KYBERNETIKA — VOLUME *55* (2019), NUMBER 6, PAGES 915–942

PROBABILISTIC PROPERTIES OF A MARKOV-SWITCHING PERIODIC *GARCH* PROCESS

Billel ALIAT and Fayçal HAMDI

In this paper, we propose an extension of a periodic $GARCH$ ($PGARCH$) model to a Markov-switching periodic $GARCH$ (MS-$PGA\ RCH$), and provide some probabilistic properties of this class of models. In particular, we address the question of strictly periodically and of weakly periodically stationary solutions. We establish necessary and sufficient conditions ensuring the existence of higher order moments. We further provide closed-form expressions for calculating the even-order moments as well as the autocovariances of the powers of a MS-$PGARCH$ process. We thus show how these moments and autocovariances can be used for estimating model parameters using GMM method.

Keywords: Markov-switching models, periodic $GARCH$ models, periodic stationarity, higher-order moments, Markov-switching $PGARCH$ models, GMM method

Classification: 62M10, 60G10

1. INTRODUCTION

Modeling volatility remains an important research area given its crucial role in modeling financial data such as stock price indices, interest rates and exchange rate data. Among volatility models, we find the $GARCH$ model proposed by Bollerslev [15] which is one of the most popular models, because it represents a powerful tool for analyzing and forecasting the volatility of financial markets. The $GARCH$ formulation explicitly describes instantaneous volatility using both conditional variances and squares of observations. After its introduction in time series literature, $GARCH$ model has captured several empirical features that characterize many financial data, referred to as stylized facts, such as lack of serial correlation, nonlinear dependence, heavy-tailed marginal distributions and volatility clustering. However, other features such as multimodality of marginal distributions and regime changes remain uncaptured by this class of models. These features are best represented by regime switching models.

Since the seminal paper by Hamilton [29], the use of Markov-switching (in short MS) models has become increasingly popular in dynamic econometrics. Introducing changes in regime into the classical time series models substantially increases their flexibility. Therefore, different research works using $ARCH$-type models have been developed in

DOI: 10.14736/kyb-2019-6-0915

that direction. Cai [18] and Hamilton and Susmel [30] introduced the Markov-switching $ARCH$ model (MS-$ARCH$). Gray [24] proposed a Markov-switching $GARCH$ (MS-$GARCH$) model assuming that the conditional variance, knowing the current regime, depends on the expectation of past conditional variances, rather than their values. Klaassen [33] and Haas et al. [26] proposed other MS-$GARCH$ formulations that differ from Gray's one. Since then, an important number of studies have been devoted to MS-$GARCH$ models. See, among others, Francq and Zakoïan [20, 21], Bauwens et al. [7], Augustyniak [5] and Billio et al. [14].

On the other hand, it is widely documented that most asset and exchange rate returns exhibit strong seasonal patterns in the form of day of the week and holiday effects (see, e. g. Bollerslev and Ghysels [16]; Franses and Paap [22]; Hamdi and Souam [28]). Hence, we can say that each day of the week constitutes a different regime. This seasonal effect can be interpreted as a deterministic regime-switching behavior. Contrary to the MS modeling, the regime that occurs at any given point in time is known with certainty in advance. As a result, financial time series analysts have, nowadays, become more convinced of the need to combine periodicity and conditional heteroskedasticity in one model. In particular, the class of periodic $GARCH$ ($PGARCH$) models introduced by Bollerslev and Ghysels [16] has shown to be appropriate for capturing periodicity in the conditional variance, a property that cannot be explained by the classical $GARCH$ formulation. Despite the interest and importance of the $PGARCH$ model since its introduction, various limitations of this class of models (see e. g. Bentarzi and Hamdi [8, 9]; Regnard and Zakoïan [37], Hamdi and Souam [28]) pushed us to propose another, more flexible, class of models, able to capture different features that characterize, usually, economic series in general, and financial time series in particular.

In this paper, we present a study of the probabilistic properties of Markov-switching periodic $GARCH$ (MS-$PGARCH$) models that constitute a very flexible and parsimonious class of nonlinear time series models of the conditional variance. The rest of this article is organized as follows. In section 2, we introduce the class of Markov-switching periodic $GARCH$ models and give some related notations and assumptions. In section 3, we give a condition under which strictly periodically stationary solution to the MS-$GARCH$ equation exists. In section 4, we focus on conditions ensuring finiteness of higher order moments, and provide explicitly the expression of these moments. We also give the autocorrelation function of the squared observations. In Section 5, we concentrate on the particular MS-$PGARCH\,(1,1)$ case that allows for rather simple explicit expressions. The first part of the Section 6 is devoted to the GMM parameter estimation problem. In the second part, we provide a simulation study of the performance of the proposed estimation method.

2. MARKOV-SWITCHING PERIODIC $GARCH$ MODEL

Let $\{\eta_t; t \in \mathbb{Z}\}$ be a sequence of independent and identically distributed ($i.i.d.$) random variables and let ω_t, $\alpha_{t,i}$ and $\beta_{t,j}$, for $1 \leq i \leq q$ and $1 \leq j \leq p$, be nonnegative periodic functions with period S, i. e. $\omega_{t+\tau S} = \omega_t > 0$, $\alpha_{t+\tau S,i} = \alpha_{t,i} \geq 0$ and $\beta_{t+\tau S,j} = \beta_{t,j} \geq 0$. Recall that a $PGARCH\,(p_t, q_t)$ process $\{\epsilon_t, t \in \mathbb{Z}\}$ with periodic volatility process

$\{h_t, t \in \mathbb{Z}\}$ is a solution to the equations

$$\begin{cases} \epsilon_t = \sqrt{h_t}\eta_t, \ t \in \mathbb{Z} \\ h_t = \omega_t + \sum\limits_{i=1}^{q_t} \alpha_{t,i}\epsilon_{t-i}^2 + \sum\limits_{j=1}^{p_t} \beta_{t,j}h_{t-j}. \end{cases} \tag{1}$$

Note that in some articles the definition of p_t and q_t for $PGARCH$ models is a S-periodic function in t. For simplicity purposes, p_t and q_t can be taken as constants in t; simply set $(p, q) = (\max p_t, \max q_t)$ and take $\alpha_{t,i} = 0$, for $i > q_t$ and $\beta_{t,j} = 0$, for $j > p_t$.

By multiplying the second equation in (1) by η_t^2, we get

$$\epsilon_t^2 = h_t\eta_t^2 = \omega_t\eta_t^2 + \sum_{i=1}^{q} \alpha_{t,i}\eta_t^2\epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_{t,j}\eta_t^2 h_{t-j},$$

and by defining the random vectors z_t and b_t of dimension $(p+q)$

$$z_t = \left(\epsilon_t^2, \epsilon_{t-1}^2, \ldots, \epsilon_{t-q+1}^2, h_t, h_{t-1}, \ldots, h_{t-p+1}\right)' \text{ and } b_t = \left(\omega_t\eta_t^2, \mathbf{0}_{1\times(q-1)}, \omega_t, \mathbf{0}_{1\times(p-1)}\right)',$$

as well as the square random matrix A_t of dimension $(p+q)$

$$A_t = \begin{pmatrix} \alpha_{t,1:q-1}\eta_t^2 & \alpha_{t,q}\eta_t^2 & \beta_{t,1:p-1}\eta_t^2 & \beta_{t,q}\eta_t^2 \\ \mathbf{I}_{q-1} & \mathbf{0}_{(q-1)\times 1} & \mathbf{0}_{(q-1)\times(p-1)} & \mathbf{0}_{(q-1)\times 1} \\ \alpha_{t,1:q-1} & \alpha_{t,q} & \beta_{t,1:p-1} & \beta_{t,q} \\ \mathbf{0}_{(p-1)\times(q-1)} & \mathbf{0}_{(q-1)\times 1} & \mathbf{I}_{p-1} & \mathbf{0}_{(p-1)\times 1} \end{pmatrix},$$

where $\alpha_{t,1:q} = (\alpha_{t,1}, \ldots, \alpha_{t,q})$ and $\beta_{t,1:p} = (\beta_{t,1}, \ldots, \beta_{t,p})$. As a result, the model (1) admits the following Markovian representation

$$Y_t = A_t Y_{t-1} + b_t. \tag{2}$$

Note that the periodic stationarity property of the model (2) can be studied by examining the stationarity of a certain appropriate transformation (see Gladyshev [23]). Indeed, it is well known that a periodically stationary process $\{Y_t, t \in \mathbb{Z}\}$ is equivalent to the stationary S-variate process $\{\mathbf{Y}_\tau, \tau \in \mathbb{Z}\}$, where $\mathbf{Y}_\tau = \left(Y'_{1+S\tau}, Y'_{2+S\tau}, \ldots, Y'_{S+S\tau}\right)$. This last process admits the generalized autoregressive representation

$$\mathbf{Y}_\tau = \mathbf{A}_\tau Y_{\tau-1} + \mathbf{b}_\tau, \tag{3}$$

where $(\mathbf{A}_\tau, \mathbf{b}_\tau)_{\tau\in\mathbb{Z}}$ is an *i.i.d.* process and $\mathbf{A}_\tau$ and $\mathbf{b}_\tau$ are defined by blocks respectively as

$$(\mathbf{A}_\tau)_{i,j} = \prod_{s=0}^{i-1} A_{i-s+\tau S}\mathbf{1}_{(j=S)}$$

and

$$(\mathbf{b}_\tau)_{i,1} = \sum_{v=1}^{i} \left(\prod_{s=0}^{i-v-1} A_{i-s+\tau S}\right)\mathbf{b}_{v+\tau S}, \text{ for } i, j = 1, \ldots, S.$$

