^d$, then

$$\xi_n \xrightarrow{\mathcal{D}} -V^{-1}U \quad \text{in } \mathbb{R}^d. \tag{23}$$

The paper of Hjort and Pollard [9] is also an unpublished manuscript, whence we refer to Theorem 7.133 of Liese and Mieschke [15], who present a proof by following the ideas of Hjort and Pollard [9]. Also notice that in contrast to Hjort and Pollard [9] we *do not* require that the matrices V_n are positive definite.

The paper is organized as follows: In section 2 we endow the function space S with the *epi-metric* e, which corresponds to *epi-convergence.* This type of convergence is known to be most suitable for minimization problems. According to Attouch [1] the metric space (S, e) is second countable (and compact). If $\mathcal{B}_e(S)$ denotes the Borel-σ-algebra induced be e, it turns out that measurability of a map $Z : (\Omega, \mathcal{A}) \to (S, \mathcal{B}_e(S))$ is exactly the same as being a *normal integrand* in the sense of Rockafellar and Wets [19]. This link to the theory of normal integrands enables us to deduce that every lsc and convex stochastic process (which has an effective domain with non-empty interior) is Borel-measurable. A first fundamental result in section 2 gives conditions under which the map A is τ_{uV}-continuous or τ_V-continuous, respectively. As a consequence we obtain that for a stochastic process Z with trajectories in S_1 and a non-negative random variable ϵ the random set $A(Z, \epsilon)$ is $\mathcal{A} - \mathcal{B}_{uV}$-measurable and in particular this holds for $\mathrm{Argmin}(Z)$. Next, for a countable and dense subset $D = \{t_i : i \in \mathbb{N}\}$ of $\mathbb{R}^d$ we consider the *projection* $\pi_D(f) := (f(t_i) : i \in \mathbb{N})$, $f \in S_0$, and show that it is a homeomorphism from (S_0, e) onto its range equipped with the metric ϱ of coordinatewise convergence. This leads to our second fundamental result in section 2, namely that $Z_n \xrightarrow{fd}_D Z$ with Z_n and Z in S_0 guarantees *epi-convergence in distribution*, i. e., $Z_n \xrightarrow{\mathcal{D}} Z$ in (S_0, e). Finally, section 3 contains the proofs of our main theorems, where we just combine the results of section 2 with the Continuous Mapping Theorem. Several technical lemmas, mainly about convex functions, are deferred in the appendix (section 4).

2. CONTINUITY OF THE FUNCTIONAL A AND EPI-CONVERGENCE IN DISTRIBUTION

A sequence $(f_n) \subseteq S$ of lsc functions *epi-converges* to some $f \in S$ ($f_n \rightarrow_{epi} f$) if at each $x \in \mathbb{R}^d$ one has

$$\liminf_{n\to\infty} f_n(x_n) \geq f(x) \quad \text{for every sequence } x_n \to x, \tag{24}$$

and

$$\limsup_{n\to\infty} f_n(x_n) \leq f(x) \quad \text{for at least one sequence } x_n \to x. \tag{25}$$

Epi-convergence can equivalently be described by convergence of the pertaining *epigraphs* in the hyperspace $(\mathcal{F}_{d+1}, \tau_F)$. To see this recall that for a function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$, the *epigraph* of f is the set

$$\text{epi}(f) := \{(x, \alpha) \in \mathbb{R}^d \times \mathbb{R} : f(x) \leq \alpha\}.$$

The crucial point is that every function f is uniquely determined by its epigraph. Indeed, we have that:

Lemma 2.1. If f and g are functions from $\mathbb{R}^d$ into $\overline{\mathbb{R}}$ with epi(f)=epi(g), then $f = g$. In other words the map with $\phi(f)$:=epi(f) is an injection from $\overline{\mathbb{R}}^{\mathbb{R}^d}$ into the power set of $\mathbb{R}^d \times \mathbb{R}$.

A proof is given at the end of the appendix. Another well-known fact says that f is lsc if and only if epi(f) is a closed subset of $\mathbb{R}^d \times \mathbb{R} \equiv \mathbb{R}^{d+1}$. Let $\mathcal{F}_{d+1} = \mathcal{F}(\mathbb{R}^{d+1})$ be equipped with the Fell-topology $\tau_F = \tau_F(\mathcal{F}_{d+1})$. The next result follows from Theorem 2.78 and Proposition 1.14 of Attouch [1].

Theorem 2.2. (Attouch [1]) For every sequence $(f_n)_{n\in\mathbb{N}}$ in S the following equivalence holds:

$$f_n \rightarrow_{epi} f \quad \Leftrightarrow \quad \text{epi}(f_n) \to \text{epi}(f) \quad \text{in} \quad (\mathcal{F}_{d+1}, \tau_F). \tag{26}$$

Let $\mathcal{E} := \{\text{epi}(f) : f \in S\}$ be the system of all epigraphs of lsc functions from $\mathbb{R}^d$ into $\overline{\mathbb{R}}$. As mentioned above $\mathcal{E} \subseteq \mathcal{F}_{d+1}$ and from Lemma 2.1 it follows that the map $\phi : S \to \mathcal{E}$ given by $\phi(f) := \text{epi}(f), f \in S$, is a bijection. Attouch [1], p.254-255, proves that $\mathcal{E}$ is compact for the Fell-topology $\tau_F(\mathcal{F}_{d+1})$ or in other words that $(\mathcal{E}, \delta)$ is a compact metric space. Recall that δ is a metrization for $\tau_F(\mathcal{F}_{d+1})$. Since $(\mathcal{F}_{d+1}, \delta)$ is second countable and therefore separable, this property applies to the subspace $(\mathcal{E}, \delta)$ as well.

Define the *epi-metric* $e : S \times S \to \mathbb{R}$ by $e(f, g) := \delta(\phi(f), \phi(g))$. Summing up we obtain from Lemma 2.1 and Theorem 2.2:

Proposition 2.3. The epi-metric e is a metric on S such that convergence in (S, e) coincides with epi-convergence, i. e., e is a metrization of epi-convergence. Moreover, $\phi : (S, e) \to (\mathcal{E}, \delta)$ is a homeomorphism, and in particular (S, e) and $(\mathcal{E}, \delta)$ are compact and separable metric spaces.

Rockafellar and Wets [19] define a *normal integrand (on* $(\Omega,\mathcal{A})$*)* as follows: it is a function-valued map $Z:(\Omega,\mathcal{A})\to\overline{\mathbb{R}}^{\mathbb{R}^d}$ such that $\phi\circ Z$ is a random closed set in $\mathbb{R}^{d+1}$. Especially it follows that $\text{epi}(Z(\omega))=\phi(Z(\omega))=\phi\circ Z(\omega)$ is closed in $\mathbb{R}^{d+1}$ and thus $Z(\omega)$ is lsc for every $\omega\in\Omega$. In fact, by Proposition 14.28 of Rockafellar and Wets [19] Z is not only lsc but actually a stochastic process (integrand). The other direction needs not to be true: Not every lsc integrand is a normal integrand. However, the following lemma gives a sufficient condition for normality.

Lemma 2.4. Assume that Z is a lsc convex stochastic process with $\text{int}(\text{dom } Z(\omega))\neq\emptyset$ for all $\omega\in\Omega$ with $\text{dom } Z(\omega)\neq\emptyset$ (as for instance when $Z\in S_0$). Then Z is a normal integrand.

