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INCOMPLETE INFORMATION AND RISK SENSITIVE ANALYSIS OF SEQUENTIAL GAMES WITHOUT A PREDETERMINED ORDER OF TURNS

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The authors introduce risk sensitivity to a model of sequential games where players don't know beforehand which of them will make a choice at each stage of the game. It is shown that every sequential game without a predetermined order of turns with risk sensitivity has a Nash equilibrium, as well as in the case in which players have types that are chosen for them before the game starts and that are kept from the other players. There are also a couple of examples that show how the equilibria might change if the players are risk prone or risk adverse.

Keywords: incomplete information, sequential game, risk sensitive, turn selection process

Classification: 91A10, 91A18, 91A25

1. INTRODUCTION

In game theory there are many types of models to study different situations. That is why there exists a basic set of characteristics that every game has and various changes or additions are made to extend the scope of these models.

In this paper a couple of variations to a model presented in [5] are studied in which players make decisions in a sequential manner, but the order of the play is not known beforehand to players: it is revealed as the game is played. The variations considered involve players having incomplete information on the type of their opponents, making it possible to consider situations in which there is uncertainty on the utilities of the opponents; and considering risk sensitivity of the players, which makes the models more accurate as players tend to consider risk in their choices, since utilities cannot be considered directly, such as the lump sum of money a player receives, or something similar. To do this, the Arrow [2] and Pratt [14] approach for risk sensitivity is used and the case in which players only consider what's happening in the game is taken into account, without any external variables interfering with their decision making. Not only is this the usual approach taken when studying Markov decision processes, for instance, in [9] and [16], but it has also been applied in [3, 4, 8, 10, 12], and [13] to study risk sensitivity in static and dynamic games including, in this last case, Markov and differential games.

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For the games presented in this work, in order to ensure the existence of Nash equilibria the authors proceed by showing that an adequately defined set of best responses satisfies the conditions of Kakutani's fixed point theorem [6] and [11] (for an extension of the Kakutani's fixed point theorem known as Kakutani-Fan-Glicksberg's fixed point theorem, see [1]) and by observing that the fixed points are exactly the Nash equilibria for the games. This is a known approach for some games and it is useful in this case to show the existence of the equilibria for the present model.

The paper is structured as follows. In Section 2 the notation used throughout the paper and some concepts that deviate from the well-known game theory are explained. In Section 3 the model of sequential games without a predetermined order of turns that will be used through the article is introduced and the risk sensitivity modification is discussed, along with the required modifications to the sequential model previously shown. In Section 4 the proofs for the results that are necessary to ensure the existence of Nash equilibria are presented. In Section 5 two examples are described and worked with to show the effect that risk sensitivity has on the equilibria of these games, both in the case of complete and incomplete information. Finally, in Section 6 some concluding remarks are given and possible extensions to the work are presented.

2. PRELIMINARIES

Throughout the article standard notation, as in [17], will be used. A *game* consists of the following elements:

- A set $I = \{1, 2, \dots, N\}$ of *players*.
- A finite set of *pure strategies* S_i for each $i \in I$.
- A real-valued *utility function* $u_i \colon \Sigma \to \mathbb{R}$ for each $i \in I$, with $\Sigma = S^T \times \dots \times S^1$ where each S^t is the finite set of all the strategies available for any of the one of the players in I at time t.

Remark 2.1. The definition of each S^t is related to the sets S_i as every strategy $s_i \in S_i$ must appear in at least one S^t for all $i \in I$, however it isn't straightforward how to determine each S^t in every game as it depends on the structure of the game that is studied: sometimes the available strategies are dependent only on each player's actions, sometimes they depend on what all players have done previously.

In this article sequential games (also called dynamic games) will be studied, therefore a *horizon* of play $T \in \mathbb{N}$ is also required, which will tell us the number of decision points in the game. A *decision point* will be the equivalent of what is known as an information set, except the game also has to choose which player will act at that point. This is because the change made in this paper to models of games is introducing uncertainty for players regarding the order in which they will make a choice. This modification is in part due to how some models are made in which it is determined that players have to decide in a given order, but in reality that might not be the case, and the order is completely different or can even be considered to be random for all intents and purposes. Therefore, this is an idea on how to make models that could be used in cases where the order of turns can be dynamic as well.

From the basic elements described above it is possible to obtain for each player $i \in I$ their *set of mixed strategies* M_i, which is made of the probability distributions that have the set S_i as their support. A *profile of mixed strategies* $x = (x_1, x_2, \ldots, x_N)$ is defined as a vector made of strategies $x_i \in M_i$ for each player i, that is, a profile of mixed strategies x describes the strategies followed by all players in the game. The set of profiles of mixed strategies is denoted here by M.

If there is a profile $x = (x_1, x_2, \ldots, x_N)$, and $\tilde{x}_i \in M_i$, it is possible to combine them to make the profile

$$(\tilde{x}_i, x_{-i}) = (x_1, x_2, \ldots, x_{i-1}, \tilde{x}_i, x_{i+1}, \ldots, x_N),$$

that is, to replace in x the strategy x_i corresponding to player i with the strategy $\tilde{x}_i$.

Let us notice that each player can be chosen to make a move at every decision point in the game, therefore forcing every player to select a strategy for each possible decision point. Players learn who has made a move in previous decision points, and it may or may not be that they also learn the actions other players have made. In either case players have *perfect recall* and as such their decision is conditioned by the actions (or possible actions) of all players. P_i is defined as the set of *plans of conditioned strategies* for player i, which can be defined as the strategies $(s_i \mid r^1, \ldots, r^{k-1}) \in P_i$ where player i observes the actions $(r^1, \ldots, r^{k-1})$ taken before the current turn $k \in T$ (or whichever actions are visible to player i). In an analogous manner it is possible to consider *mixed plans of conditioned strategies* for player i, $x_i(s_i \mid r^1, \ldots, r^{k-1})$ at each decision point $k \in T$, which consist of probability distributions that have P_i as their support; the set of mixed conditioned strategies for player i is denoted by Q_i. Finally, it is possible to define *profiles of plans of (mixed) conditioned strategies* as the vectors that consider (mixed) plans of conditioned strategies for each player $i \in I$. The set of these is denoted by P (resp. Q). When referring to the sets of profiles of plans of (mixed) conditioned strategies of players other than i, we denote them by P_{-i} (resp. Q_{-i}).

It would be desirable to define a concept of solution for the games, that is, profiles that satisfy some properties. Such a property is that once the strategies for all players are chosen, no player would like to deviate from their choice. A profile that satisfies this is called an equilibrium. These equilibria shall be defined for each of the models in the following section.

For the games that will be studied in the rest of the article, it is considered that players can be chosen to make many or few decisions in the game, anywhere from 0 to T. Therefore the information of how many decisions as well as when in the game players will be making them is hidden from every player until they reach each of the decision points in the game. Therefore, the model that will be worked on can be defined as follows:

Definition 2.2. A sequential game without a predetermined order of turns is a game with a horizon of play T and a set of probability densities $\mathcal{P} = \{p_1, \ldots, p_N\}$, where $p_i(m)$ is the probability according to player i that player m is chosen at each decision point.