The strict periodic stationarity of the $PGARCH\,(1,1)$ model was studied by Aknouche and Bentarzi [1]. For more general $PGARCH$ models, the strict periodic stationarity conditions were established by Bibi and Aknouche [10], Aknouche and Bibi [2] and Lee and Shin [35]. Bibi and Aknouche [10] and Lee and Shin [35] used the representation (3) to obtain strict periodic stationarity conditions of the $PGARCH\,(p,q)$ model. Aknouche and Bibi [2] have instead exploited the representation (2) which differs from the standard formulation studied by Bougerol and Picard [17] in that the sequence (A_t, b_t) is rather periodically stationary and periodically ergodic. Aknouche and Bibi [2] have shown that a necessary and sufficient condition for the existence of a nonanticipative and strictly periodic stationary solution is that the top Lyapunov exponent associated with the sequence of matrices $A =: \{A_t,\ t \in \mathbb{Z}\}$ is strictly negative. This nonanticipative solution is unique and periodically ergodic. A necessary and sufficient condition of second-order periodic stationarity of a particular 2-periodic $PGARCH\,(1,1)$ model was established in Bollerslev and Ghysel [16]. The general $PGARCH$ case is considered by Aknouche and Bentarzi [1], as well as the existence of a strictly periodic stationary solution having finite higher order moments (see also Aknouche and Bibi [2]; Bibi and Aknouche [10]). The calculation of some moments and the autocovariances of the squares of $PGARCH$ processes were stated by Bibi and Aknouche [10]). The asymptotic properties, namely the strong consistency and the asymptotic normality, of the quasi-maximum likelihood, the least squares and a Yule–Walker estimators were respectively established in Aknouche and Bibi [2], Bibi and Lesheb [11, 12] and Bibi and Lesheb [13].

Despite the interest and the importance of the $PGARCH$ models since their introduction, various limitations of this class of models (e. g. Bentarzi and Hamdi [8]; Regnard and Zakoïan [37]; Hamdi and Souam [27, 28]) let us to propose a new more flexible class, making possible the capture of the main stylized facts characterizing financial series.

The Markov-switching periodic $GARCH$ model (in short, MS-$PGARCH$), that we propose here can be defined as a bivariate process $\{(\epsilon_t, \Delta_t)\,; t \in \mathbb{Z}\}$. The regime, in which a $PGARCH$ process (ϵ_t) is located at a given date t, is indexed by a Markov chain (Δ_t) defined on a finite state space. Hence, a stochastic process $\{\epsilon_t, t \in \mathbb{Z}\}$ is said to have a Markov-switching $PGARCH$ representation with orders p and q and period $S \geq 1$, denoted MS-$PGARCH_S\,(p,q)$, if it is a solution of the following stochastic difference equation

$$\left\{\begin{array}{l} \epsilon_t = \sqrt{h_t}\eta_t,\ t \in \mathbb{Z} \\ h_t = \sum\limits_{k=1}^{d} \omega_t^{(k)} \mathbf{1}_{(\Delta_t = k)} + \sum\limits_{i=1}^{q} \sum\limits_{k=1}^{d} \alpha_{t,i}^{(k)} \epsilon_{t-i}^2 \mathbf{1}_{(\Delta_t = k)} + \sum\limits_{j=1}^{p} \sum\limits_{k=1}^{d} \sum\limits_{l=1}^{d} \beta_{t,j}^{(k)} h_{t-j}^{(l)} \mathbf{1}_{(\Delta_t = k, \Delta_{t-j} = l)}, \end{array}\right. \tag{4}$$

where $h_t = \mathbb{E}\left[\epsilon_t^2 \mid \mathcal{F}_{t-1}\right]$, $\mathcal{F}_t$ denotes the σ-algebra based on the information available up to time t and $\mathbf{1}_{(\cdot)}$ is the indicator function. The process (Δ_t) is a homogenous Markov chain defined on a finite state space $\mathcal{E} = \{1, 2, \ldots, d\}$, and $\{\eta_t; t \in \mathbb{Z}\}$ is a sequence of independent and identically distributed $(i.i.d.)$ random variables such that $\mathbb{E}\,(\eta_t) = 0$ and $\mathbb{E}\left(\eta_t^2\right) = 1$. The coefficients $\omega_t^{(k)}$, $\alpha_{t,i}^{(k)}$ and $\beta_{t,j}^{(k)}$, for $1 \leq k \leq d$, $1 \leq i \leq q$ and $1 \leq j \leq p$, are periodic functions with period S (i. e. $\omega_{t+\tau S}^{(k)} = \omega_t^{(k)}$, $\alpha_{t+\tau S,i}^{(k)} = \alpha_{t,i}^{(k)}$ and $\beta_{t+\tau S,j}^{(k)} = \beta_{t,j}^{(k)}$), and satisfy the constraints $\omega_t^{(k)} > 0$, $\alpha_{t,i}^{(k)} \geq 0$ and $\beta_{t,j}^{(k)} \geq 0$, for $1 \leq k \leq d$, $1 \leq i \leq q$ and $1 \leq j \leq p$.

Note that the MS-$PGARCH$ model (4) may be seen as a mixture of d components of periodic $GARCH$. At each time point t, one of them generates the observation y_t.

The stochastic process selecting the component generating the observation is supposed to be homogeneous Markov chain. Thus, our model captures, not only, the stochastic regime-switching behavior, but also the periodicity (deterministic regime-switching behavior) hidden in the autocovariance structure of some economic series in general, and of financial time series in particular. Another feature of our model, which cannot be reproduced by with the $PGARCH$ formulation (1), is time-varying skewness. However, this flexibility is unfortunately undermined by a path dependence problem which complicates the parameter estimation process (see Hamilton and Susmel, [30]).

As in the mixture $PGARCH$ framework (see Hamdi and Souam [27]), another specification which differ from (4) for the conditional variance can be considered. In this specification, each regime-specific conditional variance depends only on its own lag, that is,

$$\begin{cases} \epsilon_t = \sqrt{h_t}\eta_t, \ t \in \mathbb{Z} \\ h_t = \sum\limits_{k=1}^{d} \omega_t\left(k\right) \mathbf{1}_{(\Delta_t = k)} + \sum\limits_{i=1}^{q}\sum\limits_{k=1}^{d} \alpha_{t,i}\left(k\right) \epsilon_{t-i}^2 \mathbf{1}_{(\Delta_t = k)} + \sum\limits_{j=1}^{p}\sum\limits_{k=1}^{d} \beta_{t,j}\left(k\right) h_{t-j}^{(k)} \mathbf{1}_{(\Delta_t = k)}. \end{cases}$$

The idea behind this formulation is to model d parallel periodic $GARCH$ processes and the Markov chain determines which process is selected at each time. In addition, the current regime in the lagged variance term is thus preserved. This is, in our view, somewhat artificial to tackle the issue of eliminating the path-dependence problem by forcing the past volatilities to be in the same regime of the current volatility. Actually, the regime of our process can change between $t-1$ and $t-p$. In our opinion, model (4) appears to be the most natural formulation of MS-$PGARCH$ processes.