Proof. This is the second part of Theorem 14.39 of Rockafellar and Wets [19]. □

We shall see that a normal integrand is nothing else but a Borel-measurable map from $(\Omega,\mathcal{A})$ into the metric space (S,e).

Lemma 2.5. Let $\mathcal{B}_e(S)$ be the Borel-σ algebra on (S,e). Then Z is a normal integrand if and only if $Z:(\Omega,\mathcal{A})\to(S,\mathcal{B}_e(S))$ is measurable.

Proof. Let τ_δ denote the topology on $\mathcal{F}_{d+1}$ induced by the Painlevé–Kuratowski metric δ. We already mentioned above that τ_δ coincides with the Fell-topology τ_F on $\mathcal{F}_{d+1}$, whence the corresponding Borel-σ algebras $\mathcal{B}_\delta(\mathcal{F}_{d+1}):=\sigma(\tau_\delta)$ and $\mathcal{B}_F(\mathcal{F}_{d+1}):=\sigma(\tau_F)$ coincide as well. Further, recall that $\mathcal{E}$ is compact in $(\mathcal{F}_{d+1},\tau_F)$ and in particular $\mathcal{E}\in\mathcal{B}_F(\mathcal{F}_{d+1})$. For the Borel-$\sigma$ algebra $\mathcal{B}_\delta(\mathcal{E})$ on $(\mathcal{E},\delta)$ we therefore obtain

$$\mathcal{B}_\delta(\mathcal{E})=\sigma(\mathcal{E}\cap\tau_\delta)=\mathcal{E}\cap\sigma(\tau_\delta)=\mathcal{E}\cap\mathcal{B}_F(\mathcal{F}_{d+1})\subseteq\mathcal{B}_F(\mathcal{F}_{d+1}). \tag{27}$$

Here, the first equality holds by definition and the second one is valid according to Lemma 1.6 in Kallenberg [11]. It is well-known, see, e. g., Molchanov [16], that $\mathcal{B}_F(\mathcal{F}_{d+1})=\sigma(\{\mathcal{M}(K):K\in\mathcal{K}_{d+1}\})$, whence $\mathcal{B}_{uF}(\mathcal{F}_{d+1})=\mathcal{B}_F(\mathcal{F}_{d+1})$ and thus every random closed set C in $\mathbb{R}^{d+1}$ can alternatively be conceived as a measurable map $C:(\Omega,\mathcal{A})\to(\mathcal{F}_{d+1},\mathcal{B}_F(\mathcal{F}_{d+1}))$.

Now suppose that Z is a normal integrand. By definition and our last conclusion this means that $\phi\circ Z:(\Omega,\mathcal{A})\to(\mathcal{F}_{d+1},\mathcal{B}_F(\mathcal{F}_{d+1}))$ is measurable. But ϕ maps into $\mathcal{E}$, whence it follows from (27) that $\phi\circ Z:(\Omega,\mathcal{A})\to(\mathcal{E},\mathcal{B}_\delta(\mathcal{E}))$ is measurable. By Proposition 2.3 ϕ^{-1} is a continuous map and therefore it is $\mathcal{B}_\delta(\mathcal{E})-\mathcal{B}_e(S)$ measurable. Since $Z=\phi^{-1}\circ(\phi\circ Z)$ we can infer that Z is $\mathcal{A}-\mathcal{B}_e(S)$ measurable as a composition of measurable maps.

For the other direction notice that by Proposition 2.3 ϕ is continuous and hence

$$\phi \text{ is } \mathcal{B}_e(S)-\mathcal{B}_\delta(\mathcal{E}) \text{ measurable.} \tag{28}$$

Let $\mathbf{B}\in\mathcal{B}_F(\mathcal{F}_{d+1})$ be an arbitrary Borel-set. It has the inverse image $\phi^{-1}(\mathbf{B})=\phi^{-1}(\mathcal{E}\cap\mathbf{B})$, where $\mathcal{E}\cap\mathbf{B}\in\mathcal{B}_\delta(\mathcal{E})$ by the equalities in (27). Thus $\phi^{-1}(\mathbf{B})\in\phi^{-1}(\mathcal{B}_\delta(\mathcal{E}))\subseteq\mathcal{B}_e(S)$ by (28), whence $(\phi\circ Z)^{-1}(\mathbf{B})=Z^{-1}(\phi^{-1}(\mathbf{B}))\in\mathcal{A}$ for Z is $\mathcal{A}-\mathcal{B}_e(S)$ measurable by assumption. This shows that $\phi\circ Z$ is a random closed set and hereby Z is a normal integrand. □

Corollary 2.6. Fix a subspace U of (S, e) and assume that Z is a normal integrand on $(\Omega, \mathcal{A})$ with trajectories in U. Then $Z : (\Omega, \mathcal{A}) \to (U, \mathcal{B}_e(U))$ is measurable.

Proof. Let $B \in \mathcal{B}_e(U)$. Since $\mathcal{B}_e(U) = U \cap \mathcal{B}_e(S)$ by Lemma 1.6 in Kallenberg [11] it follows that $B = U \cap \tilde{B}$ for some $\tilde{B} \in \mathcal{B}_e(S)$. We thus can infer that

$$Z^{-1}(B) = Z^{-1}(U) \cap Z^{-1}(\tilde{B}) = \Omega \cap Z^{-1}(\tilde{B}) = Z^{-1}(\tilde{B}) \in \mathcal{A}$$

by Lemma 2.5. □

Notice: If Z is a stochastic process with trajectories in S_0, then it is a normal integrand by Lemma 2.4, whence by Corollary 2.6 it is $\mathcal{A} - \mathcal{B}_e(S_0)$ measurable, which in turn is equivalent to $\mathcal{A} - \mathcal{B}_e(S)$-measurability. Therefore, given a sequence (Z_n) of stochastic processes with values in S_0, the measurability requirement in the definition of distributional convergence $Z_n \xrightarrow{\mathcal{D}} Z$ in (S_0, e) and in (S, e) is fulfilled.

The following lemma gives an equivalent description for convergence in the Vietoris-topology τ_V and in the upper-Vietoris topology τ_{uV}.

Lemma 2.7. Let F and $F_n, n \in \mathbb{N}$, be closed subsets of $\mathbb{R}^d$.
(1) The following statements (a) and (b) are equivalent:

(a) $F_n \to F$ in $(\mathcal{F}_d, \tau_V)$.

(b) The *miss-criterion* (b1) and the *hit-criterion* (b2) are satisfied, where

(b1) For every $H \in \mathcal{F}_d$ with $F \cap H = \emptyset$ there exists a natural number n_0 such that $F_n \cap H = \emptyset$ for all $n \geq n_0$,

(b2) For every $G \in \mathcal{G}_d$ with $F \cap G \neq \emptyset$ there exists a natural number n_1 such that $F_n \cap G \neq \emptyset$ for all $n \geq n_1$.

(2) $F_n \to F$ in $(\mathcal{F}_d, \tau_{uV})$ if and only if the miss-criterion (b1) holds.

Proof. Both equivalences follow immediately from the definitions of the respective topologies upon noticing that for checking convergence it suffices to consider subbase-neighbourhoods. □

With the help of Lemma 2.7 we prove continuity of the map A. This plays a fundamental role in our paper. Here we deal with the superset $U_0 := \{f \in S : f \text{ convex and proper } \} \supseteq S_0$.