In other words, the games that will be studied all have the characteristic that before the game, it is not known what player will decide when. As the game goes on and it is required for a decision to be made, the game decides, via a randomized process (or what at least seems to be for the players), who gets to move and then the player chosen acts. Notice that it is allowed for each player to have their own model for the *turn selection process*, which are known to every player, which also means that, if it is not explicitly required that no player has an advantage, if such advantage exists, it is not known to the other players, so each considers their density to be the most adequate model for the turn selection process. As such, it can be assumed that however players come up with their probability distribution model, they consider it to be the most adequate. For the purposes of this paper's examples in Section 5, the way the turn selection process randomizes will be of common knowledge.

In order to work with these games, and be able to give a reasonable idea of how players should choose their strategies, the expected utility is the way to go. The *expected utility* of each player i when the profile of plans of mixed conditioned strategies $x \in Q$ is played is given by

$$E_i(x) = \sum_{n^1 \in I} \sum_{s^1 \in S_{n^1}} \cdots \sum_{n^T \in I} \sum_{s^T \in S_{n^T}} u_i(s^T, \dots, s^1) \\ \times x_n^T(s^T \mid s^{T-1}, \dots, s^1) p_i(n^T) \cdots x_{n^1}(s^1) p_i(n^1).$$

As it stands, fixed probabilities for selecting players are considered but it is possible to adapt the model to accept variable probabilities at each stage. One thing that should be noticed is that these probabilities are known by every player, so it is known how each player views the turn selection process. An *equilibrium*, therefore, is a profile $x^* \in Q$ such that for every player $i \in I$ and every plan of mixed conditioned strategies $x_i \in Q_i$ it holds that

$$E_i(x^*) \geq E_i(x_i, x^*_{-i}).$$

3. MODELS

3.1. Risk neutral models

First the case with incomplete information is studied, that is, there is the possibility of players having different types, which can change the behavior of each player by modifying their utility functions. Each player has as well a determined type before the game starts and the type of each player is only known by the player himself, but it is general knowledge which types are possible for each player and the utility functions associated with each of these types. That is, players do not know the others' types, but the set of possible types for the other players is known, and each of these types carries a utility function associated with it.

This way, for each player i, there is a finite set of *types* Θ_i. Before the game starts, a type $\theta_i \in \Theta_i$ is chosen for each i. Each player has an a priori distribution $b_i(\cdot)\colon \Theta \to [0, 1]$, where $\Theta = \times_{i \in I} \Theta_i$, which can be refined once player i knows what his type θ_i is. In order to do so, it is required for the distribution b_i to have *Bayesian updating*, that is, for every $i \in I$

$$b_i(\theta_{-i} = c \mid \theta_i = a) = \frac{b_i(\theta_i = a \text{ and } \theta_{-i} = c)}{b_i^m(\theta_i = a)}$$

for every $a \in \Theta_i$ and $c \in \Theta_{-i} = \times_{j \neq i} \Theta_j$, where $b_i^m(\theta_i = a)$ is the marginal distribution for player i.

As stated before, the utility function of i also needs to be dependent on the type of i, that is, $u_i(\cdot \mid \theta_i) \colon \Sigma \to \mathbb{R}$.

Given all of the above, the *ex-ante expected utility* [15] for player i when the profile $x \in Q$ has been chosen is defined as:

$$E_i(x) = \sum_{\theta \in \Theta} \sum_{n^1 \in I} \sum_{s^1 \in S_{n^1}} \cdots \sum_{n^T \in I} \sum_{s^T \in S_{n^T}} b_i(\theta) u_i(s^T, \ldots, s^1 \mid \theta_i) \\ \times x_{n^T}(s^T \mid s^{T-1}, \ldots, s^1; \theta_{n^T}) p_i(n^T) \cdots x_{n^1}(s^1 \mid \theta_{n^1}) p_i(n^1).$$

A *Bayes-Nash equilibrium* for this model is a profile $x^* \in Q$ such that

$$E_i(x^*) \geq E_i(x_i, x_{-i}^*)$$

for every $x_i \in Q_i$ and every $i \in I$.

Instead of dealing with all the sources of uncertainty in the game at once, it is possible to do it in steps by defining other expected utilities according to what is known at the moment. In order to do so, the *ex-post expected utility* for player i when the profile $x \in Q$ has been chosen and the vector of types θ of all players is known can be defined as:

$$E_i(x, \theta) = \sum_{n^1 \in I} \sum_{s^1 \in S_{n^1}} \cdots \sum_{n^T \in I} \sum_{s^T \in S_{n^T}} u_i(s^T, \ldots, s^1 \mid \theta_i) \\ \times x_{n^T}(s^T \mid s^{T-1}, \ldots, s^1; \theta_{n^T}) p_i(n^T) \cdots x_{n^1}(s^1 \mid \theta_{n^1}) p_i(n^1).$$

The *ex-interim expected utility* for player i when $x \in Q$ has been chosen and the player knows their type θ_i is given by

$$E_i(x, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} \sum_{n^1 \in I} \sum_{s^1 \in S_{n^1}} \cdots \sum_{n^T \in I} \sum_{s^T \in S_{n^T}} b_i(\theta_i \mid \theta_{-i}) u_i(s^T, \ldots, s^1 \mid \theta_i) \\ \times x_{n^T}(s^T \mid s^{T-1}, \ldots, s^1; \theta_{n^T}) p_i(n^T) \cdots x_{n^1}(s^1 \mid \theta_{n^1}) p_i(n^1),$$

that is, the ex-interim expected utility can also be rewritten as

$$E_i(x, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} b_i(\theta_i \mid \theta_{-i}) E_i(x, (\theta_i, \theta_{-i}))$$

which in turn makes it possible to rewrite the ex-ante expected utility as either

$$E_i(x) = \sum_{\theta \in \Theta} b_i(\theta) E_i(x, \theta)$$

or

$$E_i(x) = \sum_{\theta_i \in \Theta_i} b_i^m(\theta_i) E_i(x, \theta_i)$$

where b_i^m is the marginal distribution of b_i for player i.

3.2. Risk sensitive models

Additionally to the references [2] and [14] given for the risk sensitivity, the book by L. Eeckhoudt, C. Gollier, and H. Schlesinger [7] is provided. In particular, in Chapter 1 of [7], a complete treatise on the risk-sensitivity is analyzed.

Concerning games, it is important to mention the seminal paper by Nowak [13] in which a detailed discussion on the risk-sensitive Nash equilibrium concept in static non-cooperative games and two-stage stochastic games of resource extraction is given. Moreover, several detailed examples to show the meaning of the risk-sensitive Nash equilibrium in such models is presented.

Now a modification to the model of sequential games without a predetermined order of turns, which was presented in the previous section, is made in order to study the behavior of risk prone and risk sensitive players. To do this a risk sensibility variable λ_i is introduced, such that the utilities of each player are affected adequately.

For starters, the *risk aversion coefficient* is defined according to Arrow [2] and Pratt [14] as

$$r_i(z) = -\frac{{U_i^{\lambda_i}}''(z)}{{U_i^{\lambda_i}}'(z)}, \quad z \in (-\infty, \infty)$$

where $U_i^{\lambda_i}$ is the modified utility function for player i. When $r_i(z) > 0$, player i is *risk averse*, whereas if $r_i(z) < 0$, then player i is *risk prone*. The case in which $r_i(z) = 0$ corresponds to player i being *risk neutral*, and the equation can be easily solved to find that $U_i^{\lambda_i}(z)$ is linear.