To rewrite our proposed model (4) in a simple form, we make the following notations. Let

$$\begin{aligned} \omega_t\left(\Delta_t\right) &:= \sum_{k=1}^{d} \omega_t^{(k)} \mathbf{1}_{(\Delta_t = k)}, \\ \alpha_{t,i}\left(\Delta_t\right) &:= \sum_{k=1}^{d} \alpha_{t,i}^{(k)} \mathbf{1}_{(\Delta_t = k)}, \\ \beta_{t,j}\left(\Delta_t\right) &:= \sum_{k=1}^{d} \beta_{t,j}^{(k)} \mathbf{1}_{(\Delta_t = k)}, \end{aligned}$$

and

$$h_{t-j} := \sum_{l=1}^{d} h_{t-j}^{(l)} \mathbf{1}_{(\Delta_{t-j} = l)},$$

then model (4) can be written as

$$\begin{cases} \epsilon_t = \sqrt{h_t}\eta_t, \\ h_t = \omega_t\left(\Delta_t\right) + \sum\limits_{i=1}^{q} \alpha_{t,i}\left(\Delta_t\right) \epsilon_{t-i}^2 + \sum\limits_{j=1}^{p} \beta_{t,j}\left(\Delta_t\right) h_{t-j}, \end{cases} \quad t \in \mathbb{Z}. \tag{5}$$

In what follows, we consider that the processes (η_t) and (Δ_t) are assumed to be independent. In addition, (Δ_t) is a homogenous, stationary, irreducible and aperiodic

Markov chain. The stationary probabilities of (Δ_t) are denoted by $\pi(k) = P(\Delta_1 = k)$, the transition probability matrix is denoted by $\mathbb{P}$ and written in the following way

$$\mathbb{P} = (p(k,l))_{k,l=1,\ldots,d} = \begin{pmatrix} p(1,1) & p(2,1) & \cdots & p(d,1) \\ p(1,2) & p(2,2) & \cdots & p(d,2) \\ \vdots & \vdots & \ddots & \vdots \\ p(1,d) & p(2,d) & \cdots & p(d,d) \end{pmatrix},$$

where $p(k,l) = P(\Delta_t = l \mid \Delta_{t-1} = k)$, and the i-step transition probabilities are denoted by $p^{(i)}(k,l) = P(\Delta_t = l \mid \Delta_{t-i} = k)$, for $k,l \in \mathcal{E}$ and $i \geq 1$.

It is worth mentioning that our proposed *MS-PGARCH* formulation (5) includes, as special cases, various models such as:

- If $p = 0$, we have

$$\epsilon_t = \sqrt{h_t}\eta_t, \text{ and } h_t = \omega_t(\Delta_t) + \sum_{i=1}^{q} \alpha_{t,i}(\Delta_t)\epsilon_{t-i}^2, \ t \in \mathbb{Z},$$

  and the process is called a Markov-switching periodic *ARCH* (in short, *MS-PARCH*).

- *MS-GARCH* model which can be obtained by assuming that the functions $\omega_t(k)$, $\alpha_{t,i}(k)$ and $\beta_{t,j}(k)$ are constant in t (see Haas and Paolella [25] for a survey of mixture and regime-switching *GARCH* models).

- Mixture *PGARCH* model which can be obtained by assuming that (Δ_t) is an independent process. This class of models has been proposed and studied by Hamdi and Souam [28].

- *PGARCH* model, introduced by Bollerslev and Ghysels [16], which is obtained by assuming that the Markov-chain (Δ_t) has a single regime.

- Standard *GARCH* model of Bollerslev [15] which can be obtained assuming that the functions $\omega_t(k)$, $\alpha_{t,i}(k)$ and $\beta_{t,j}(k)$ constant over time and regimes.

3. STRICT PERIODIC STATIONARITY OF *MS-PGARCH* MODEL

In this section we are interested in the existence of a unique strictly periodically stationary (henceforth *s.p.s.*) solution of equation (5) in the sense given, for example, by Aknouche and Guerbyenne [3]. So, we derive a necessary and sufficient condition under which our process be *s.p.s.* The definition (5) is difficult to deal with when we want to study the probabilistic properties of the *MS-PGARCH* model written in this form. For this reason, it will be useful to rewrite the model (5) in an equivalent Markovian representation. Along the lines of the work drawn up by Aknouche and Guerbyenne [3] for random coefficient periodic autoregressions, we consider the following representation

$$z_t = A_t z_{t-1} + b_t, \tag{6}$$

where z_t and b_t are random vectors of dimension $r = p + q$ defined as follows

$$\begin{aligned} z_t &= \left(\epsilon_t^2, \epsilon_{t-1}^2, \ldots, \epsilon_{t-q+1}^2, h_t, h_{t-1}, \ldots, h_{t-p+1}\right)', \\ b_t &= \left(\omega_t\left(\Delta_t\right)\eta_t^2, \mathbf{0}_{1\times(q-1)}, \omega_t\left(\Delta_t\right), \mathbf{0}_{1\times(p-1)}\right)', \end{aligned}$$

and A_t is a square random matrix of dimension r, such that

$$A_t = \begin{pmatrix} \alpha_{t,1:q-1}\left(\Delta_t\right)\eta_t^2 & \alpha_{t,q}\left(\Delta_t\right)\eta_t^2 & \beta_{t,1:p-1}\left(\Delta_t\right)\eta_t^2 & \beta_{t,q}\left(\Delta_t\right)\eta_t^2 \\ \mathbf{I}_{q-1} & \mathbf{0}_{(q-1)\times 1} & \mathbf{0}_{(q-1)\times(p-1)} & \mathbf{0}_{(q-1)\times 1} \\ \alpha_{t,1:q-1}\left(\Delta_t\right) & \alpha_{t,q}\left(\Delta_t\right) & \beta_{t,1:p-1}\left(\Delta_t\right) & \beta_{t,q}\left(\Delta_t\right) \\ \mathbf{0}_{(p-1)\times(q-1)} & \mathbf{0}_{(q-1)\times 1} & \mathbf{I}_{p-1} & \mathbf{0}_{(p-1)\times 1} \end{pmatrix},$$

where $\alpha_{t,1:q}\left(\Delta_t\right) = \left(\alpha_{t,1}\left(\Delta_t\right), \ldots, \alpha_{t,q}\left(\Delta_t\right)\right)$ and $\beta_{t,1:p}\left(\Delta_t\right) = \left(\beta_{t,1}\left(\Delta_t\right), \ldots, \beta_{t,p}\left(\Delta_t\right)\right)$. Here, $\mathbf{I}_n$ and $\mathbf{0}_{n\times m}$ are, respectively, the $n \times n$ identity matrix and the $n \times m$ matrix whose elements are zeros. Note that there exists a one-to-one correspondence between the solutions (ϵ_t) of (5) and the positive solutions (z_t) of (6).

As in many periodic time series models (e. g., Aknouche and Guerbyenne [3]; Hamdi and Souam [28]), the main tool for studying strict periodic stationarity is the top Lyapunov exponent associated to independent and periodically distributed (*i.p.d.*) random matrices. Let $\|\cdot\|$ be an arbitrary norm operator in $\mathcal{M}_{r\times r}(\mathbb{R})$, the space of real matrices of dimension r. Then, the top Lyapunov exponent associated with the *i.p.d.* sequence of matrices $A =: \{A_t,\ t \in \mathbb{Z}\}$ is defined by

$$\gamma^S(A) := \inf_{n\in\mathbb{N}^*} \frac{1}{n} \mathbb{E}\left\{\log \|A_{nS}A_{nS-1}\ldots A_1\|\right\}, \tag{7}$$

whenever $\sum_{s=1}^{S} \mathbb{E}\left(\log^+ \|A_s\|\right) < \infty$, where for $x > 0$, $\log^+(x) = \max\left(\log(x), 0\right)$.

We state now the following theorem that provides a necessary and sufficient condition ensuring the existence of a unique *s.p.s.* solution of (5) which is also periodically ergodic.

Theorem 3.1. Model (5) admits a nonanticipative *s.p.s.* solution given by the first component of

$$z_t = b_t + \sum_{i=1}^{\infty} \left(\prod_{j=0}^{i-1} A_{t-i}\right) b_{t-j},\ t \in \mathbb{Z}, \tag{8}$$

if and only if $\gamma^S(A)$ given by (7) is strictly negative, where the series (8) converges almost surely for all $t \in \mathbb{Z}$. Moreover, this solution is unique and periodically ergodic.

Proof. The proof is similar to that of Aknouche and Guerbyenne ([3], Theorem 2.1 and Remark 2.1) and hence it will be omitted. □

Remark 3.2. We give the reduced conditions obtained in some special cases, which coincides with some known results in literature.

1. It is well known that for $S=1$, $\gamma^S(A)$ defined in (7) reduces to the definition of the top Lyapunov exponent for *i.i.d.* matrices (Bougerol and Picard [17]
$$\gamma(A)=\inf_{n\in\mathbb{N}^*}\mathbb{E}\left\{\frac{1}{t}\|A_tA_{t-1}\dots A_1\|\right\}.$$
Thus, the non-periodic model (5) (when $S=1$), admits a unique nonanticipative strictly stationary and ergodic solution if and only if $\gamma(A)<0$. Note that this condition is the same one obtained by Francq et al. (2001, Theorem 1) for non-periodic MS-$GARCH$ models.

2. If $d=1$, the previous theorem coincides with the result provided in Aknouche and Bibi ([2], Theorem 1) for the strict periodic stationarity of $PGARCH$ models.

3. When the process (Δ_t) is an *i.i.d.* random variables sequence, the previous theorem coincides with the result stated in Hamdi and Souam ([28], Theorem 1) for the case of mixture periodic $GARCH$ models.

4. EXISTENCE AND CALCULATION OF HIGHER ORDER MOMENTS OF A MS-$PGARCH$ PROCESS

It is important to know whether the *s.p.s.* solution has moments of higher order. In this section, one is interested in conditions ensuring finiteness of higher order moments, the most important case being finiteness of $\mathbb{E}\left(\epsilon_t^2\right)$ and $\mathbb{E}\left(\epsilon_t^4\right)$, under which we study the autocovariance structure of the squared MS-$PGARCH$ process.