Theorem 2.8. Let u be the usual metric on $\mathbb{R}_+$ and $e \times u$ be the product-metric on $S \times \mathbb{R}_+$. Then:
(1) $A : (U_0 \times \mathbb{R}_+, e \times u) \to (\mathcal{F}_d, \tau_{uV})$ is continuous on $U \times \mathbb{R}_+$, where

$$U = \{f \in S : f \text{ convex, proper and level-bounded}\}.$$

(2) $A : (U_0 \times \mathbb{R}_+, e \times u) \to (\mathcal{F}_d, \tau_V)$ is continuous on $U^* \times \{0\}$, where

$$U^* = \{f \in S : f \text{ convex, proper with unique minimizer}\} \subseteq U.$$

P r o o f. (1) Let $(f,r) \in U \times \mathbb{R}_+$ and $(f_n, r_n)_{n\in\mathbb{N}}$ be a sequence in $U_0 \times \mathbb{R}_+$ with $(f_n, r_n) \to_{e\times d} (f,r)$. Convergence by components and Proposition 2.3 yield that $f_n \to_{epi} f$ and $r_n \to r$. By Exercise 7.32(c) in Rockafellar and Wets [19] the sequence $(f_n)_{n\in\mathbb{N}}$ is *eventually level-bounded*, that means there exists some $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$ we have that:

$$\forall\, \alpha \in \mathbb{R}\ \exists\, K = K_\alpha \in \mathcal{K}_d \text{ such that } \{f_n \le \alpha\} \subseteq K. \tag{29}$$

Notice that $K = K_\alpha$ does not depend on n. Lemma 4.3 in the appendix ensures that

$$A(f_n, r_n) \neq \emptyset \quad \text{for all } n \ge n_0. \tag{30}$$

Furthermore Theorem 7.33 in Rockafellar and Wets [19] guarantees the convergence

$$\inf_{t\in\mathbb{R}^d} f_n(t) \to \inf_{t\in\mathbb{R}^d} f(t) \in \mathbb{R}. \tag{31}$$

Let $t \in A(f_n, r_n)$. Then $f_n(t) \le \inf_{t\in\mathbb{R}^d} f_n(t) + r_n$ and by (31) there exists an integer $n_1 \in \mathbb{N}$ such that $\inf_{t\in\mathbb{R}^d} f_n(t) \le \inf_{t\in\mathbb{R}^d} f(t) + 1$ for all $n \ge n_1$. Moreover, since $r_n \to r$ we have that $r_n \le r+1$ for all $n \ge n_2$ for some $n_2 \in \mathbb{N}$. Thus $f_n(t) \le \inf_{t\in\mathbb{R}^d} f(t)+2+r =: \alpha \in \mathbb{R}$ for all $n \ge n_3 := n_1 \vee n_2 \in \mathbb{N}$. Conclude that $A(f_n, r_n) \subseteq \{f_n \le \alpha\}$ for all $n \ge n_3$. With K and n_0 as in (29) plus $n_4 := n_0 \vee n_3 \in \mathbb{N}$ we obtain that

$$A(f_n, r_n) \subseteq K \quad \forall\, n \ge n_4. \tag{32}$$

In order to verify the miss-criterion (b1) of Lemma 2.7 let us consider an arbitrary closed set $H \in \mathcal{F}_d$ with $A(f,r) \cap H = \emptyset$. Then a fortiori

$$A(f,r) \cap H \cap K = \emptyset. \tag{33}$$

We shall show that

$$A(f_n, r_n) \cap H \cap K = \emptyset \quad \forall\, n \ge n_5 \text{ for some } n_5 \in \mathbb{N}. \tag{34}$$

Assume that (34) is not true, i. e., there exists a subsequence $(n_j)_{j\in\mathbb{N}} \subseteq \mathbb{N}$ such that $A(f_{n_j}, r_{n_j}) \cap H \cap K \neq \emptyset$ for all $j \in \mathbb{N}$. Then one can find a sequence $(x_{n_j})_{j\in\mathbb{N}} \subseteq H \cap K$ such that $x_{n_j} \in A(f_{n_j}, r_{n_j})$ for each $j \in \mathbb{N}$. Since $H \cap K$ is compact, the sequence $(x_{n_j})_{j\in\mathbb{N}}$ has a subsequence $(x_{n_{j_l}})_{l\in\mathbb{N}}$ with $x_{n_{j_l}} \to x \in H \cap K$ as $l \to \infty$. For notational convenience we take $x_{n_j} \to x, j \to \infty$ for granted. It follows that

$$x \in A(f,r). \tag{35}$$

Indeed, assume that $x \notin A(f,r)$, i. e., $f(x) > \inf_{t\in\mathbb{R}^d} f(t) + r$, whence

$$f(x) > f(y) + r \text{ for some } y \in \mathbb{R}^d. \tag{36}$$

Recall that $f_n \to_{epi} f$. Thus by (24) and (25) there exists a sequence (y_n) with $y_n \to y$ such that $f(y) = \lim_{n\to\infty} f_n(y_n)$. Conclude from (36) that

$$f(x) > \lim_{n\to\infty} f_n(y_n) + r = \liminf_{j\to\infty} f_{n_j}(y_{n_j}) + r. \tag{37}$$

Now $x_{n_j} \in A(f_{n_j}, r_{n_j})$ entails $f_{n_j}(y_{n_j}) \geq f_{n_j}(x_{n_j}) - r_{n_j}$ and so

$$\liminf_{j\to\infty} f_{n_j}(y_{n_j}) \geq \liminf_{j\to\infty} f_{n_j}(x_{n_j}) - r \geq f(x) - r, \tag{38}$$

where the last inequality holds by (24), because $x_{n_j} \to x$ and by Proposition 2.3 the subsequence $(f_{n_j})_{j\in\mathbb{N}}$ epi-converges to f as well. Combining (37) and (38) leads to $f(x) > (f(x) - r) + r = f(x)$, a contradiction. Thus relation (35) is true and since $x \in H \cap K$ we arrive at $A(f, r) \cap H \cap K \neq \emptyset$, which is a contradiction to (33). This shows (34).

For $n_6 := n_4 \vee n_5 \in \mathbb{N}$ observe that $A(f_n, r_n) = A(f_n, r_n) \cap K$ by (32) for all $n \geq n_6$, whence (34) yields:

$$A(f_n, r_n) \cap H = A(f_n, r_n) \cap H \cap K = \emptyset \quad \forall\, n \geq n_6,$$

whence the miss-criterion (b1) in Lemma 2.7(2) is fulfilled and therefore $A(f_n, r_n) \to A(f, r)$ in $(\mathcal{F}_d, \tau_{uV})$. This shows continuity of A at every point $(f, r) \in U \times \mathbb{R}_+$.