A general case to study the risk sensitivity is dependent on the current state of each player, namely their current "wealth". However, in this case this assumption will be dismissed and therefore $r_i(z) = \lambda_i$ will be considered, where each λ_i is constant.

The other important property that will be used is what has been referred to as the Δ-property (see [9]), that is, when every reward is increased by Δ, then the certain equivalent of the player (see [9]) has to be increased exactly by Δ whether the player is risk averse, neutral or prone. From these two considerations, it is possible to find that if $r_i(z) = \lambda_i$, then the modified utility function must be of the form:

$$U_i^{\lambda_i}(z) = \begin{cases} -\exp(-\lambda_i z) & \text{if } \lambda_i > 0 \\ z & \text{if } \lambda_i = 0 \\ \exp(-\lambda_i z) & \text{if } \lambda_i < 0 \end{cases}$$

up to an affine transformation. The previous expression can be succinctly written for $\lambda_i \neq 0$ as

$$U_i^{\lambda_i}(z) = -(\operatorname{sgn} \lambda_i) \exp(-\lambda_i z),$$

whose inverse function is given by

$${U_i^{\lambda_i}}^{-1}(w) = -\frac{1}{\lambda_i} \log(-(\operatorname{sgn} \lambda_i) w).$$

Since the *certain equivalent* is defined by

$$U_i^{\lambda_i}(c(\tilde{z})) = E(U_i^{\lambda_i}(\tilde{z})),$$

it is obtained that

$$\begin{aligned}c(\tilde{z}) &= {U_i^{\lambda_i}}^{-1}(E(U_i^{\lambda_i}(\tilde{z}))) \\ &= -\frac{1}{\lambda_i}\log(-(\operatorname{sgn}\lambda_i)E(-(\operatorname{sgn}\lambda_i)\exp(-\lambda\tilde{z}))) \\ &= -\frac{1}{\lambda_i}\log(E(\exp(-\lambda_i\tilde{z}))).\end{aligned}$$

In order to define the expected utility and an equilibrium with the risk sensitive modification, it is necessary to find an expected utility operator $\mathcal{E}_i^{\lambda_i}$ such that $\mathcal{E}_i^{\lambda_i}(z) = E(\exp(-\lambda_i\tilde{z}))$. This follows easily as shown below.

To analyze the risk sensitive case, a *Nash equilibrium* is defined as a profile $x^* \in Q$ such that

$$\mathcal{E}_i^{\lambda_i}(x^*) \geq \mathcal{E}_i^{\lambda_i}(x_i, x_{-i}^*)$$

for every $x_i \in Q_i$ and every $i \in I$, where the expected utility for player i when the profile of plans of mixed conditioned strategies $x \in Q$ is played, is defined as

$$\begin{aligned}\mathcal{E}_i^{\lambda_i}(x) = \sum_{n^1\in I}\sum_{s^1\in S_{n^1}}\cdots\sum_{n^T\in I}\sum_{s^T\in S_{n^T}} & e^{-\lambda_i u_i(s^T,\ldots,s^1)} x_{n^T}(s^T \mid s^{T-1},\ldots,s^1) \\ & \times p_i(n^T)\cdots x_{n^1}(s^1)p_i(n^1).\end{aligned}$$

The case in which there is incomplete information and the players are risk sensitive is also studied. The types of each player, as before, are only associated with their utility function, and are completely independent of the sensitivity to risk of each player. A *Bayes-Nash equilibrium* is defined as a profile $x^* \in Q$ such that

$$\mathcal{E}_i^{\lambda_i}(x^*) \geq \mathcal{E}_i^{\lambda_i}(x_i, x_{-i}^*)$$

for every $x_i \in Q_i$ and every $i \in I$, where the ex-ante expected utility of playing the profile of plans of mixed conditioned strategies $x \in Q$ is given as:

$$\begin{aligned}\mathcal{E}_i^{\lambda_i}(x) = \sum_{\theta\in\Theta}\sum_{n^1\in I}\sum_{s^1\in S_{n^1}}\cdots\sum_{n^T\in I}\sum_{s^T\in S_{n^T}} & b_i(\theta)e^{-\lambda_i u_i(s^T,\ldots,s^1\mid\theta_i)} \\ & \times x_{n^T}(s^T \mid s^{T-1},\ldots,s^1;\theta_{n^T})p_i(n^T)\cdots x_{n^1}(s^1\mid\theta_{n^1})p_i(n^1);\end{aligned}$$

the ex-interim expected utility of playing the profile $x \in Q$ when player i is of type $\theta_i \in \Theta_i$ is given by:

$$\begin{aligned}\mathcal{E}_i^{\lambda_i}(x,\theta_i) = \sum_{\theta_{-i}\in\Theta_{-i}}\sum_{n^1\in I}\sum_{s^1\in S_{n^1}}\cdots\sum_{n^T\in I}\sum_{s^T\in S_{n^T}} & b_i(\theta_{-i}\mid\theta_i)e^{-\lambda_i u_i(s^T,\ldots,s^1\mid\theta_i)} \\ & \times x_{n^T}(s^T \mid s^{T-1},\ldots,s^1;\theta_{n^T})p_i(n^T)\cdots x_{n^1}(s^1\mid\theta_{n^1})p_i(n^1);\end{aligned}$$

and the ex-post utility of playing the profile $x \in Q$ when the vector of types $\theta \in \Theta$ is known, can be defined as:

$$\mathcal{E}_i^{\lambda_i}(x,\theta) = \sum_{n^1\in I}\sum_{s^1\in S_{n^1}}\cdots\sum_{n^T\in I}\sum_{s^T\in S_{n^T}} e^{-\lambda_i u_i(s^T,\ldots,s^1\mid\theta_i)}$$

$$\times\, x_{n^T}(s^T \mid s^{T-1}, \ldots, s^1; \theta_{n^T}) p_i(n^T) \cdots x_{n^1}(s^1 \mid \theta_{n^1}) p_i(n^1),$$

and the relations shown previously between these expected utilities still hold.

In order to study the existence of Nash equilibria, the best response correspondences are defined and it is shown that each of them is nonempty, convex and has a closed graph.

The *best response correspondence* for each $i \in I$, $x_{-i} \in Q_{-i}$ and $\lambda_i \neq 0$, is given by

$$BR_i(x_{-i}) = \{x_i \in Q_i \mid E_i(x_i, x_{-i}) \geq E_i(\tilde{x}_i, x_{-i}) \text{ for all } \tilde{x}_i \in Q_i\}.$$

For the risk sensitive cases, we consider that the risk sensitivity of each player is common knowledge, that is, the values λ_i are known to everyone. With this in mind it is possible to define the best response correspondences in terms of the certain equivalent, according to whether the player is risk averse or risk prone. If $\lambda_i < 0$, that is, if the player is *risk prone*, the *best response correspondences* are defined as:

$$BR_i(x_{-i}) = \left\{ x_i \in Q_i \;\middle|\; -\frac{1}{\lambda_i} \log(\mathcal{E}_i^{\lambda_i}(x_i, x_{-i})) \geq -\frac{1}{\lambda_i} \log(\mathcal{E}_i^{\lambda_i}(\tilde{x}_i, x_{-i})) \text{ for all } \tilde{x}_i \in Q_i \right\}$$

whereas if $\lambda_i > 0$, that is, if the player is *risk averse*, the *best response correspondences* are defined as:

$$BR_i(x_{-i}) = \left\{ x_i \in Q_i \;\middle|\; -\frac{1}{\lambda_i} \log(\mathcal{E}_i^{\lambda_i}(x_i, x_{-i})) \leq -\frac{1}{\lambda_i} \log(\mathcal{E}_i^{\lambda_i}(\tilde{x}_i, x_{-i})) \text{ for all } \tilde{x}_i \in Q_i \right\}.$$

4. PROOFS

As a preamble, it can be noticed that the case with complete information about the players utilities is a special case of the incomplete information variant, in which the set of types for each player is a singleton. Therefore, only the incomplete information case will be considered in the rest of the article.