4.1. Conditions of existence of higher order moments

It may be pointed out that if the symmetry assumption is made on the distribution of (η_t), the odd-order moments of (ϵ_t) are null when they exist. In this section, only even-order moments are considered. Before stating the result ensuring the existence of moments of orders $2m$, where m is a strictly positive integer, we give some notations that we will need afterwards. Denote by $\rho(A)$ the spectral radius of any square matrix A. Let $\otimes$ denote the Kronecker product. For any matrix A and any strictly positive integer m, let $A^{[m]}$ be the Kronecker power $A\otimes A\otimes\cdots\otimes A$ of m factors.

Recall that, a matrix A in $\mathcal{M}_{n\times m}(\mathbb{R})$ is said to be positive (resp. strictly positive), that we note $A\geq 0$ (resp. $A>0$), if no element of A is negative (resp. negative or null). For two matrices A and B of the same size, the notation $A\succeq B$ (resp. $A\succ B$) means that $A-B$ is a positive matrix (resp. $A-B$ is strictly positive).

Let us define $\underline{A}_t^{[m]}(k)$ as the conditional expectation of the matrix $A_t^{[m]}$ given Δ_t equal to k. We similarly define the function $\underline{b}_t^{[m]}$ by $\underline{b}_t^{[m]}(\cdot)=\mathbb{E}\left[b_t^{[m]}\middle|\Delta_t=\cdot\right]$. Let us also define the following matrices
$$\mathbb{P}_{f_t}=\begin{pmatrix}p(1,1)f_t(1)&\cdots&p(d,1)f_t(1)\\ \vdots&\ddots&\vdots\\ p(1,d)f_t(d)&\cdots&p(d,d)f_t(d)\end{pmatrix}\text{ and }\Pi_{f_t}=\begin{pmatrix}\pi(1)f_t(1)\\ \vdots\\ \pi(d)f_t(d)\end{pmatrix},$$

for any periodic function $f_t : \mathcal{E} \to \mathcal{M}_{n\times n'}(\mathbb{R})$, where n and n' are strictly positive integers.

Theorem 4.1. Suppose that $\mathbb{E}\left(\eta_t^{2m}\right) < \infty$ and

$$\rho\left(\prod_{s=0}^{S-1} \mathbb{P}_{\underline{A}_{S-s}^{[m]}}\right) < 1, \tag{9}$$

then, for all $t \in \mathbb{Z}$, the series defined by (8) converges in $\mathcal{L}^m$ and the process $\{\epsilon_t, t \in \mathbb{Z}\}$, defined as the first component of $\{z_t, t \in \mathbb{Z}\}$, is *s.p.s.* and admits moments up to order m.

Conversely, if $\prod_{s=0}^{S-1} \mathbb{P}_{\underline{A}_{S-s}^{[m]}}$ is irreducible and $\rho\left(\prod_{s=0}^{S-1} \mathbb{P}_{\underline{A}_{S-s}^{[m]}}\right) \geq 1$, then the model (5) has no *s.p.s.* solution such that $\mathbb{E}\left(\epsilon_t^{2m}\right) < \infty$.

Proof. The proof is an adaptation of that provided in Francq and Zakoïan ([20], Theorem 1) to the periodic case.

(i) Let

$$z_t = \sum_{k=0}^{\infty} z_{t,k}, \tag{10}$$

with $z_{t,0} = b_t$ and $z_{t,k} = \left(\prod_{l=0}^{k-1} A_{t-l}\right) b_{t-k}$, for $k \geq 1$.

By the fact that the matrices $A_t, \ldots, A_{t-k+1}, b_{t-k}$ are independent conditional on Δ_t, we have

$$\begin{aligned}
\mathbb{E}\left[z_{t,k}^{[m]}\right] &= \mathbb{E}\left[A_t^{[m]} A_{t-1}^{[m]} \ldots A_{t-k+1}^{[m]} b_{t-k}^{[m]}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[A_t^{[m]} A_{t-1}^{[m]} \ldots A_{t-k+1}^{[m]} b_{t-k}^{[m]} \middle| \Delta_t, \ldots, \Delta_{t-k}\right]\right] \\
&= \mathbb{E}\left[\underline{A}_t^{[m]}(\Delta_t)\, \underline{A}_{t-1}^{[m]}(\Delta_{t-1}) \ldots \underline{A}_{t-k+1}^{[m]}(\Delta_{t-k+1})\, \underline{b}_{t-k}^{[m]}(\Delta_{t-k})\right].
\end{aligned}$$

Let $t = s + S\tau$ and $k = \nu + S\delta$ such that $\tau \in \mathbb{Z}$, $\delta \in \mathbb{N}$ and $s, \nu \in \{1, 2, \ldots, S\}$. Using the relation (7) given in Aliat and Hamdi ([4], Lemma 1), we obtain

$$\mathbb{E}\left(z_{t,k}^{[m]}\right) = \mathbb{I}_{r^m}\left(\prod_{l=0}^{S-1} \mathbb{P}_{\underline{A}_{s-l}^{[m]}}\right)^{\delta}\left(\prod_{l=0}^{\nu-1} \mathbb{P}_{\underline{A}_{s-l}^{[m]}}\right) \Pi_{\underline{b}_{s-\nu}^{[m]}}, \tag{11}$$

where $\mathbb{I}_n = (\mathbf{I}_n, \ldots, \mathbf{I}_n)$ is a $n \times nd$ matrix.

Let $\|\cdot\|$ denote the matrix norm such that $\|A\| = \sum_{i,j} |a_{i,j}|$, where $a_{i,j}$ denotes the generic element of a matrix A. Using some Kronecker product properties, we get

$$\begin{aligned}
\|z_{t,k}\|_{\mathcal{L}^m} &= \left(\mathbb{E}\left[\|z_{t,k}\|^m\right]\right)^{1/m} \\
&= \left(\left\|\mathbb{E}\left[z_{t,k}^{[m]}\right]\right\|\right)^{1/m} \\
&= \left\|\mathbb{I}_{r^m}\left(\prod\nolimits_{l=0}^{S-1}\mathbb{P}_{\underline{A}_{s-l}^{[m]}}\right)^{\delta}\left(\prod\nolimits_{l=0}^{\nu-1}\mathbb{P}_{\underline{A}_{s-l}^{[m]}}\right)\Pi_{\underline{b}_{s-\nu}^{[m]}}\right\|^{1/m} \\
&\leq \|\mathbb{I}_{r^m}\|^{1/m}\left\|\left(\prod\nolimits_{l=0}^{S-1}\mathbb{P}_{\underline{A}_{s-l}^{[m]}}\right)^{\delta}\right\|^{1/m} \\
&\quad\times\left\|\left(\prod\nolimits_{l=0}^{\nu-1}\mathbb{P}_{\underline{A}_{s-l}^{[m]}}\right)\right\|^{1/m}\left\|\Pi_{\underline{b}_{s-\nu}^{[m]}}\right\|^{1/m}.
\end{aligned}$$

If the spectral radius of the matrix $\prod_{l=0}^{S-1}\mathbb{P}_{\underline{A}_{s-l}^{[m]}}$ is strictly less than 1, then $\left\|\left(\prod_{l=0}^{S-1}\mathbb{P}_{\underline{A}_{s-l}^{[m]}}\right)^{\delta}\right\|$ converges to zero at an exponential rate as $\delta \longrightarrow \infty$. Since for all s, $\left\|\prod_{l=0}^{\nu-1}\mathbb{P}_{\underline{A}_{s-l}^{[m]}}\right\|$ is uniformly bounded by $\max_{1\leq s\leq S}\left\|\left(\prod_{l=0}^{\nu-1}\mathbb{P}_{\underline{A}_{s-l}^{[m]}}\right)\right\|$, which is finite, then $z_t = \sum_{k=0}^{\infty} z_{t,k}$ is almost surely finite and belongs to $\mathcal{L}^m$. Moreover, by the circular property of the spectral radius, it can be easily seen that

$$\rho\left(\prod\nolimits_{l=0}^{S-1}\mathbb{P}_{\underline{A}_{s-l}^{[m]}}\right) = \rho\left(\prod\nolimits_{s=0}^{S-1}\mathbb{P}_{\underline{A}_{S-s}^{[m]}}\right), \text{ for all } s,\ 1 \leq s \leq S.$$

It is clear that the norm of z_t is greater than that of its first component ϵ_t^2. Hence, a sufficient condition for the existence of $\mathbb{E}\left(\epsilon_t^{2m}\right)$ is then $\rho\left(\prod_{s=0}^{S-1}\mathbb{P}_{\underline{A}_{S-s}^{[m]}}\right) < 1$.