(2) Let $(f_n, r_n) \to_{e\times d} (f, 0)$ with $f \in U^*$. It follows from Lemma 4.2 in the appendix that f is level-bounded and thus $U^* \subseteq U$. In particular, the missing-criterion (b1) is fulfilled by (1) above. Therefore it remains to show the hit-criterion (b2) in Lemma 2.7. For that purpose let $G \in \mathcal{G}_d$ with $A(f, 0) \cap G \neq \emptyset$. We have to show that

$$A(f_n, r_n) \cap G \neq \emptyset \quad \text{for eventually all } n \in \mathbb{N}. \tag{39}$$

Assume that (39) does not hold, i. e., there exists a subsequence $(n_j)_{j\in\mathbb{N}}$ of the natural numbers such that $A(f_{n_j}, r_{n_j}) \cap G = \emptyset$ or equivalently $A(f_{n_j}, r_{n_j}) \subseteq G^c$ for every $j \in \mathbb{N}$, where $G^c := E \setminus G$ denotes the complement of G in E. From (32) we can deduce that there exist a compact set K and a $j_0 \in \mathbb{N}$ such that $A(f_{n_j}, r_{n_j}) \subseteq K$ for all $j \geq j_0$ and consequently $A(f_{n_j}, r_{n_j}) \subseteq G^c \cap K$ for all $j \geq j_0$. By (30) there exists a $j_1 \in \mathbb{N}$ such that $A(f_{n_j}, r_{n_j}) \neq \emptyset$ for all $j \geq j_1$. Put $j_2 = j_0 \vee j_1 \in \mathbb{N}$. Then for every $j \geq j_2$ there exists some $z_{n_j} \in A(f_{n_j}, r_{n_j}) \subseteq G^c \cap K$. Since G is open $G^c \cap K$ is compact, whence w.l.o.g. we may assume that $z_{n_j} \to z \in G^c \cap K$ as $j \to \infty$. Now, $z_{n_j} \in A(f_{n_j}, r_{n_j})$ means that $f_{n_j}(z_{n_j}) \leq \inf_{s\in\mathbb{R}^d} f_{n_j}(s) + r_{n_j}$ for all $j \geq j_2$. From $f_{n_j} \to_{epi} f$ it follows with (24) that

$$f(z) \leq \liminf_{j\to\infty} f_{n_j}(z_{n_j}) \leq \liminf_{j\to\infty} \inf_{s\in\mathbb{R}^d} f_{n_j}(s) + \liminf_{j\to\infty} r_{n_j} = \inf_{s\in\mathbb{R}^d} f(s),$$

where the last equality holds by (31) and $r_n \to 0$. Conclude that $z \in A(f, 0)$, where by $f \in U^*$ the argmin-set $A(f, 0) =$Argmin(f) is a singleton. Hence $A(f, 0) = \{z\}$. However, recall that $A(f, 0) \cap G \neq \emptyset$, which results in $z \in G$ in contradiction to $z \in G^c \cap K$. □

Proposition 2.9. Let Z be a stochastic process with trajectories in S_1 and let ϵ be a $\mathbb{R}_+$-valued random variable both defined on $(\Omega, \mathcal{A})$. Then $A(Z, \epsilon) = A \circ (Z, \epsilon)$ is a $\mathcal{A} - \mathcal{B}_{uV}$ measurable map from Ω into $\mathcal{F}_d$.

P r o o f . By Lemma 2.4 Z is a normal integrand and therefore by Corollary 2.6 it is a $\mathcal{A} - \mathcal{B}_e(S_1)$ measurable map from Ω into S_1. Thus $(Z, \epsilon) : (\Omega, \mathcal{A}) \to (S_1 \times \mathbb{R}_+, \mathcal{B}_e(S_1) \otimes \mathcal{B}_u(\mathbb{R}_+))$ is measurable. It follows from Proposition 2.3 that the subspace (S_1, e) is separable, and clearly $(\mathbb{R}_+, u)$ is also separable. Consequently

$$\mathcal{B}_e(S_1) \otimes \mathcal{B}_u(\mathbb{R}_+) = \mathcal{B}_{e\times u}(S_1 \times \mathbb{R}_+). \tag{40}$$

By $S_1 \subseteq U$ Theorem 2.8 ensures that the restriction $A : (S_1 \times \mathbb{R}_+, e \times u) \to (\mathcal{F}_d, \tau_{uV})$ is continuous and consequently $\mathcal{B}_{e\times u}(S_1 \times \mathbb{R}_+) - \mathcal{B}_{uV}$ measurable. The assertion now follows from (40), which shows that $A \circ (Z, \epsilon)$ is a composition of measurable maps. □

In view of $\mathrm{Argmin}(Z) = A(Z, 0)$ we immediately obtain

Corollary 2.10. If Z is a stochastic process with trajectories in S_1, then $\mathrm{Argmin}(Z)$ is $\mathcal{A} - \mathcal{B}_{uV}$ measurable.

Next, we seek conditions under which a random closed set $C : \Omega \to \mathcal{F}_d$ is actually $\mathcal{A} - \mathcal{B}_{uV}$ measurable. An answer is given in

Proposition 2.11. If the random closed set $C : \Omega \to \mathcal{F}_d$ is convex and bounded with $\mathrm{int}(C) \neq \emptyset$, then it is $\mathcal{A} - \mathcal{B}_{uV}$ measurable.

P r o o f . Consider the special indicator function

$$Z(\omega, t) := \delta_{C(\omega)}(t) := \begin{cases} 0, & t \in C(\omega) \\ \infty, & t \neq C(\omega). \end{cases}$$

Observe that for each fixed $t \in \mathbb{R}^d$ and every $\alpha \in \mathbb{R}$ the set $\{\omega \in \Omega : Z(\omega, t) \leq \alpha\}$ is equal to $\{\omega \in \Omega : t \in C(\omega)\}$, if $\alpha \geq 0$ and it is equal to $\emptyset$, if $\alpha < 0$. Recall that $\mathcal{B}_F := \mathcal{B}_F(\mathcal{F}_d) = \sigma(\{\mathcal{M}(K) : K \in \mathcal{K}_d\})$. Therefore $\{\omega \in \Omega : t \in C(\omega)\} = \{\omega \in \Omega : C(\omega) \cap \{t\} \neq \emptyset\} \in \mathcal{A}$, since $\{t\} \in \mathcal{K}_d$. This shows that Z is an integrand (stochastic process). Similarly, one sees that for each fixed $\omega \in \Omega$ the level-set $\{t \in \mathbb{R}^d : Z(\omega, t) \leq \alpha\}$ is equal to $C(\omega)$ or $\emptyset$ according as $\alpha \geq 0$ or $\alpha < 0$. Consequently Z is level-bounded, because C is bounded by assumption. Furthermore, $\mathrm{epi}(Z(\omega)) = C(\omega) \times [0, \infty)$ is a closed subset of $\mathbb{R}^d \times \mathbb{R}$, whence Z is lsc. It is easy to check that Z is also convex and proper with dom $Z = C$. To sum up, Z is a an integrand with trajectories in S_1. Thus Corollary 2.10 yields the assumption upon noticing that $\mathrm{Argmin}(Z) = C$. □

Let $D = \{t_i : i \in \mathbb{N}\}$ be a countable and dense subset of $\mathbb{R}^d$. We define the projection-map $\pi_D : S_0 \to \overline{\mathbb{R}}^\infty$ by $\pi_D(f) := (f(t_i) : i \in \mathbb{N})$, $f \in S_0$. Let ϱ be the metric of coordinatewise convergence on $\overline{\mathbb{R}}^\infty$ or in other words ϱ is the product-metric pertaining to the metric on $\overline{\mathbb{R}}$. Further let $R := \pi_D(S_0) \subseteq \overline{\mathbb{R}}^\infty$ be the range of π_D. We obtain:

Theorem 2.12. For every countable and dense subset D the corresponding projection-map $\pi_D : (S_0, e) \to (R, \varrho)$ is bijective and its inverse $\pi_D^{-1} : (R, \varrho) \to (S_0, e)$ is continuous.