Remark 4.1. Each of the propositions that is proven next for the ex-post expected utility trickles down because of the chained definitions of our different expected utilities, and therefore gives the same characteristic to the ex-ante expected utility.

Proposition 4.2. For each player i, the ex-post expected utility function is a continuous function in i's plan of conditioned strategies.

P r o o f. Notice that the ex-post expected utility function can be split in the following way:

$$E_i(x, \theta) = \left(\sum_{s^1 \in S_i} \cdots \sum_{s^T \in S_i} u_i(s^T, \ldots, s^1 \mid \theta_i) x_i(s^T \mid s^{T-1}, \ldots, s^1; \theta_i) \right.$$
$$\left. \times\, p_i(i) \cdots x_i(s^1 \mid \theta_i) p_i(i) \right)$$

$$+\left(\sum_{n^1\in I\setminus\{i\}}\sum_{s^1\in S_{n^1}}\sum_{s^2\in S_i}\cdots\sum_{s^T\in S_i}u_i(s^T,\dots,s^1\mid\theta_i)x_i(s^T\mid s^{T-1},\dots,s^1;\theta_i)\right.$$

$$\times\, p_i(i)\cdots x_i(s^2\mid s^1;\theta_i)p_i(i)x_{n^1}(s^1\mid\theta_{n^1})p_i(n^1)+\cdots$$

$$+\sum_{s^1\in S_i}\cdots\sum_{s^{T-1}\in S_i}\sum_{n^T\in I\setminus\{i\}}\sum_{s^T\in S_{n^T}}u_i(s^T,\dots,s^1\mid\theta_i)x_{n^T}(s^T\mid s^{T-1},\dots,s^1;\theta_{n^T})$$

$$\left.\times\, p_i(n^T)x_i(s^{T-1}\mid s^{T-2},\dots,s^1;\theta_i)p_i(i)\cdots x_i(s^1\mid\theta_i)p_i(i)\right)+\cdots$$

$$+\left(\sum_{n^1\in I\setminus\{i\}}\sum_{s^1\in S_{n^1}}\cdots\sum_{n^T\in I\setminus\{i\}}\sum_{s^T\in S_{n^T}}u_i(s^T,\dots,s^1\mid\theta_i)\right.$$

$$\left.\times\, x_{n^T}(s^T\mid s^{T-1},\dots,s^1;\theta_{n^T})p_i(n^T)\cdots x_{n^1}(s^1\mid\theta_{n^1})p_i(n^1)\right),$$

where the terms in the first bracket are those in which player i was selected T times, the terms in the second bracket are those in which player i was selected $T-1$ times, and so on, ending with the terms in the last bracket in which player i was selected 0 times. It can be easily seen that each of these terms is continuous in the plan of strategies for player i, so the whole sum is continuous in the plan of strategies of player i. For the case of risk prone and risk averse players, in the expression above, $e^{-\lambda_i u_i(s^T,\dots,s^1|\theta_i)}$ is the term that appears instead of only $u_i(s^T,\dots,s^1\mid\theta_i)$, but the continuity of the expected utility is conserved by the same arguments. □

The following proposition can be directly proved, see Theorem 2 in [5].

Proposition 4.3. The set of profiles of conditioned strategies Q is a non-empty, compact and convex subset of some $\mathbb{R}^q$.

Proposition 4.4. The best response correspondence $BR\colon Q\to Q$ defined as

$$BR(x)=(BR_1(x_{-1}),\dots,BR_N(x_{-N}))$$

is non-empty and has a closed graph.

Proof. Since the expected utility function is a continuous function defined on a compact set, for each i and each $x_{-i}\in Q_{-i}$ there must exist x_i^* for which it achieves its maximum. For the case of risk prone and averse players, it is necessary to analyze each case separately. For risk prone players, the expected utility function is bounded below by 0, so $-\frac{1}{\lambda_i}\log(\mathcal{E}_i^{\lambda_i}(x_i^*,x_{-i}))$ is well-defined and continuous, and therefore it must attain its maximum at some x_i^*. For risk averse players, the same holds for $-\frac{1}{\lambda_i}\log(\mathcal{E}_i^{\lambda_i}(x_i^*,x_{-i}))$, so it must attain its minimum at some x_i^*.

Therefore, for each player i, BR_i is non-empty for every $x_{-i} \in Q_{-i}$, so BR is non-empty for every $x \in Q$.

Let $(x_k)_{k=1}^{\infty}$ be a sequence of strategy profiles and $(y_k)_{k=1}^{\infty}$ be the sequence of responses derived from the previous sequence, that is, let $y_k \in BR(x_k)$ for every k. Assume that both sequences are convergent to x^* and y^*, respectively. For each player i, therefore, it is obtained that $y_{k,i} \in BR(x_{k,-i})$, that is, for every $\tilde{x}_i \in Q_i$ it holds that

$$E_i(y_{k,i}, x_{k,-i}) \geq E_i(\tilde{x}_i, x_{k,-i})$$

for the risk neutral cases, whereas for the risk sensitive version an analogous result can be seen:

$$-\frac{1}{\lambda_i}\log(\mathcal{E}_i^{\lambda_i}(y_{k,i}, x_{k,-i})) \geq -\frac{1}{\lambda_i}\log(\mathcal{E}_i^{\lambda_i}(\tilde{x}_i, x_{k,-i}))$$

for the risk prone case, and

$$-\frac{1}{\lambda_i}\log(\mathcal{E}_i^{\lambda_i}(y_{k,i}, x_{k,-i})) \leq -\frac{1}{\lambda_i}\log(\mathcal{E}_i^{\lambda_i}(\tilde{x}_i, x_{k,-i}))$$

for the risk averse case. In all instances it is possible to take limits on both sides, so

$$\begin{aligned}
\lim_{k\to\infty} E_i(y_{k,i}, x_{k,-i}) &\geq \lim_{k\to\infty} E_i(\tilde{x}_i, x_{k,-i})\\
\lim_{k\to\infty} -\frac{1}{\lambda_i}\log(\mathcal{E}_i^{\lambda_i}(y_{k,i}, x_{k,-i})) &\geq \lim_{k\to\infty} -\frac{1}{\lambda_i}\log(\mathcal{E}_i^{\lambda_i}(\tilde{x}_i, x_{k,-i}))\\
\lim_{k\to\infty} -\frac{1}{\lambda_i}\log(\mathcal{E}_i^{\lambda_i}(y_{k,i}, x_{k,-i})) &\leq \lim_{k\to\infty} -\frac{1}{\lambda_i}\log(\mathcal{E}_i^{\lambda_i}(\tilde{x}_i, x_{k,-i}))
\end{aligned}$$