Furthermore, for each K, $\sum_{i=0}^{K} z_{t,k}$ is a S-periodic measurable function of the *i.p.d.* sequence $\{(A_t, b_t), t \in \mathbb{Z}\}$. Hence, the solution z_t is *s.p.s.* and periodically ergodic (see e. g. Aknouche and Guerbyenne [3]).

(*ii*) The proof of uniqueness is similar to that of Francq and Zakoïan ([20], Theorem 1) and hence omitted.

(*iii*) Conversely, suppose that $\mathbb{E}\left(\epsilon_t^{2m}\right) < \infty$. From (6) the nonnegativity of all the elements of the vector z_t and the matrices A_t, we have for any $k \geq 0$

$$z_t = z_{t,0} + \cdots + z_{t,k} + A_t A_{t-1} \ldots A_{t-k} z_{t-k-1} \succeq \sum_{i=0}^{k} z_{t,i}.$$

Thus, $z_t \succeq \sum_{i=0}^{\infty} z_{t,i}$ and

$$\begin{aligned}
\mathbb{E}\left[z_t^{[m]}\right] &\succeq \mathbb{E}\left[\left(\sum_{k=0}^{\infty} z_{t,k}\right)^{[m]}\right] \succeq \sum_{k=0}^{\infty} \mathbb{E}\left[z_{t,k}^{[m]}\right] \\
&= \sum_{k=0}^{\infty} \mathbb{I}_{r^m}\left(\prod_{j=0}^{k-1} \mathbb{P}_{\underline{A}_{t-j}^{[m]}}\right) \Pi_{\underline{b}_{t-k}^{[m]}} \\
&= \sum_{\nu=0}^{S-1}\sum_{\delta=0}^{\infty} \mathbb{I}_{r^m}\left(\prod_{j=0}^{S-1} \mathbb{P}_{\underline{A}_{s-j}^{[m]}}\right)^{\delta}\left(\prod_{j=0}^{\nu-1} \mathbb{P}_{\underline{A}_{s-j}^{[m]}}\right) \Pi_{\underline{b}_{s-\nu}^{[m]}}.
\end{aligned}$$

Using some nonnegative matrices properties (see e. g. Lancaster and Tismenetsky [34], Chapter 15), we get

$$\begin{aligned}
\left\|\mathbb{E}\left[z_t^{[m]}\right]\right\| &\geq \left\|\sum_{\nu=0}^{S-1}\sum_{\delta=0}^{\infty} \mathbb{I}_{r^m}\left(\prod_{j=0}^{S-1} \mathbb{P}_{\underline{A}_{s-j}^{[m]}}\right)^{\delta}\left(\prod_{j=0}^{\nu-1} \mathbb{P}_{\underline{A}_{s-j}^{[m]}}\right) \Pi_{\underline{b}_{s-\nu}^{[m]}}\right\| \\
&= \sum_{\nu=0}^{S-1}\sum_{\delta=0}^{\infty}\left\| \mathbb{I}_{r^m}\left(\prod_{j=0}^{S-1} \mathbb{P}_{\underline{A}_{s-j}^{[m]}}\right)^{\delta}\left(\prod_{j=0}^{\nu-1} \mathbb{P}_{\underline{A}_{s-j}^{[m]}}\right) \Pi_{\underline{b}_{s-\nu}^{[m]}}\right\| \\
&= \sum_{\nu=0}^{S-1}\sum_{\delta=0}^{\infty}\left\| \left(\prod_{j=0}^{S-1} \mathbb{P}_{\underline{A}_{s-j}^{[m]}}\right)^{l\delta}\left[\mathbf{I}_{dr^m} + \left(\prod_{j=0}^{S-1} \mathbb{P}_{\underline{A}_{s-j}^{[m]}}\right) + \cdots\right.\right. \\
&\qquad \left.\left. + \left(\prod_{j=0}^{S-1} \mathbb{P}_{\underline{A}_{s-j}^{[m]}}\right)^{l-1}\right] \times \left(\prod_{j=0}^{\nu-1} \mathbb{P}_{\underline{A}_{s-j}^{[m]}}\right) \Pi_{\underline{b}_{s-\nu}^{[m]}}\right\| \\
&\geq \frac{1}{l!}\sum_{\nu=0}^{S-1}\sum_{\delta=0}^{\infty}\left\| \left(\prod_{j=0}^{S-1} \mathbb{P}_{\underline{A}_{s-j}^{[m]}}\right)^{l\delta}\left[\mathbf{I}_{dr^m} + \left(\prod_{j=0}^{S-1} \mathbb{P}_{\underline{A}_{s-j}^{[m]}}\right)\right]^{l-1}\right. \\
&\qquad \left. \times \left(\prod_{j=0}^{\nu-1} \mathbb{P}_{\underline{A}_{s-j}^{[m]}}\right) \Pi_{\underline{b}_{s-\nu}^{[m]}}\right\| \\
&\geq \frac{Sc}{l!}\sum_{\delta=0}^{\infty}\left\| \left(\prod_{j=0}^{S-1} \mathbb{P}_{\underline{A}_{s-j}^{[m]}}\right)^{l\delta}\right\| \\
&\geq \frac{Sc}{l!}\sum_{\delta=0}^{\infty}\left\{\rho\left(\prod_{j=0}^{S-1} \mathbb{P}_{\underline{A}_{s-j}^{[m]}}\right)\right\}^{l\delta},
\end{aligned}$$

where c is the smallest element of

$$\left[\mathbf{I}_{dr^m} + \left(\prod_{j=0}^{S-1} \mathbb{P}_{\underline{A}_{s-j}^{[m]}}\right)\right]^{l-1}\left(\prod_{j=0}^{\nu-1} \mathbb{P}_{\underline{A}_{s-j}^{[m]}}\right) \Pi_{\underline{b}_{s-\nu}^{[m]}},$$

and l is a fixed strictly positive integer. Finally, it is easy to see that since $\mathbb{E}\left[\epsilon_t^{2m}\right] < \infty$, we also have $\left\|\mathbb{E}\left[z_t^{[m]}\right]\right\| < \infty$ and consequently the condition (9) must hold which completes the proof of the theorem. □

Remark 4.2.

1. Note that, if there exist only one regime, i. e. $d = 1$, the condition (9) is reduced to
$$\rho\left(\prod_{s=0}^{S-1}\mathbb{E}\left(A_{S-s}^{[m]}\right)\right) < 1.$$
This condition is the same obtained by Aknouche and Bentarzi ([1], Proposition 3.1) and Bibi and Aknouche ([10], Theorem 4.2), in the case of $PGARCH$ models.

2. When $S = 1$, the condition (9) can be reduced to
$$\rho\left(\mathbb{P}_{\underline{A}^{[m]}}\right) < 1,$$
with $\underline{A}^{[m]}(\cdot) = \mathbb{E}\left(A_t^{[m]}\middle|\Delta_t = \cdot\right)$. This condition was established in Francq and Zakoïan ([20], Theorem 1) for non-periodic MS-$GARCH$ models.

3. In the case where (Δ_t) is a sequence of $i.i.d.$ random variables, the condition (9) can be written as follows
$$\rho\left(\prod_{s=0}^{S-1}\mathbb{P}_{\underline{A}_{S-s}^{[m]}}\right) < 1,$$
where $\underline{A}^{[m]}(\cdot) = \mathbb{E}\left(A_t^{[m]}\middle| Z_t^{(\cdot)} = 1\right)$, and the variable $Z_t^{(k)}$ (for $k = 1,\ldots,d$) is equal to 1 means that the observation ϵ_t comes from the kth component of the mixture. This condition is the one obtained by Hamdi and Souam ([28], Theorem 2) for the mixture periodic $GARCH$ models.