P r o o f. Write $\pi = \pi_D$ for short. For the first assertion it suffices to show that π is injective. So, assume $\pi(f) = \pi(g)$, that is $f(s) = g(s)$ for all $s \in D$. We firstly show that

$$\text{int(dom f)=int(dom g)}. \tag{41}$$

For that purpose consider $t \in$ int(dom f). Since D lies dense in $\mathbb{R}^d$ there exists a sequence $(s_m)_{m\in\mathbb{N}}$ in D such that $s_m \to t$. Observe that int(dom f) is an open neighborhood of t, whence there exists a $m_0 \in \mathbb{N}$ such that $s_m \in$ int(dom f) for all $m \geq m_0$. Now, $f \in S_0$ implies that f is finite on dom $f \neq \emptyset$, which is convex. Recall that the nonempty interior of a convex set in $\mathbb{R}^d$ is convex as well, see Theorem 2.33 of Rockafellar and Wets [19]. Thus f is a finite convex function on the open and convex set $O :=$ int(dom f). Since by assumption $O \neq \emptyset$ Corollary 2.36 in Rockafellar and Wets [19] says that f is continuous on O. This makes us to infer that

$$\infty > f(t) = \lim_{m\to\infty} f(s_m) = \lim_{m\to\infty} g(s_m) = \liminf_{m\to\infty} g(s_m) \geq g(t),$$

where the last inequality holds because g is lsc. Conclude that $g(t) < \infty$, whence $t \in$ dom g. This shows that int(dom f) $\subseteq$ dom g, which in turn gives int(dom f) $\subseteq$ int(dom g) for int(dom g) is the largest open set contained in dom g. Using the same arguments with f and g reversing their roles yields int(dom g) $\subseteq$ int(dom f) and thus the equality (41).

For every $t \in$ int(dom f) as above we obtain that

$$f(t) = \lim_{m\to\infty} f(s_m) = \lim_{m\to\infty} g(s_m) = g(t),$$

because f and g are continuous on $O :=$ int(dom f)=int(dom g). This means that f and g coincide on O, which as nonempty set agrees with the relative interiors rint(dom f) and rint(dom g). Thus Exercise 2.46(a) in Rockafellar and Wets [19] guarantees that $f = g$ upon noticing that f and g are lsc. Consequently, π is injective.

For proving continuity of the inverse π^{-1} let (y_n) be a sequence in the range R with

$$y_n \to_\varrho y \in R, \quad \text{that is} \quad \varrho(y_n, y) \to 0. \tag{42}$$

Observe that $y_n = \pi(f_n) = (f_n(t_i) : i \in \mathbb{N})$ and $y = \pi(f) = (f(t_i) : i \in \mathbb{N})$ with $f_n = \pi^{-1}(y_n)$ and $f = \pi^{-1}(y)$ by the first part. Then by definition of ϱ the convergence (42) means that

$$f_n(t_i) \to f(t_i) \quad \text{for all } i \in \mathbb{N}.$$

Since $D = \{t_i : i \in \mathbb{N}\}$ is a dense subset of $\mathbb{R}^d$, Theorem 7.17 of Rockafellar and Wets [19] yields that $f_n \to_{epi} f$, which by Proposition 2.3 is equivalent to $f_n \to_e f$ and thus $\pi^{-1}(y_n) \to_e \pi^{-1}(y)$. This shows continuity of the inverse. □

With our last theorem we can prove that for lsc and convex stochastic processes convergence of the finite dimensional distributions entails *epi-convergence in distribution.* More precisely we have

Proposition 2.13. Fix some countable and dense subset $D = \{t_1, t_2, \ldots\}$ of $\mathbb{R}^d$. Let Z and $Z_n, n \in \mathbb{N}$, be integrands with trajectories in S_0.

If $Z_n \xrightarrow{fd}_D Z$ then

$$Z_n \xrightarrow{\mathcal{D}} Z \quad \text{in } (S_0, e) \tag{43}$$

and

$$Z_n \xrightarrow{\mathcal{D}} Z \quad \text{in } (S, e). \tag{44}$$

Proof. Again let $\pi = \pi_D$. Since $\overline{\mathbb{R}}$ is separable, the assumption $Z_n \xrightarrow{fd}_D Z$ in combination with Theorem 3.29 in Kallenberg [11] yields that $\pi(Z_n) \xrightarrow{\mathcal{D}} \pi(Z)$ in $(\overline{\mathbb{R}}^\infty, \varrho)$. By the Subspace-Lemma 3.26 in Kallenberg [11] this is equivalent to $\pi(Z_n) \xrightarrow{\mathcal{D}} \pi(Z)$ in (R, ϱ). By Theorem 2.12 the inverse $\pi^{-1} : (R, \varrho) \to (S_0, e)$ is continuous, whence the Continuous Mapping Theorem ensures (43), because $Z_n = \pi^{-1}(\pi(Z_n))$. Another application of the Subspace-Lemma gives (44). □

3. PROOFS

In this section we prove our results in section 1. With the preparations made in section 2 the proofs reduce to a few lines.

Proof. (of Theorem 1.1) By Proposition 2.13 we have that $Z_n \xrightarrow{\mathcal{D}} Z$ in (S, e). Since (S, e) and $(\mathbb{R}_+, u)$ are separable Slutsky's theorem yields $(Z_n, \epsilon_n) \xrightarrow{\mathcal{D}} (Z, c)$ in $(S \times \mathbb{R}_+, e \times u)$, which in particular entails $(Z_n, \epsilon_n) \twoheadrightarrow^{\sim} (Z, c)$ in $(S \times \mathbb{R}_+, e \times u)$. Theorem 2.8 (1) says that A is τ_{uV}-continuous on $U \times \mathbb{R}_+ \supseteq S_1 \times \mathbb{R}_+$, whence the set of discontinuity-points $D_A := \{(f, r) \in S \times \mathbb{R}_+ :$ A is not τ_{uV}-continuous at $(f, r)\}$ of A is contained in $(S \setminus S_1) \times \mathbb{R}_+$. Consequently, $\mathbb{P}_*((Z, c) \in D_A) \leq \mathbb{P}_*(Z \notin S_1) = \mathbb{P}(Z \notin S_1) = 0$ and the Continuous Mapping Theorem for $\twoheadrightarrow^{\sim}$, see Lemma 4.5 yields the desired result (3). The second part follows from Lemma 4.3. □

Proof. (of Theorem 1.2) By Proposition 2.9 the random sets $A(Z_n, \epsilon_n), n \in \mathbb{N}$, and $A(Z, c)$ are $\mathcal{A} - \mathcal{B}_{uV}$ measurable. Therefore

$$\bigcap_{F \in \mathcal{F}'} \{A(Z_n, \epsilon_n) \cap F \neq \emptyset\} = \{A(Z_n, \epsilon_n) \in \bigcap_{F \in \mathcal{F}'} \mathcal{H}(F)\} \in \mathcal{A}$$

and

$$\bigcap_{F \in \mathcal{F}'} \{A(Z, c) \cap F \neq \emptyset\} = \{A(Z, c) \in \bigcap_{F \in \mathcal{F}'} \mathcal{H}(F)\} \in \mathcal{A},$$

because $\bigcap_{F \in \mathcal{F}'} \mathcal{H}(F)$ is τ_{uV}-closed and in particular a Borel-set in $\mathcal{B}_{uV}$. Thus (5) follows from Theorem 1.1, because (4) reduces to (6), since $\mathbb{P}^* = \mathbb{P} = \mathbb{P}_*$ on $\mathcal{A}$. Finally, by the Portmanteau-Theorem (6) is equivalent to (5). Again the second part is a consequence of Lemma 4.3 □