and by continuity of log and interchanging the limits and the sums, it follows that

$$\begin{aligned}
E_i(y_i^*, x_{-i}^*) &\geq E_i(\tilde{x}_i, x_{-i}^*)\\
-\frac{1}{\lambda_i}\log(\mathcal{E}_i^{\lambda_i}(y_i^*, x_{-i}^*)) &\geq -\frac{1}{\lambda_i}\log(\mathcal{E}_i^{\lambda_i}(\tilde{x}_i, x_{-i}^*))\\
-\frac{1}{\lambda_i}\log(\mathcal{E}_i^{\lambda_i}(y_i^*, x_{-i}^*)) &\leq -\frac{1}{\lambda_i}\log(\mathcal{E}_i^{\lambda_i}(\tilde{x}_i, x_{-i}^*))
\end{aligned}$$

holds for every $\tilde{x}_i \in Q_i$. Therefore, $y_i^* \in BR(x_{-i}^*)$ for each player i. This implies that $y^* \in BR(x^*)$. □

For these models, the convexity of the best response correspondence can be easily proved.

Proposition 4.5. The best response correspondence BR is convex.

P r o o f. Let $x_i, x_i' \in BR_i(x_{-i})$, then the ex-post expected utility for the convex combination $\mu x_i + (1-\mu)x_i'$ for $\mu \in [0,1]$ against x_{-i} can be written in the following way:

$$E_i(((\mu x_i + (1-\mu)x_i'), x_{-i}), \theta) = \left(\sum_{s_i \in S_i} \cdots \sum_{s^T \in S_i} u_i(s^T, \ldots, s^1 \mid \theta_i) \right.$$

$$\times (\mu x_i(s^T \mid s^{T-1}, \ldots, s^1; \theta_i) + (1-\mu)x_i'(s^T \mid s^{T-1}, \ldots, s^1; \theta_i))p_i(i) \cdots$$

$$\left. \times\ (\mu x_i(s^1 \mid \theta_i) + (1-\mu)x_i'(s^1 \mid \theta_i))p_i(i) \right)$$

$$+ \left(\sum_{n^1 \in I \setminus \{i\}} \sum_{s^1 \in S_{n^1}} \sum_{s^2 \in S_i} \cdots \sum_{s^T \in S_i} u_i(s^T, \ldots, s^1 \mid \theta_i) \cdots \right.$$

$$\times (\mu x_i(s^T \mid s^{T-1}, \ldots, s^1; \theta_i) + (1-\mu)x_i'(s^T \mid s^{T-1}, \ldots, s^1; \theta_i))p_i(i) \cdots$$

$$\times (\mu x_i(s^2 \mid s^1; \theta_i) + (1-\mu)x_i'(s^2 \mid s^1; \theta_i))p_i(i)x_{n^1}(s^1 \mid \theta_{n^1})p_i(n^1) + \cdots$$

$$+ \sum_{s^1 \in S_i} \cdots \sum_{s^{T-1} \in S_i} \sum_{n^T \in I \setminus \{i\}} \sum_{s^T \in S_{n^T}} u_i(s^T, \ldots, s^T \mid \theta_i)x_{n^T}(s^T \mid s^{T-1}, \ldots, s^1; \theta_{n^T})p_i(n^T) \cdots$$

$$\times (\mu x_i(s^{T-1} \mid s^{T-2}, \ldots, s^1; \theta_i) + (1-\mu)x_i'(s^{T-1} \mid s^{T-2}, \ldots, s^1; \theta_i))p_i(i) \cdots$$

$$\left. \times (\mu x_i(s^1 \mid \theta_i) + (1-\mu)x_i'(s^1 \mid \theta_i))p_i(i) \right) + \cdots$$

$$+ \left(\sum_{n^1 \in I \setminus \{i\}} \sum_{s^1 \in S_{n^1}} \cdots \sum_{n^T \in I \setminus \{i\}} \sum_{s^T \in S_{n^T}} u_i(s^T, \ldots, s^1 \mid \theta_i) \right.$$

$$\left. \times\ x_{n^T}(s^T \mid s^{T-1}, \ldots, s^1; \theta_{n^T})p_i(n^T) \cdots x_{n^1}(s^1 \mid \theta_{n^1})p_i(n^1) \right)$$

$$= \left(\sum_{s_i \in S_i} \cdots \sum_{s^T \in S_i} u_i(s^T, \ldots, s^1 \mid \theta_i) \right.$$

$$\times (\mu x_i(s^T \mid s^{T-1}, \ldots, s^1; \theta_i) + (1-\mu)x_i'(s^T \mid s^{T-1}, \ldots, s^1; \theta_i))p_i(i) \cdots$$

$$\left. \times\ (\mu x_i(s^1 \mid \theta_i) + (1-\mu)x_i'(s^1 \mid \theta_i))p_i(i) \right)$$

$$+ \left(\sum_{n^1 \in I \setminus \{i\}} \sum_{s^1 \in S_{n^1}} \sum_{s^2 \in S_i} \cdots \sum_{s^T \in S_i} u_i(s^T, \ldots, s^1 \mid \theta_i) \cdots \right.$$

$$\times\, (\mu x_i(s^T \mid s^{T-1},\ldots,s^1;\theta_i) + (1-\mu)x_i'(s^T \mid s^{T-1},\ldots,s^1;\theta_i))p_i(i)\cdots$$

$$\times\, (\mu x_i(s^2 \mid s^1;\theta_i) + (1-\mu)x_i'(s^2 \mid s^1;\theta_i))p_i(i)$$

$$\times\, (\mu x_{n^1}(s^1 \mid \theta_{n^1}) + (1-\mu)x_{n^1}(s^1 \mid \theta_{n^1}))p_i(n^1) + \cdots$$

$$+ \sum_{s^1\in S_i}\cdots\sum_{s^{T-1}\in S_i}\sum_{n^T\in I\setminus\{i\}}\sum_{s^T\in S_{n^T}} u_i(s^T,\ldots,s^T \mid \theta_i)$$

$$\times\, (\mu x_{n^T}(s^T \mid s^{T-1},\ldots,s^1;\theta_{n^T}) + (1-\mu)x_{n^T}(s^T \mid s^{T-1},\ldots,s^1;\theta_{n^T}))p_i(n^T)$$

$$\times\, (\mu x_i(s^{T-1} \mid s^{T-2},\ldots,s^1;\theta_i) + (1-\mu)x_i'(s^{T-1} \mid s^{T-2},\ldots,s^1;\theta_i))p_i(i)\cdots$$

$$\left.\times(\mu x_i(s^1 \mid \theta_i) + (1-\mu)x_i'(s^1 \mid \theta_i))p_i(i)\right) + \cdots$$

$$+ \left(\sum_{n^1\in I\setminus\{i\}}\sum_{s^1\in S_{n^1}}\cdots\sum_{n^T\in I\setminus\{i\}}\sum_{s^T\in S_{n^T}} u_i(s^T,\ldots,s^1 \mid \theta_i)\right.$$