4.2. Calculation of moments

In this section, we will characterize explicitly the moments for the MS-$PGARCH$ model (4). Suppose that (9) holds and $\mathbb{E}\left(\eta_t^{2m}\right) < \infty$. It can be shown that
$$(A_t z_{t-1} + b_t)^{[m]} = \sum_{l=0}^{m}\sum_{\substack{l_i\in\{0,1\}\\ \sum_{i=1}^m l_i = l}}\left\{\left(A_t^{[l_1]}\otimes b_t^{[1-l_1]}\right)\otimes\cdots\otimes\left(A_t^{[l_m]}\otimes b_t^{[1-l_m]}\right)\right\} z_{t-1}^{[l]},$$
and by taking the expectation with respect to $\Delta_t = k$, we get
$$\begin{aligned}\pi(k)\,\mathbb{E}\left(z_t^{[m]}\middle|\Delta_t = k\right) &= \pi(k)\,\underline{b}_t^{[m]}(k)\\ &+\pi(k)\sum_{l=1}^{m-1}\sum_{\substack{l_i\in\{0,1\}\\ \sum_{i=1}^m l_i = l}}\sum_{j=1}^{d}\left\{\left(\underline{A}_t^{[l_1]}(k)\otimes\underline{b}_t^{[1-l_1]}(k)\right)\otimes\cdots\otimes\left(\underline{A}_t^{[l_m]}(k)\otimes\underline{b}_t^{[1-l_m]}(k)\right)\right\}\\ &\times\mathbb{E}\left(z_{t-1}^{[l]}\middle|\Delta_{t-1} = j\right)p(j,k)\,\pi(j)\\ &+\pi(k)\,\underline{A}_t^{[m]}(k)\sum_{j=1}^{d}\mathbb{E}\left(z_{t-1}^{[m]}\middle|\Delta_{t-1} = j\right)p(j,k)\,\pi(j).\end{aligned}$$

As in the MS-$PARMA$ model (Aliat and Hamdi [4]), let us consider the dr^m-variate S-periodic vectors

$$M_{t,m}=\begin{pmatrix} M_{t,m}(1)\\ \vdots \\ M_{t,m}(d)\end{pmatrix}:=\begin{pmatrix} \pi(1)\,\mathbb{E}\left(z_t^{[m]}\middle|\,\Delta_t=1\right)\\ \vdots \\ \pi(d)\,\mathbb{E}\left(z_t^{[m]}\middle|\,\Delta_t=d\right)\end{pmatrix} \text{ and } C_{t,m}=\begin{pmatrix} c_m(1)\\ \vdots \\ c_m(d)\end{pmatrix},$$

where

$$\begin{aligned} c_m(k) &= \pi(k)\,\underline{b}_t^{[m]}(k)+\pi(k)\sum_{l=1}^{m-1}\sum_{\substack{l_i\in\{0,1\}\\ \sum_{i=1}^m l_i=l}}\sum_{j=1}^{d}\left\{\left(\underline{A}_t^{[l_1]}(k)\otimes\underline{b}_t^{[1-l_1]}(k)\right)\otimes\cdots\right.\\ &\quad\left.\otimes\left(\underline{A}_t^{[l_m]}(k)\otimes\underline{b}_t^{[1-l_m]}(k)\right)\right\}\mathbb{E}\left(z_{t-1}^{[l]}\middle|\,\Delta_{t-1}=j\right)p(j,k)\,\pi(j), \end{aligned}$$

from which we can easily show that

$$M_{t,m}=C_{t,m}+\mathbb{P}_{\underline{A}_t^{[m]}}M_{t-1,m},$$

After $S-1$ successive replacements in the latter equation and taking into account the periodicity of $M_{t,m}$, we get

$$M_{t,m}=\sum_{i=0}^{S-1}\left(\prod_{j=0}^{i-1}\mathbb{P}_{\underline{A}_{t-j}^{[m]}}\right)C_{t-i,m}+\left(\prod_{j=0}^{S-1}\mathbb{P}_{\underline{A}_{t-j}^{[m]}}\right)M_{t,m},$$

and as a result, we obtain

$$M_{t,m}=\left[\mathbf{I}_{r^m}-\left(\prod_{j=0}^{S-1}\mathbb{P}_{\underline{A}_{t-j}^{[m]}}\right)\right]^{-1}\left[\sum_{i=0}^{S-1}\left(\prod_{j=0}^{i-1}\mathbb{P}_{\underline{A}_{t-j}^{[m]}}\right)C_{t-i,m}\right].$$

From the definitions of $M_{t,m}$ and z_t, the unconditional moments $\mathbb{E}\left(\epsilon_t^{2m}\right)$ can be obtained as

$$\mathbb{E}\left(\epsilon_t^{2m}\right)=H_{r^m}\mathbb{I}_{r^m}M_{t,m},$$

where $H_n=\left(1,\mathbf{0}_{1\times(n-1)}\right)$ is a row-vector of size n with 1 in the first position and zero elsewhere.

4.3. Autocovariances of the squares of a MS-$PGARCH$

To calculate the autocovariances of the squares of the MS-$PGARCH$ model (5), we can use its Markovian representation (6). In the following, we shall show how to compute these autocovariances explicitly.

Suppose that $\rho\left(\prod_{s=0}^{S-1}\mathbb{P}_{\underline{A}_{S-s}^{[2]}}\right)<1$, and $\mathbb{E}\left(\eta_t^4\right)<\infty$. It is easy to see that for all $h\geq 0$, we have

$$z_t\otimes z_{t-h}=\left(A_t\otimes\mathbf{I}_r\right)\left(z_{t-1}\otimes z_{t-h}\right)+\left(b_t\otimes\mathbf{I}_r\right)z_{t-h},$$

and for $k = 1, 2, \dots, d$,

$$\begin{aligned}\pi(k)\,\mathbb{E}\left[z_t \otimes z_{t-h} \middle| \Delta_t = k\right] &= \sum_{j=1}^{d} \Upsilon_t(k)\,\mathbb{E}\left[z_{t-1} \otimes z_{t-h} \middle| \Delta_{t-1} = j\right] p(j,k)\,\pi(j) \\ &\quad + \sum_{j=1}^{d} \Xi_t(k)\,\mathbb{E}\left[z_{t-h} \middle| \Delta_{t-h} = j\right] p^{(h)}(j,k)\,\pi(j),\end{aligned}$$

where $\Upsilon_t(\cdot) = \underline{A}_t^{[1]}(\cdot) \otimes \mathbf{I}_r$ and $\Xi_t(\cdot) = \underline{b}_t^{[1]}(\cdot) \otimes \mathbf{I}_r$. Hence, we obtain the following system

$$\mathbf{W}_h^{(t)} = \begin{cases} M_{t,2} & \text{if } h = 0, \\ \mathbb{P}_{\Upsilon_t} \mathbf{W}_{h-1}^{(t-1)} + \mathbb{P}_{\Xi_t}^{(h)} M_{t-h,1} & \text{if } h \geq 1, \end{cases} \tag{12}$$

where $\mathbf{W}_h^{(t)} = \left(\pi(1)\,\mathbb{E}\left[z_t \otimes z_{t-h} \middle| \Delta_t = 1\right], \dots, \pi(d)\,\mathbb{E}\left[z_t \otimes z_{t-h} \middle| \Delta_t = d\right]\right)$, $t, h \in \mathbb{Z}$. Consequently, we get for all $h \in \mathbb{N}$

$$\mathbb{E}\left(z_t \otimes z_{t-h}\right) = \sum_{k=1}^{d} \mathbf{W}_h^{(t)}(k).$$

Finally, the autocovariance function of lag h and period s of the squares of a *MS-PGARCH* process defined by (5) is given by

$$\gamma_{\epsilon^2,h}^{(t)} = H_{r^2} \mathbb{I}_{r^2} \mathbf{W}_h^{(t)} - \left(H_r \mathbb{I}_r M_{t,1}\right)\left(H_r \mathbb{I}_r M_{t-h,1}\right). \tag{13}$$

5. STUDY OF THE $MS\text{-}PGARCH_S(1,1)$

5.1. Periodic stationarity and existence of higher order moments

We have already studied the periodic stationarity and we have calculates higher order moments and autocorrelations of the squares of the specification (5). In this section, we shall thus concentrate on the particular $MS\text{-}PGARCH_S(1,1)$ case that allows for rather simple explicit expressions.

When $p = q = 1$, the model (5) will be written as follows

$$\begin{cases} \epsilon_t = \sqrt{h_t}\,\eta_t, \\ h_t = \omega_t(\Delta_t) + \alpha_t(\Delta_t)\,\epsilon_{t-1}^2 + \beta_t(\Delta_t)\,h_{t-1}, \end{cases} \quad t \in \mathbb{Z}. \tag{14}$$

From this representation, it is easy to see that the volatility process h_t can be written as

$$h_t = a_t h_{t-1} + b_t, \tag{15}$$

where $b_t = \omega_t(\Delta_t)$ and $a_t = \alpha_t(\Delta_t)\,\eta_{t-1}^2 + \beta_t(\Delta_t)$. Iterating equation (15) l time and letting l goes to infinity, we thus obtain

$$h_t = b_t + \sum_{k=1}^{\infty} \left(\prod_{i=0}^{k-1} a_{t-i}\right) b_{t-k}.$$