Proof. (of Corollary 1.3) First observe that by complementation the sequence (C_n) satisfies

$$\limsup_{n\to\infty} \mathbb{P}^*(C_n \not\subseteq A(Z_n, \epsilon_n)) = 0. \tag{45}$$

Now, since $\{C_n \cap F \neq \emptyset\} \cap \{C_n \subseteq A(Z_n, \epsilon_n)\} \subseteq \{A(Z_n, \epsilon_n) \cap F \neq \emptyset\}$, a decomposition of the set $\bigcap_{F\in\mathcal{F}'}\{C_n \cap F \neq \emptyset\}$ results in

$$\begin{aligned} &\limsup_{n\to\infty} \mathbb{P}^*\Big(\bigcap_{F\in\mathcal{F}'}\{C_n \cap F \neq \emptyset\}\Big) \\ \leq\ &\limsup_{n\to\infty} \mathbb{P}^*\Big(\bigcap_{F\in\mathcal{F}'}\{A(Z_n,\epsilon_n) \cap F \neq \emptyset\}) + \limsup_{n\to\infty} \mathbb{P}^*(C_n \not\subseteq A(Z_n, \epsilon_n)\Big). \end{aligned}$$

Here, by (45) the second summand vanishes and by Theorem 1.1 the first summand can be estimated as in (4). Thus we obtain

$$\limsup_{n\to\infty} \mathbb{P}^*\Big(\bigcap_{F\in\mathcal{F}'}\{C_n \cap F \neq \emptyset\}\Big) \leq \mathbb{P}_*\Big(\bigcap_{F\in\mathcal{F}'}\{A(Z,c) \cap F \neq \emptyset\}\Big) \quad \text{for all } \mathcal{F}' \subseteq \mathcal{F}_d, \tag{46}$$

which by the Borel law Portmanteau Theorem gives the assertion (9). In case of measurable C_n's we can argue analogously as in the above proof to conclude that (46) holds without the asteriks $*$, which by the Portmanteau-Theorem results in (10). □

P r o o f. (of Theorem 1.4) First notice that by Theorem 1.1 $\text{Argmin}(Z) = A(Z, 0)$ is a.s. non-empty, whence $\text{Argmin}(Z) = \{\xi\}$ a.s. by (11).

From the proof of Theorem 1.1 we know that $(Z_n, \epsilon_n) \twoheadrightarrow^{\sim} (Z, 0)$ in $(S \times \mathbb{R}_+, e \times u)$. Theorem 2.8 (2) yields that $D_A := \{(f, r) \in S\times\mathbb{R}_+ :\ \text{A is not } \tau_V\text{-continuous at } (f, r)\} \subseteq (S \times \mathbb{R}_+) \setminus (U^* \times \{0\}) = ((S \setminus U^*) \times \mathbb{R}_+) \cup (S \times (\mathbb{R}_+ \setminus \{0\}))$. Thus it follows that $\mathbb{P}_*((Z, 0) \in D_A) \leq \mathbb{P}_*(Z \notin U^*) = 0$ by (12) and so the CMT (Lemma 4.5) gives (13). □

P r o o f. (of Theorem 1.5) The first part (12) follows from Theorem 1.4. From the proof of Theorem 1.1 we know that $(Z_n, \epsilon_n) \xrightarrow{\mathcal{D}} (Z, 0)$ in $(S \times \mathbb{R}_+, e \times u)$, whence by the subspace-lemma

$$(Z_n, \epsilon_n) \xrightarrow{\mathcal{D}} (Z, 0) \ \text{ in } (S_1 \times \mathbb{R}_+, e \times u). \tag{47}$$

Since $S_1 \subseteq U$ and $\tau_{uF} \subseteq \tau_{uV}$ it follows from Theorem 2.8(1) that $A : (S_1 \times \mathbb{R}_+, e \times u) \to (\mathcal{F}_d, \tau_{uF})$ is continuous and herewith A is $\mathcal{B}_{e\times u}(S_1 \times \mathbb{R}_+) - \mathcal{B}_{uF}$ measurable. Recall that $\mathcal{B}_{uF} = \mathcal{B}_F$, see the proof of Proposition 2.5. Therefore $A : (S_1 \times \mathbb{R}_+, e \times u) \to (\mathcal{F}_d, \tau_F)$ is Borel-measurable. From $\tau_F \subseteq \tau_V$ and Theorem 2.8(2) we can infer that A is τ_F-continuous on $U^* \times \{0\}$. Thus the assertion (14) follows from (47) and the CMT. □

P r o o f. (of Theorem 1.6) $C_n := \{\xi_n\}$ is $\mathcal{A} - \mathcal{B}_{uV}$ measurable by Lemma 4.6 in the appendix and so Corollary 1.3 yields the distributional convergence (15) of the singletons. By the Portmanteau-Theorem this is equivalent to

$$\limsup_{n\to\infty} \mathbb{P}\Big(\bigcap_{F\in\mathcal{F}'}\{\{\xi_n\} \cap F \neq \emptyset\}\Big) \leq \mathbb{P}\Big(\bigcap_{F\in\mathcal{F}'}\{A(Z,c) \cap F \neq \emptyset\}\Big) \ \text{ for all } \mathcal{F}' \subseteq \mathcal{F}_d,$$

which with $\mathcal{F}' = \{F\}$ simplifies to (16) since $\{\{\xi_n\} \cap F \neq \emptyset\} = \{\xi_n \in F\}$. $\square$

Proof. (of Theorem 1.7) By Theorem 1.4 $\mathrm{Argmin}(Z) = \{\xi\}$ a.s. From Corollary 1.3 with $C_n := \{\xi_n\}$ and $c = 0$ we know that $\{\xi_n\} \twoheadrightarrow^{\sim} A(Z,0) = \mathrm{Argmin}(Z)$ in $(\mathcal{F}_d, \tau_{uV})$. Use (4) with $\mathcal{F}' := \{F\}$ to infer that

$$\limsup_{n\to\infty} \mathbb{P}^*(\xi_n \in F) \leq \mathbb{P}_*(\mathrm{Argmin}(Z) \cap F \neq \emptyset) = \mathbb{P}_*(\{\xi\} \cap F \neq \emptyset) = \mathbb{P}_*(\xi \in F)\ \forall F \in \mathcal{F}.$$

Since the ξ_n's and ξ are random variables it follows that $\{\xi_n \in F\} \in \mathcal{A}$ for every $n \in \mathbb{N}$ and $\{\xi \in F\} \in \mathcal{A}$ as well. Consequently, the above inequalities hold without the asterisks leftmost and rightmost and therefore

$$\limsup_{n\to\infty} \mathbb{P}(\xi_n \in F) \leq \mathbb{P}(\xi \in F)\ \forall F \in \mathcal{F},$$

which by the Portmanteau-Theorem gives $\xi_n \stackrel{\mathcal{D}}{\to} \xi$ in $\mathbb{R}^d$. $\square$

Proof. (of Corollary 1.8) By assumption $\mathrm{Argmin}(Z) = \{\xi\}$ a.s. and thus $Z \in S_1$ a.s. according to Lemma 4.2 and in particular (18) is fulfilled. Then the assertion follows from Theorem 1.7 with $\epsilon_n := c_n$. $\square$