$$\times\, (\mu x_{n^T}(s^T \mid s^{T-1},\ldots,s^1;\theta_{n^T}) + (1-\mu)x_{n^T}(s^T \mid s^{T-1},\ldots,s^1;\theta_{n^T}))p_i(n^T)\cdots$$

$$\left.\times\, (\mu x_{n^1}(s^1 \mid \theta_{n^1}) + (1-\mu)x_{n^1}(s^1 \mid \theta_{n^1}))p_i(n^1)\right)$$

$$= \mu E_i((x_i,x_{-i}),\theta) + (1-\mu)E_i((x_i',x_{-i}),\theta).$$

Now, it can be seen that $E_i((x_i,x_{-i}),\theta) = E_i((x_i',x_{-i}),\theta)$. This is because, by definition, since x_i is a best response to x_{-i}, then $E_i((x_i,x_{-i}),\theta) \geq E_i((\tilde{x}_i,x_{-i}),\theta)$ for all $\tilde{x}_i \in Q_i$, particularly for $\tilde{x}_i = x_i'$. The same can be done reversing the roles of x_i and x_i'. Therefore, it follows that

$$E_i(((\mu x_i + (1-\mu)x_i'),x_{-i}),\theta) = \mu E_i((x_i,x_{-i}),\theta) + (1-\mu)E_i((x_i,x_{-i}),\theta) = E_i((x_i,x_{-i}),\theta)$$

and then, since this follows for the ex-ante expected utility, implies that $\mu x_i + (1-\mu)x_i'$ is a best response to x_{-i}. Therefore BR is a convex correspondence.

Following similar arguments, we may show that in the risk sensitive case the best response correspondence is convex. □

The previous results show that the best response correpondence BR satisfies Kakutani's fixed-point theorem [6], therefore guaranteeing the existence of at least one fixed point for the best response correspondence. It is easy to show that the fixed points of the best response correspondence are exactly the Nash equilibria (or Bayes-Nash equilibria, respectively) of each of the models presented above, since the definition of the ex-ante expected utility can be given in terms of the ex-interim and this one in turn can be defined in terms of the ex-post utility, which means the best responses for the ex-ante expected utility are an optimization over the "best responses" for the ex-interim expected utility, and these are an optimization over the "best responses" for the ex-post case. Therefore the ex-ante best responses are also optimal for the other two expected utilities. Finally, this implies that finding a fixed point on the set of best responses is equivalent to a Bayes-Nash equilibrium, since it is optimal in the ex-post case, where the uncertainty over the types of the players has been removed. Therefore the main result of this paper is obtained.

Theorem 4.6. Every game that can be modelled using any of the previous four frameworks has at least one Nash (or Bayes-Nash, respectively) equilibrium.

5. EXAMPLES

Example 5.1. There are two players, denoted 1 and 2. To show how risk averse and risk prone players change their behavior from risk neutral players the following example is presented. There is a two stage game in which the player that chooses in the second stage doesn't know what the player in the first stage chose, unless the same player chooses in both the first and the second stages. In each of these, players have two choices, which in the first stage have been denoted as A and B and for the second stage as C and D. At each decision point, each player has equal probabilities of being chosen. The diagram for this game with the corresponding utilities for each possible pair of choices is shown in Figure 1.

It can be noticed that there are no dominated strategies as there are instances in which A and B are preferred in the first stage by both players according to what happens in the second stage, while the same occurs for C and D. As previously stated, the player who is chosen at the second and the last period can make a completely informed choice if and only if the previous period he was chosen as well. Otherwise, the choice is dependent on what the other player's strategy is. Therefore what happens at the last period is studied first and then the full expected utility for each player is considered.

If player 2 is chosen at the first period and player 1 is chosen at the second period, then the expected utility of player 1 is:

$$\begin{aligned}
E_1(x) &= \sum_{s^1 \in \{A,B\}} \sum_{s^2 \in \{C,D\}} u_1(s^2, s^1) p_1(1) x_1(s^2 \mid s^1) p_1(2) x_2(s^1) \\
&= 0.25(u_1(C,A)x_1(C)x_2(A) + u_1(C,B)x_1(C)(1 - x_2(A)) \\
&\quad + u_1(D,A)x_1(D)x_2(A) + u_1(D,B)x_1(D)(1 - x_2(A))) \\
&= 0.25(5 - 2x_2(A))x_1(C) + (2x_2(A) + 4)(1 - x_1(C))) \\
&= 0.25((1 - 4x_2(A))x_1(C) + 2x_2(A) + 4),
\end{aligned}$$

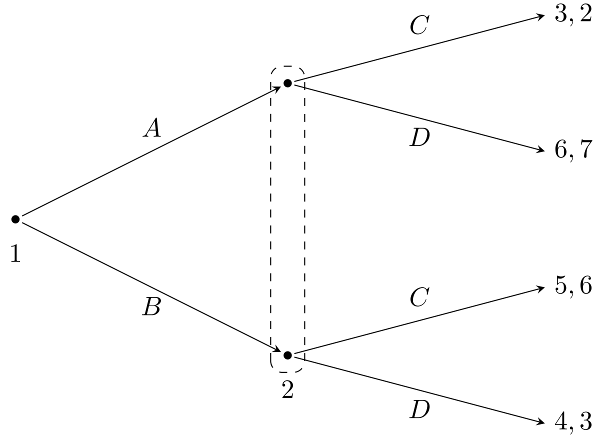

Fig. 1. A two-stage sequential game.

where player 1 doesn't know whether A or B has been chosen at the first stage, so player 1's strategies are unconditioned.

Therefore it is obtained that

- if $x_2(A) > \frac{1}{4}$, then $x_1(C) = 0$ and $x_1(D) = 1$,
- if $x_2(A) < \frac{1}{4}$, then $x_1(C) = 1$ and $x_1(D) = 0$, and
- if $x_2(A) = \frac{1}{4}$, then $x_1(C) \in [0, 1]$ and $x_1(D) = 1 - x_2(C)$.

A similar process can be used for the case in which player 1 is chosen at the first period and player 2 is chosen at the second period, as the expected utility of player 2 is:

$$\begin{aligned}
&0.25(u_2(C, A)x_2(C)x_1(A) + u_2(C, B)x_2(C)(1 - x_1(A)) \\
&+ u_2(D, A)x_2(D)x_1(A) + u_2(D, B)x_2(D)(1 - x_1(A))) \\
&= 0.25((6 - 4x_1(A))x_2(C) + (4x_1(A) + 3)(1 - x_2(C))) \\
&= 0.25((3 - 8x_1(A))x_2(C) + 4x_1(A) + 3)
\end{aligned}$$

which gives the following conditions:

- if $x_1(A) > \frac{3}{8}$, then $x_2(C) = 0$ and $x_2(D) = 1$,
- if $x_1(A) < \frac{3}{8}$, then $x_2(C) = 1$ and $x_2(D) = 0$, and
- if $x_1(A) = \frac{3}{8}$, then $x_2(C) \in [0, 1]$ and $x_2(D) = 1 - x_2(C)$.