For the series defined in the previous relation, to belong to $\mathcal{L}^m$, it suffices that $h_{t,k} = \left(\prod_{i=0}^{k-1} a_{t-i}\right) b_{t-k}$ converges to zero in $\mathcal{L}^m$. It is not difficult to show from the definition of $h_{t,k}$ that

$$\begin{aligned}
\mathbb{E}\left(h_{t,k}^m\right) &= \mathbb{E}\left\{\mathbb{E}\left[\left(\prod_{i=0}^{k-1} a_{t-i}\right)^m b_{t-k}^m \middle| \Delta_t, \ldots, \Delta_{t-k}\right]\right\} \\
&= \mathbb{E}\left\{\left(\prod_{i=0}^{k-1} a_{t-i,m}\left(\Delta_{t-i}\right)\right) b_{t,m}\left(\Delta_{t-k}\right)\right\} \\
&= \mathbb{I}_1\left(\prod_{l=0}^{S-1} \mathbb{P}_{a_{s-l,m}}\right)^{\delta}\left(\prod_{l=0}^{\nu-1} \mathbb{P}_{a_{s-l,m}}\right)\Pi_{b_{s-\nu,m}},
\end{aligned}$$

where $a_{t,m}(\cdot) = \mathbb{E}\left(a_t^m \mid \Delta_t = \cdot\right)$ and $b_{t,m}(\cdot) = \mathbb{E}\left(b_t^m \mid \Delta_t = \cdot\right)$. It follows that

$$\begin{aligned}
\|h_{t,k}\|_{\mathcal{L}^m} &= \left(\mathbb{E}\left[\|h_{t,k}\|^m\right]\right)^{1/m} \\
&= \left\|\mathbb{I}_1\left(\prod_{l=0}^{S-1} \mathbb{P}_{a_{s-l,m}}\right)^{\delta}\left(\prod_{l=0}^{\nu-1} \mathbb{P}_{a_{s-l,m}}\right)\Pi_{b_{s-\nu,m}}\right\|^{1/m} \\
&\leq \|\mathbb{I}_1\|^{1/m}\left\|\left(\prod_{l=0}^{S-1} \mathbb{P}_{a_{s-l,m}}\right)^{\delta}\right\|^{1/m}\left\|\left(\prod_{l=0}^{\nu-1} \mathbb{P}_{a_{s-l,m}}\right)\right\|^{1/m}\left\|\Pi_{b_{s-\nu,m}}\right\|^{1/m}.
\end{aligned}$$

If $\rho\left(\prod_{l=0}^{S-1} \mathbb{P}_{a_{s-l,m}}\right) < 1$, then $\left\|\left(\prod_{l=0}^{S-1} \mathbb{P}_{a_{s-l,m}}\right)^{\delta}\right\|$ converges to zero at an exponential rate as $\delta \longrightarrow \infty$. Since for all s, $\left\|\left(\prod_{l=0}^{\nu-1} \mathbb{P}_{a_{s-l,m}}\right)\right\|$ is uniformly bounded by $\max_{1\leq s\leq S}\left\|\left(\prod_{l=0}^{\nu-1} \mathbb{P}_{a_{s-l,m}}\right)\right\|$, which is finite, then $h_t = \sum_{k=0}^{\infty} h_{t,k}$ is almost surely finite and belongs to $\mathcal{L}^m$. So, we conclude that, $\rho\left(\prod_{s=0}^{S-1} \mathbb{P}_{a_{S-s,m}}\right) < 1$, is a sufficient condition for the existence of the moments up to order $2m$ of the process $MS\text{-}PGARCH_S\,(1,1)$ defined in (14).

An analogous reasoning to that of part (iii) of the proof of Theorem 2, with

$$\mathbb{E}\left(h_t^m\right) \geq \sum_{i=0}^{\infty} \mathbb{I}_1\left(\prod_{j=0}^{i-1} \mathbb{P}_{a_{t-j,m}}\right)\Pi_{b_{t-i,m}},$$

will lead us to conclude that in the case of $MS\text{-}PGARCH_S\,(1,1)$ model, the condition, $\rho\left(\prod_{s=0}^{S-1} \mathbb{P}_{a_{S-s,m}}\right) < 1$, is also necessary.

We now compute the variance of the process $\{\epsilon_t; t \in \mathbb{Z}\}$. If $\rho\left(\prod_{s=0}^{S-1} \mathbb{P}_{a_{S-s,1}}\right) < 1$, we have

$$\mathbb{E}\left(\epsilon_t^2\right) = \sum_{k=1}^{d} \pi(k)\,\mathbb{E}\left(\epsilon_t^2 \mid \Delta_t = k\right) = \sum_{k=1}^{d} N_{t,2}(k),$$

where $N_{t,m}(\cdot) = \pi(\cdot)\mathbb{E}[\epsilon_t^m \mid \Delta_t = \cdot]$. From (14), we have

$$\epsilon_t^2 = \omega_t(\Delta_t)\eta_t^2 + \alpha_t(\Delta_t)\eta_t^2\epsilon_{t-1}^2 + \beta_t(\Delta_t)\eta_t^2 h_{t-1},$$

and for $k = 1, 2, \ldots, d$

$$\begin{aligned}\pi(k)\mathbb{E}\left[\epsilon_t^2 \middle| \Delta_t = k\right] &= \pi(k)\omega_t(k) + \pi(k)\alpha_t(k)\mathbb{E}\left[\epsilon_{t-1}^2 \middle| \Delta_t = k\right] \\ &\quad + \pi(k)\beta_t(k)\mathbb{E}\left(h_{t-1} \middle| \Delta_t = k\right).\end{aligned}$$

Furthermore, since $\mathbb{E}[h_{t-1} \mid \Delta_t = k] = \mathbb{E}\left[\epsilon_{t-1}^2 \mid \Delta_t = k\right]$, thus the previous equation can be written as follows

$$\begin{aligned}\pi(k)\mathbb{E}\left[\epsilon_t^2 \middle| \Delta_t = k\right] &= \pi(k)\omega_t(k) + \pi(k)\left[\alpha_t(k) + \beta_t(k)\right]\mathbb{E}\left[\epsilon_{t-1}^2 \middle| \Delta_t = k\right] \\ &= \pi(k)\omega_t(k) + \sum_{j=1}^{d} a_{t,1}(k)\mathbb{E}\left[\epsilon_{t-1}^2 \middle| \Delta_{t-1} = j\right] p(j,k)\pi(j),\end{aligned}$$

from which we obtain the following system

$$N_{t,2} = \mathbb{P}_{a_{t,1}} N_{t-1,2} + \Pi_{\omega_t},$$

where $N_{t,m} = \left(\pi(1)\mathbb{E}[\epsilon_t^m \mid \Delta_t = 1], \ldots, \pi(d)\mathbb{E}(\epsilon_t^m \mid \Delta_t = d)\right)'$. After $S-1$ successive replacements in the previous equation and taking into account the periodicity of $N_{t,2}$, we get

$$N_{t,2} = \left[\mathbf{I}_d - \prod_{i=0}^{S-1}\mathbb{P}_{a_{t-i,1}}\right]^{-1}\sum_{j=0}^{S-1}\left[\prod_{i=0}^{j-1}\mathbb{P}_{a_{t-i,1}}\right]\Pi_{\omega_{t-j}}. \tag{16}$$

Therefore, the second order moment of $\{\epsilon_t; t \in \mathbb{Z}\}$ is given by

$$\mathbb{E}\left(\epsilon_t^2\right) = \sum_{k=1}^{d} N_{t,2}(k). \tag{17}$$

On the other hand, we know that $\epsilon_t^4 = h_t^2\eta_t^4$. So

$$\begin{aligned}\epsilon_t^4 &= \eta_t^4\left[\omega_t(\Delta_t) + \alpha_t(\Delta_t)\epsilon_{t-1}^2 + \beta_t(\Delta_t)h_{t-1}\right]^2 \\ &= \eta_t^4\left[\omega_t^2(\Delta_t) + 2\omega_t(\Delta_t)\alpha_t(\Delta_t)\epsilon_{t-1}^2 + 2\omega_t(\Delta_t)\beta_t(\Delta_t)h_{t-1}\right. \\ &\quad \left. + \alpha_t^2(\Delta_t)\epsilon_{t-1}^4 + 2\alpha_t(\Delta_t)\beta_t(\Delta_t)\epsilon_{t-1}^2 h_{t-1} + \beta_t^2(\Delta_t)h_{t-1}^2\right].\end{aligned}$$

Suppose that $\rho\left(\prod_{s=0}^{S-1}\mathbb{P}_{a_{S-s,2}}\right) < 1$, and $\mu_4 = \mathbb{E}\left(\eta_t^4\right) < \infty$, then

$$\mathbb{E}\left(\epsilon_t^4\right) = \sum_{k=1}^{d}\pi(k)\mathbb{E}\left[\epsilon_t^4 \mid \Delta_t = k\right].$$

But

$$\begin{aligned}\mathbb{E}\left(\epsilon_t^4 \middle| \Delta_t = k\right) &= \mu_4\left\{\omega_t^2(k) + 2\omega_t(k)\alpha_t(k)\mathbb{E}\left[\epsilon_{t-1}^2 \middle| \Delta_t = k\right]\right. \\ &\quad + 2\omega_t(k)\beta_t(k)\mathbb{E}\left[h_{t-1} \middle| \Delta_t = k\right] + \alpha_t^2(k)\mathbb{E}\left[\epsilon_{t-1}^4 \middle| \Delta_t = k\right] \\ &\quad \left. + 2\alpha_t(k)\beta_t(k)\mathbb{E}\left[\epsilon_{t-1}^2 h_{t-1} \middle| \Delta_t = k\right] + \beta_t^2(k)\mathbb{E}\left[h_{t-1}^2 \middle| \Delta_t = k\right]\right\}.\end{aligned}$$