Proof. (of Corollary 1.9) First notice that every convex and real-valued function $f : \mathbb{R}^d \to \mathbb{R}$ is lsc (actually even continuous) and proper with $\mathrm{dom}(f) = \mathbb{R}^d$. Therefore, the processes Z and $Z_n, n \in \mathbb{N}$, especially have trajectories in S_0. Moreover, by the subspace-lemma (21) entails $Z_n \stackrel{fd}{\to}_D Z$, whence the proposition follows from Corollary 1.8 with $c_n := 0$. $\square$

Proof. (of Corollary 1.10) $Z_n(t) = U_n't + D_n(t)$ with $D_n(t) := \frac{1}{2}t'V_nt + r_n(t)$. Let $t_1, \ldots, t_k \in \mathbb{R}^k$. Then $(U_n't_1, \ldots, U_n't_k) \stackrel{\mathcal{D}}{\to} (U't_1, \ldots, U't_k)$ by the Continuous Mapping Theorem. By continuity $D_n(t) \stackrel{\mathbb{P}}{\to} D(t) = \frac{1}{2}t'Vt$ and stochastic convergence by components gives $(D_n(t_1), \ldots, D_n(t_k)) \stackrel{\mathbb{P}}{\to} (D(t_1), \ldots, D(t_k))$. Thus Slutsky's lemma yields the finite dimensional convergence (21), where $Z(t) = U't + \frac{1}{2}t'Vt$. Since Z has unique minimizer $-V^{-1}U$ the statement follows from Corollary 1.9. $\square$

4. APPENDIX

Lemma 4.1. For every $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ lsc and every real $r \geq 0$ we have that

$$A(f, r) = \{t \in \mathbb{R}^d : f(t) \leq \inf_{s \in \mathbb{R}^d} f(s) + r\}$$

is a closed subset of $\mathbb{R}^d$.

P r o o f. If $\inf_{s\in\mathbb{R}^d} f(s) = +\infty$ then $A(f,r) = \mathbb{R}^d \in \mathcal{F}(\mathbb{R}^d)$ and if $\inf_{s\in\mathbb{R}^d} f(s) = -\infty$ then $A(f,r) = \{f \le -\infty\}$, which is closed, because for each sequence $t_n \to t \in \mathbb{R}^d$ with $f(t_n) \le -\infty$ it follows by lower-semicontinuity of f that $f(t) \le \liminf_{n\to\infty} f(t_n) \le -\infty$, whence $t \in \{f \le -\infty\}$. Finally, assume that $\inf_{s\in\mathbb{R}^d} f(s) \in \mathbb{R}$. Then $\alpha := \inf_{s\in\mathbb{R}^d} f(s) + r \in \mathbb{R}$ and $A(f,r) = \{f \le \alpha\} \in \mathcal{F}(\mathbb{R}^d)$, since f lsc means that $\{f > \alpha\}$ is open for each real α. □

Lemma 4.2. Let $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ be lsc, convex and proper. Then Argmin(f) is non-empty and bounded if and only if f is level-bounded.

P r o o f. The if-part follows from Theorem 1.9 in Rockafellar and Wets [19]. For the other direction first observe that by Argmin$(f) \neq \emptyset$ there exists $t_0 \in \mathbb{R}^d$ such that $f(t_0) = \inf_{\in\mathbb{R}^d} f(t)$. Since f is proper $f(t) > -\infty$ for all $t \in \mathbb{R}^d$ and so in particular $f(t_0) > -\infty$. Moreover, there exists $s \in \mathbb{R}^d$ such that $f(s) < \infty$, whence $f(t_0) \le f(s) < \infty$. Consequently, $\alpha_0 := f(t_0) \in \mathbb{R}$. It follows that $\{f \le \alpha_0\} = \{f = \alpha_0\} =$Argmin$(f)$. Thus by assumption on Argmin(f) the level-set $\{f \le \alpha_0\}$ is non-empty and bounded and hence compact, because $\{f \le \alpha_0\}$ is closed by lower-semicontinuity of f. Now, the assertion that f is level-bounded follows from Proposition 2.3.1 of Bertsekas [2], Convex Analysis and Optimization. □

Lemma 4.3. If f is lsc, convex, proper and level-bounded, then $A(f,r)$ is non-empty and compact for every real $r \ge 0$.

P r o o f. Conclude from Argmin$(f) = A(f,0) \subseteq A(f,r)$ and Lemma 4.2 that $A(f,r) \neq \emptyset$ for all real $r \ge 0$. As in the proof of Lemma 4.2 we see that $A(f,r) = \{f \le \alpha_0 + r\}$, where $\alpha_0 = \inf_{t\in\mathbb{R}^d} f(t) \in \mathbb{R}$ and another application of Proposition 2.3.1 of Bertsekas [2] yields that $\{f \le \alpha_0 + r\}$ is compact as desired. □

Lemma 4.4. Let $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ be convex. Then the following statements are equivalent:

(1) f is proper and int(dom$f$) $\neq \emptyset$.

(2) f is finite on some nonempty open set

P r o o f. If (1) holds then f is finite on dom$f$ and in particular on int(dom$f$) $\neq \emptyset$. For the reverse let $G \neq \emptyset$ be open such that $\infty < f(x) < \infty$ for all $x \in G$. Then $G \subseteq$ dom$f$ and thus $G \subseteq$ int(dom$f$), whence int(dom$f$) $\neq \emptyset$. Next, assume that f is not proper. By Exercise 2.5 in Rockafellar and Wets [19] it follows that $f(x) = -\infty$ for all $x \in$ int(dom$f$) $\supset G$, which contradicts $f > -\infty$ on G. □

Lemma 4.5. (CMT for $\twoheadrightarrow^{\sim}$) Let $(X_1, \mathcal{O}_1)$ and $(X_2, \mathcal{O}_2)$ be topological spaces and let $h : X_1 \to X_2$ be a mapping with pertaining set $D_h := \{x \in X_1 : h \text{ is not continuous at } x\}$ of all discontinuity-points of h. For mappings $Y_n : (\Omega, \mathcal{A}) \to X_1$ and $Y : (\Omega, \mathcal{A}) \to X_1$ assume that

$$Y_n \twoheadrightarrow^{\sim} Y \text{ in } (X_1, \mathcal{O}_1).$$

If $\mathbb{P}_*(Y \in D_h) = 0$, then

$$h(Y_n) \twoheadrightarrow^{\sim} h(Y) \text{ in } (X_2, \mathcal{O}_2).$$

P r o o f. Let F be closed in $(X_2, \mathcal{O}_2)$. Check that

$$\mathrm{cl}_1(h^{-1}(F)) \subseteq h^{-1}(F) \cup D_h, \tag{48}$$

where $\mathrm{cl}_1(A)$ denotes the closure of $A \subset (X_1, \mathcal{O}_1)$. It follows that

$$\limsup_{n\to\infty} \mathbb{P}^*(h(Y_n) \in F) = \limsup_{n\to\infty} \mathbb{P}^*(Y_n \in h^{-1}(F)) \leq \limsup_{n\to\infty} \mathbb{P}^*(Y_n \in \mathrm{cl}_1(h^{-1}(F)))$$
$$\leq \mathbb{P}_*(Y \in \mathrm{cl}_1(h^{-1}(F))) \leq \mathbb{P}_*(Y \in h^{-1}(F)) + \mathbb{P}_*(Y \in D_h) = \mathbb{P}_*(h(Y) \in F).$$

Here, in the second row the first inequality follows from the Borel-Portmanteau-Theorem, the second inequality from (48) and the subsequent equality from the requirement $\mathbb{P}_*(Y \in D_h) = 0$. Another application of the Borel-Portmanteau-Theorem yields the assertion. □

Lemma 4.6. Let $\xi_1, \ldots, \xi_n$ be finitely many random variables defined on a measurable space $(\Omega, \mathcal{A})$ with values in $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Then $C := \{\xi_1, \ldots, \xi_n\}$ is $\mathcal{A} - \mathcal{B}_{uV}$-measurable.