Now the first period is studied altogether. For player 1, the expected utility can be written as

$$E_1(x) = \sum_{n^1 \in \{1,2\}} \sum_{s^1 \in \{A,B\}} \sum_{n^2 \in \{1,2\}} \sum_{s^2 \in \{C,D\}} u_1(s^2, s^1)p_1(n^2)x_{n^2}(s^2 \mid s^1)p_1(n^1)x_{n^1}(s^1)$$

$$
\begin{aligned}
&= 0.25\{[2u_1(C,B) + u_1(C,B)x_1(C \mid B_2) + u_1(C,B)x_2(C \mid B_1) \\
&\quad + u_1(C,B) + u_1(D,B)x_1(D \mid B_2) + u_1(D,B)x_2(D \mid B_1)] \\
&\quad + [-u_1(C,B) + u_1(C,A)x_2(C \mid A_1) - u_1(C,B)c_2(C \mid B_1) \\
&\qquad + u_1(D,A) + u_1(D,A)x_2(D \mid A_1) - u_1(D,B)x_2(D \mid B_1)]x_1(A) \\
&\quad + [u_1(C,A)x_1(C \mid A_2) - u_1(C,B)x_1(C \mid B_2) - u_1(C,B) \\
&\qquad + u_1(D,A)x_1(D \mid A_2) - u_1(D,B)x_1(D \mid B_2) + u_1(D,A)]x_2(A)\}
\end{aligned}
$$

from which it is possible to only focus on the factor that multiplies $x_1(A)$ as it determines how player 1 should play if he is chosen at the first turn. Therefore, it is obtained that

$$-5 + 3x_2(C \mid A_1) - 5x_2(C \mid B_1) + 6 + 6x_2(D \mid A_1) - 4x_2(D \mid B_1) = -1 + 4x_2(D)$$

since, given that players don't know whether A or B was chosen by their opponent, it is possible to consider $x_2(C \mid A_1)$ and $x_2(C \mid B_1)$ as simply $x_2(C)$, and the same occurs for $x_2(D)$. Then the following conditions are obtained:

- If $x_2(D) > \frac{1}{4}$, then $x_1(A) = 1$ and $x_1(B) = 0$,
- if $x_2(D) < \frac{1}{4}$, then $x_1(A) = 0$ and $x_1(B) = 1$, and
- if $x_2(D) = \frac{1}{4}$, then $x_1(A) \in [0,1]$ and $x_1(B) = 1 - x_1(A)$,

which, together with the conditions when player 1 is chosen in the first period and player 2 is chosen in the second period, gives the equilibria:

1. $x_1(A) = 1$, $x_1(B) = 0$, $x_2(C) = 0$ and $x_2(D) = 1$,
2. $x_1(A) = 0$, $x_1(B) = 1$, $x_2(C) = 1$ and $x_2(D) = 0$, and
3. $x_1(A) = \frac{3}{8}$, $x_1(B) = \frac{5}{8}$, $x_2(C) = \frac{3}{4}$ and $x_2(D) = \frac{1}{4}$.

By a similar analysis as the one made above, we get the equilibria for the case in which player 2 is chosen first and player 1 is chosen second:

1. $x_1(C) = 0$, $x_1(D) = 1$, $x_2(A) = 1$ and $x_2(B) = 0$,
2. $x_1(C) = 1$, $x_1(D) = 0$, $x_2(A) = 0$ and $x_2(B) = 1$, and
3. $x_1(C) = \frac{5}{8}$, $x_1(D) = \frac{3}{8}$, $x_1(A) = \frac{1}{4}$ and $x_2(B) = \frac{3}{4}$.

Now the players are risk averse or risk prone and it is to be studied how the equilibria are affected. To see the effect, one of the players is kept risk neutral and the behavior of the other player is changed. If player 1's behavior is changed, his expected utility is now

$$
\begin{aligned}
\mathcal{E}_1^\lambda(x) &= \sum_{n^1 \in \{1,2\}} \sum_{s^1 \in \{A,B\}} \sum_{n^2 \in \{1,2\}} \sum_{s^2 \in \{C,D\}} e^{-\lambda u_1(s^2,s^1)} x_{n^2}(s^2 \mid s^1) p_1(n^2) x_{n^1}(s^1) p_1(n^1) \\
&= 0.25((e^{-\lambda u_1(C,B)}(2 + x_1(C \mid B_2) + x_2(C \mid B_1)) \\
&\quad + e^{-\lambda u_1(D,B)}(x_1(D \mid B_2) + x_2(D \mid B_1)))
\end{aligned}
$$

$$
\begin{aligned}
&+ [-e^{-\lambda u_1(C,B)}(1 + x_1(C \mid B)) + e^{-\lambda u_1(C,A)} x_2(C \mid A_1) \\
&\quad + e^{-\lambda u_1(D,A)}(1 + x_2(D \mid A_1) - e^{-\lambda u_1(D,B)} x_2(D \mid B_1)] x_1(A) \\
&+ \{e^{-\lambda u_1(C,A)} x_1(C \mid A_2) - e^{-\lambda u_1(C,B)}(1 + x_1(C \mid B_2)) \\
&\quad + e^{-\lambda u_1(D,A)}(1 + x_1(D \mid A_2)) - e^{-\lambda u_1(D,B)} x_1(D \mid B_2)\} x_2(A))
\end{aligned}
$$

and only the factor inside square brackets is important to determine $x_1(A)$, for which the numerical values for the utilities are substituted, and consider that players do not know what happened in the second period if they were not chosen, i. e. both $x_2(C \mid B_1)$ and $x_2(C \mid A_1)$ can be regarded as the same, as player 2 would not have the information of the first period. This way the square brackets can be reduced to

$$2e^{-6\lambda} - e^{-5\lambda} - e^{-4\lambda} + (e^{-3\lambda} + e^{-4\lambda} - e^{-5\lambda} - e^{-6\lambda}) x_2(C),$$

since $x_2(C) = 1 - x_2(D)$. Therefore, the equilibrium is obtained when

$$x_2(C) = \frac{e^{-4\lambda} + e^{-5\lambda} - 2e^{-6\lambda}}{e^{-3\lambda} + e^{-4\lambda} - e^{-5\lambda} - e^{-6\lambda}}.$$

For values of $\lambda < 0$, that is when player 1 is risk prone, if $x_2(C)$ is regarded as the belief about player 2's choice, it can be seen that $x_2(C) > 3/4$, that is, in order to play with positive probability the choice A, player 1 can risk player 2 choosing C with a higher probability, as long as it is possible that D is chosen, since (A, D) would give the highest utility to player 1. Similarly, for $\lambda > 0$, when player 1 is risk averse, it is obtained that $x_2(C) < 3/4$, implying that player 1 requires C being chosen with less probability in order to risk playing A, otherwise, it is better to play B as it has safer utilities. Similarly, it is possible to get an expression for the expected utility of player 2:

$$
\begin{aligned}
\mathcal{E}_1^\lambda(x) = 0.25((&e^{-\lambda u_2(C,B)}(2 + x_1(C \mid B_2) + x_2(C \mid B_1)) \\
&+ e^{-\lambda u_2(D,B)}(x_1(D \mid B_2) + x_2(D \mid B_1))) \\
&+ [-e^{-\lambda u_2(C,B)}(1 + x_1(C \mid B)) + e^{-\lambda u_2(C,A)} x_2(C \mid A_1) \\
&\quad + e^{-\lambda u_2(D,A)}(1 + x_2(D \mid A_1) - e^{-\lambda u_2(D,B)} x_2(D \mid B_1)] x_1(A) \\
&+ \{e^{-\lambda u_2(C,A)} x_1(C \mid A_2) - e^{-\lambda u_2(C,B)}(1 + x_1(C \mid B_2)) \\
&\quad + e^{-\lambda u_2(D,A)}(1 + x_1(D \mid A_2)) - e^{-\lambda u_2(D,B)} x_1(D \mid B_2)\} x_2(A))
\end{aligned}
$$

and now the important part is between curly braces, which can be reduced to

$$2e^{-7\lambda} - e^{-6\lambda} - e^{-3\lambda} + (e^{-2\lambda} + e^{-3\lambda} - e^{-6\lambda} - e^{-7\lambda}) x_1(C)$$

implying that equilibrium is reached with

$$x_1(C) = \frac{e^{-3\lambda} + e^{-6\lambda} - 2e^{-7\lambda}}{e^{-2\lambda} + e^{-3\lambda} - e^{-6\lambda} - e^{-7\lambda}}.$$

As before, it is obtained for $\lambda < 0$ that $x_1(C) > 5/8$, so player 2 would risk playing A even if player 2 would choose C with higher probability than in the risk neutral case, and for $\lambda > 0$ to play A the value should be $x_1(C) < 5/8$, so being risk averse would make player 2 choose A only if the chances of player 2 choosing D were higher than in the risk neutral case.

Example 5.2. For the next example, an addition is made onto Example 1. Player 2 will have two types which will reverse his payouts. For type α the payouts are the same as in Figure 1, whereas for type β the game has the structure shown in Figure 2. Player 2 is type α with probability 0.4 and type β with probability 0.6.

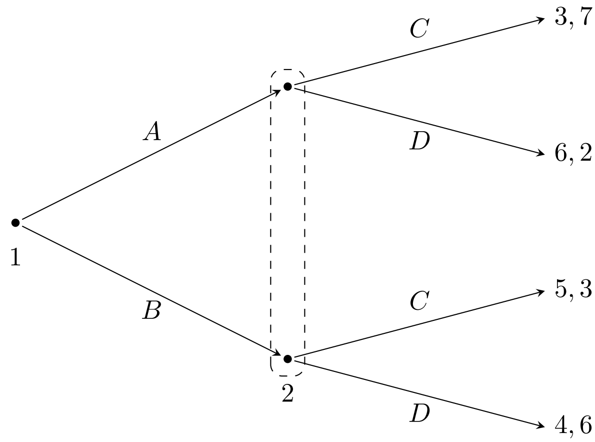

Fig. 2. Variation of the game where player 2 is of type β.

The case in which player 1 is chosen at turn 1 and player 2 is chosen at turn 2 is considered. The expected utility for player 1 (or rather the factor which concerns player 1's choice at the first turn) is

$$(3 - 4(0.4x_2(C;\alpha) + 0.6x_2(C;\beta)))x_1(A);$$

whereas player 2's expected utility (the factor that determines player 2's choice at the second turn) is:

$$9 + x_2(C;\alpha)[-8x_1(A) + 3] + x_2(C;\beta)[8x_1(A) - 3]$$

which means that the equilibria are given by:

- $x_1(A) =1$, $x_1(B) = 0$, $x_2(C;\alpha) = 0$, $x_2(D;\alpha) = 1$, $x_2(C;\beta) = 1$, $x_2(D;\beta) = 0$.
- $x_1(A) = \frac{3}{8}$, $x_1(B) = \frac{5}{8}$, and for player 2, any mixed strategies such that $0.4x_2(C;\alpha) + 0.6x_2(C;\beta) = \frac{3}{4}$.

Now if player 1 is risk sensitive, his expected utility is given by:

$$[e^{-6\lambda} + e^{-6\lambda} - e^{-5\lambda} - e^{-4\lambda} + (e^{-3\lambda} + e^{-4\lambda} - e^{-5\lambda} - e^{-6\lambda})(0.4x_2(C;\alpha) + 0.6x_2(C;\beta))]x_1(A)$$

which changes the second equilibrium described above to $x_1(A) = \frac{3}{8}$, $x_1(B) = \frac{5}{8}$ and for player 2 any mixed strategies such that:

$$0.4x_2(C;\alpha) + 0.6x_2(C;\beta) = \frac{e^{-4\lambda} + e^{-5\lambda} - 2e^{-6\lambda}}{e^{-3\lambda} + e^{-4\lambda} - e^{-5\lambda} - e^{-6\lambda}}. \qquad (1)$$

We plot the right side of this last equation in order to figure out its behavior for different values of λ.

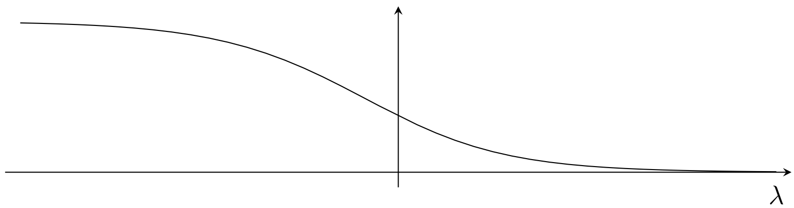

Fig. 3. Plot of the right side of Equation 1.

As it can be seen from Figure 3 the solutions to the left side of Equation 1 for $\lambda < 0$ are such that player 1 is willing to risk player 2 choosing C with a higher probability than in the risk neutral case for at least one of its types (or maybe both). In the same way, when $\lambda > 0$, player 1 is risk averse, so he's playing A only if player 2 would choose C with a smaller probability than what happens in the risk neutral case for at least one of its types (if not both).

6. CONCLUSIONS

A modification to consider games with risk sensitivity was shown above, particularly for games in which the order of the players is not known from the beginning, while also considering games where players have partial information about the utilities of the other players. To do so, it has been shown that the sets of best responses for each player which are obtained via a best response correspondence satisfy the conditions of Kakutani's fixed point theorem, and such fixed points are the equilibria for the games.

Moreover, the examples showed first in a game with unobserved past choices the equilibria in which introducing risk sensitivity made the equilibria move such that it makes risk prone players go for the choices that lead towards the higher utilities, and risk averse players go for a safer option with less variability. The degree of risk proneness or averseness can also be observed as λ is farther from 0. Afterwards a similar example with incomplete information is presented, in which the equilibria are presented for a risk neutral player and then a risk sensitive modification is made, in which it is possible to observe once again that players will move towards options that give a larger utility but with a higher risk if they're risk prone, and vice versa. In the example shown the behavior is visible but it could get more entangled if the utilities change in a more drastic way.

A possible extension to this work would be to change the way the risk aversion coefficient is defined by allowing it to depend as well on the wealth of the player so that $r_i(z)$ is no longer constant. In this way, players would not only make their choice according to the utility but also according to the differences of the utilities in each possible choice, and the relative difference of these with the current wealth, making it possible to model situations where external variables influence how choices in real life are made.

Another possible line is to consider when the turn selection process is affected by the choices made by players, therefore making it depend as well on the actions selected. Therefore, players also have to take into account whether the choices they make are worth it given that this may modify their chances of being picked to make another move sooner or later. In a similar line, it would also be interesting to study the case in which the turn selection process depends as well on the types of all players. Though we only consider the case in which types affect the utility function of players, this would introduce some incomplete information as well on the turn selection process, making it more general, though at the same time, more difficult to study.

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