P r o o f. Clearly, C maps into $\mathcal{F}_d$. Let $\mathbf{O} \in \tau_{uV}$. Since $\mathcal{S}_{uV}$ is actually a base for τ_{uV}, there exists a family $(F_i)_{i\in I} \subseteq \mathcal{F}_d$ with some index-set $I \neq \emptyset$ such that $\mathbf{O} = \bigcup_{i\in I} \mathcal{M}(F_i)$. If $\pi_l : (\mathbb{R}^d)^n \to \mathbb{R}^d$ with $l \in \{1, \ldots, n\}$ denotes the $l-$th projection, i. e., $\pi_l(x_1, \ldots, x_n) = x_l$ for $(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$, then $\{C \in \mathbf{O}\} = \{(\xi_1, \ldots, \xi_n) \in V\}$, where $V = \bigcup_{i\in I} \bigcap_{l=1}^n \pi_l^{-1}(F_i^c)$ is open in $(\mathbb{R}^d)^n$. In particular, $V \in \mathcal{B}((\mathbb{R}^d)^n)$. Now, $\mathcal{B}((\mathbb{R}^d)^n) = (\mathcal{B}(\mathbb{R}^d))^n$, whence $\{(\xi_1, \ldots, \xi_n) \in V\} \in \mathcal{A}$. Thus $\{C \in \mathbf{O}\} \in \mathcal{A}$ for every open $\mathbf{O} \in \tau_{uV}$, which yields that C is $\mathcal{A} - \mathcal{B}_{uV}$-measurable. □

P r o o f. (of Lemma 2.1) Let $x \in \mathbb{R}^d$. In case 1 assume that $f(x) = -\infty$. Then $(x, \alpha) \in \mathrm{epi}(f) = \mathrm{epi}(g)$ for every $\alpha \in \mathbb{R}$, and so $g(x) \leq \alpha$ for every $\alpha \in \mathbb{R}$, which means that $g(x) = -\infty = f(x)$. In case 2 let $-\infty < f(x) < \infty$. Then $(x, f(x)) \in \mathrm{epi}(f) = \mathrm{epi}(g)$, whence $(\star)$ $g(x) \leq f(x) < \infty$. Assume that $g(x) = -\infty$. Then as in case 1 (exchange f for g) it followed that $f(x) = -\infty$, a contradiction. Therefore $g(x) \in \mathbb{R}$ and consequently $(x, g(x)) \in \mathrm{epi}(g) = \mathrm{epi}(f)$ resulting in $f(x) \leq g(x)$ and by $(\star)$ we obtain that $f(x) = g(x)$. Finally, let $f(x) = \infty$. Assume that $g(x) < \infty$. Then either $g(x) = -\infty$ and as in case 1 it followed that $f(x) = -\infty$ (contradiction) or $-\infty < g(x) < \infty$ and as in case 2 it followed that $f(x) < \infty$ (contradiction). Consequently, $g(x) = \infty = f(x)$. □

(Received February 11, 2021)

REFERENCES

[1] H. Attouch: Variational Convergence for Functions and Operators. Applicable Mathematics Series, Pitmann, London 1984.

[2] D. P. Bertsekas: Convex Analysis and Optimization. Athena Scientific, Belmont, Massachusetts 2003.

[3] V. Chernozhukov: Extremal quantile regression. Ann. Statist. *33* (2005), 806–839. DOI:10.1214/009053604000001165

[4] R. A. Davis, K. Knight, and J. Liu: M-estimation for autoregressions with infinite variance. Stochastic Process. Appl. *40* (1992), 145–180. DOI:10.1016/0304-4149(92)90142-D

[5] D. Ferger: Weak convergence of probability measures to Choquet capacity functionals. Turkish J. Math. *42* (2018), 1747–1764. DOI:10.3906/mat-1705-106

[6] P. Gaenssler and W. Stute: Wahrscheinlichkeitstheorie. Springer-Verlag, Berlin, Heidelberg, New York 1977.

[7] C. J. Geyer: On the asymptotics of convex stochastic optimization. Unpublished manuscript (1996).

[8] S. J. Haberman: Concavity and estimation. Ann. Statist. *17* (1989), 1631–1661. DOI:10.1214/aos/1176347385

[9] N. L. Hjort and D. Pollard: Asymptotic for minimizers of convex processes. Preprint, Dept. of Statistics, Yale University (1993). arxiv:1107.3806v1

[10] J. Hoffmann-Jørgensen: Convergence in law of random elements and random sets. In: High Dimensional Probability (E. Eberlein, M. Hahn and M. Talagrand, eds.), Birkhuser Verlag, Basel 1998, pp. 151–189.

[11] O. Kallenberg: Foundations of Modern Probability. Springer-Verlag, New York 1997.

[12] K. Knight: Limiting distributions for L_1 regression estimators under general conditions. Ann. Statist. *26* (1998), 755–770. DOI:10.1214/aos/1028144858

[13] K. Knight: Limiting distributions of linear programming estimators. Extremes *4* (2001), 87–103. DOI:10.1023/A:1013991808181

[14] K. Knight: What are the limiting distributions of quantile estimators? In: Statistical Data Analysis Based on the L_1-Norm and Related Methods (Y. Dodge, ed.), Series Statistics for Industry and Technology, Birkhäuser Verlag, Basel pp. 47–65. DOI:10.1007/978-3-0348-8201-9_5

[15] F. Liese and K-J. Mieschke: Statistical Decision Theory. Springer Science and Business Media, LLC, New York 2008. DOI:10.1007/978-0-387-73194-0_3

[16] I. Molchanov: Theory of Random Sets. Second Edition. Springer-Verlag, New York 2017. DOI:10.1007/978-1-4471-7349-6

[17] G. Ch. Pflug: Asymptotic dominance and confidence regions for solutions of stochastic programs. Czechoslovak J. Oper. Res. *1* (1992), 21–30.

[18] G. Ch. Pflug: Asymptotic stochastic programs. Math. Oper. Res. *20* (1995), 769–789. DOI:10.1287/moor.20.4.769

[19] R. T. Rockefellar and R. J.-B. Wets: Variational Analysis. Springer-Verlag, Berlin, Heidelberg 1998.

[20] N. V. Smirnov: Limiting distributions for the terms of a variational series. Amer. Math. Soc. Trans. *67* (1952), 82–143.

[21] F. Topsøe: Topology and Measure. Lecture Notes in Mathematics. Springer-Verlag, Berlin, Heidelberg, New York 1970.

[22] J. Wagener and H. Dette: Bridge estimators and the adaptive Lasso under heteroscedasticity. Math. Methods Statist. *21* (2012), 109–126. DOI:10.1287/moor.20.4.769

Dietmar Ferger, Technische Universität Dresden, Fakultät Mathematik, Zellescher Weg 12-14, D-01069 Dresden. Germany.

e-mail: dietmar.ferger@tu-dresden.de