By reason of

$$\begin{aligned}\mathbb{E}\left[\epsilon_{t-1}^2 h_{t-1}\middle|\,\Delta_t=k\right] &= \mathbb{E}\left[h_{t-1}^2\middle|\,\Delta_t=k\right]\\ \mathbb{E}\left[h_{t-1}\middle|\,\Delta_t=k\right] &= \mathbb{E}\left[\epsilon_{t-1}^2\middle|\,\Delta_t=k\right],\end{aligned}$$

and

$$\mathbb{E}\left[\epsilon_{t-1}^4\middle|\,\Delta_t=k\right]=\mu_4\mathbb{E}\left[h_{t-1}^2\middle|\,\Delta_t=k\right],$$

the conditional expectation $\mathbb{E}\left(\epsilon_t^4\middle|\,\Delta_t=k\right)$ can be written in the following form

$$\begin{aligned}\mathbb{E}\left[\epsilon_t^4\middle|\,\Delta_t=k\right] &= \mu_4\omega_t^2(k)+2\mu_4\omega_t(k)\{\alpha_t(k)+\beta_t(k)\}\,\mathbb{E}\left[\epsilon_{t-1}^2\middle|\,\Delta_t=k\right]\\ &\quad+\left\{\mu_4\alpha_t^2(k)+2\alpha_t(k)\beta_t(k)+\beta_t^2(k)\right\}\mathbb{E}\left[\epsilon_{t-1}^4\middle|\,\Delta_t=k\right].\end{aligned}$$

Let for $k=1,\ldots,d$

$$\phi_{t,2}(k)=2\mu_4\omega_t(k)\left(\alpha_t(k)+\beta_t(k)\right)\ \text{ and }\ \lambda_{t,2}(k)=\mu_4\omega_t^2(k).$$

Using these notations, then we have

$$\begin{aligned}\pi(k)\,\mathbb{E}\left[\epsilon_t^4\middle|\,\Delta_t=k\right] &= \pi(k)\lambda_t(k)+\sum_{j=1}^{d}\phi_{t,2}(k)\,\mathbb{E}\left[\epsilon_{t-1}^2\middle|\,\Delta_{t-1}=j\right]p(j,k)\,\pi(j)\\ &\quad+\sum_{j=1}^{d}a_{t,2}(k)\,\mathbb{E}\left[\epsilon_{t-1}^4\middle|\,\Delta_{t-1}=j\right]p(j,k)\,\pi(j),\end{aligned}$$

which gives the following system

$$N_{t,4}=\mathbb{P}_{a_{t,2}}N_{t-1,4}+C_{t,2},$$

where $C_{t,2}=\Pi_{\lambda_{t,2}}+\mathbb{P}_{\phi_{t,2}}N_{t-1,2}$. After $S-1$ successive replacements, we obtain

$$N_{t,4}=\left[\mathbf{I}_d-\prod_{i=0}^{S-1}\mathbb{P}_{a_{t-i,2}}\right]^{-1}\sum_{j=0}^{S-1}\left[\prod_{i=0}^{j-1}\mathbb{P}_{a_{t-i,2}}\right]C_{t-j,2}. \tag{18}$$

Consequently, the fourth order moment of $\{\epsilon_t;t\in\mathbb{Z}\}$ is given by

$$\mathbb{E}\left(\epsilon_t^4\right)=\sum_{k=1}^{d}N_{t,4}(k). \tag{19}$$

Now, let us consider the calculation of $\mathbb{E}\left[\epsilon_t^{2m}\right]$. We have, for any positive integer $m\geq 1$, $\epsilon_t^{2m}=\eta_t^{2m}h_t^m$. However,

$$\begin{aligned}h_t^m &= \left\{\omega_t(\Delta_t)+\left[\alpha_t(\Delta_t)\eta_{t-1}^2+\beta_t(\Delta_t)\right]h_{t-1}\right\}^m\\ &= \omega_t^m(\Delta_t)+\sum_{i=1}^{m-1}\binom{m}{i}\omega_t^{m-i}(\Delta_t)\left[\alpha_t(\Delta_t)\eta_{t-1}^2+\beta_t(\Delta_t)\right]^i h_{t-1}^i\\ &\quad+\left[\alpha_t(\Delta_t)\eta_{t-1}^2+\beta_t(\Delta_t)\right]^m h_{t-1}^m.\end{aligned}$$

Under the assumption $\rho\left(\prod_{s=0}^{S-1}\mathbb{P}_{a_{S-s,m}}\right)<1$ and $\mu_{2m}=\mathbb{E}\left[\eta_t^{2m}\right]<\infty$, we have

$$\mathbb{E}\left(\epsilon_t^{2m}\right)=\sum_{k=1}^{d}N_{t,2m}\left(k\right)=\mu_{2m}\sum_{k=1}^{d}\pi\left(k\right)\mathbb{E}\left[h_t^m\mid\Delta_t=k\right].$$

But

$$\begin{aligned}\mathbb{E}\left[h_t^m\mid\Delta_t=k\right] &= \omega_t^m\left(k\right)+\sum_{i=1}^{m-1}\binom{m}{i}\frac{\omega_t^{m-i}\left(k\right)a_{t,i}\left(k\right)}{\mu_{2i}}\mathbb{E}\left[\epsilon_{t-1}^{2i}\mid\Delta_t=k\right]\\ &\quad+\frac{a_{t,m}}{\mu_{2m}}\mathbb{E}\left[\epsilon_{t-1}^{2m}\mid\Delta_t=k\right].\end{aligned}$$

Thus,

$$\begin{aligned}\pi\left(k\right)\mathbb{E}\left[\epsilon_t^{2m}\mid\Delta_t=k\right]&=\pi\left(k\right)\mu_{2m}\omega_t^m\left(k\right)\\ &+\sum_{i=1}^{m-1}\sum_{l=1}^{d}\tbinom{m}{i}\tfrac{\mu_{2m}}{\mu_{2i}}\omega_t^{m-i}\left(k\right)a_{t,i}\left(k\right)p\left(l,k\right)\pi\left(l\right)\mathbb{E}\left[\epsilon_{t-1}^{2i}\mid\Delta_{t-1}=l\right]\\ &+\sum_{l=1}^{d}a_{t,m}p\left(l,k\right)\pi\left(l\right)\mathbb{E}\left[\epsilon_{t-1}^{2m}\mid\Delta_{t-1}=l\right],\end{aligned}$$

which can be stacked as follows

$$N_{t,2m}=\mathbb{P}_{a_{t,m}}N_{t-1,2m}+C_{t,m},$$

where

$$\begin{aligned}C_{t,m}&=\sum_{i=1}^{m-1}\mathbb{P}_{\phi_{t,2i}}N_{t-1,2i}+\Pi_{\lambda_{t,m}},\\ \phi_{t,2i}\left(\cdot\right)&=\binom{m}{i}\frac{\mu_{2m}}{\mu_{2i}}\omega_t^{m-i}\left(\cdot\right)a_{t,i}\left(\cdot\right),\end{aligned}$$

and

$$\lambda_{t,m}\left(\cdot\right)=\mu_{2m}\omega_t^m\left(\cdot\right).$$

After $S-1$ successive replacements

$$N_{t,2m}=\left[\mathbf{I}_d-\prod_{i=0}^{S-1}\mathbb{P}_{a_{t-i,m}}\right]^{-1}\sum_{j=0}^{S-1}\left[\prod_{i=0}^{j-1}\mathbb{P}_{a_{t-i,m}}\right]C_{t-j,m}.\tag{20}$$

Hence, the moment of order $2m$ of $\{\epsilon_t;t\in\mathbb{Z}\}$ can be computed as follows

$$\mathbb{E}\left[\epsilon_t^{2m}\right]=\sum_{k=1}^{d}N_{t,2m}\left(k\right).$$

Proposition 5.1. Suppose that $\mathbb{E}\left(\eta_t^{2m}\right) < \infty$. A necessary and sufficient condition under which the *MS-PGARCH* process defined by (14) is *s.p.s.* and $\mathbb{E}\left[\epsilon_t^{2m}\right] < \infty$, is given by

$$\rho\left(\prod_{s=0}^{S-1} \mathbb{P}_{a_{S-s,m}}\right) < 1. \tag{21}$$

In that case, the closed-form expression of the moment of order $2m$ of the process $\{\epsilon_t; t \in \mathbb{Z}\}$, for any positive integer $m \geq 1$, is given by

$$\mathbb{E}\left(\epsilon_t^{2m}\right) = \sum_{k=1}^{d} N_{t,2m}(k),$$

where $N_{t,2m}$ can be obtained from (20). Therefore, the variance and the kurtosis of the distribution of $\epsilon_{s+\tau S}$ are

$$\sigma_\epsilon^{(s)2} := \mathbb{E}\left(\epsilon_{s+\tau S}^2\right) = \sum_{k=1}^{d} N_{s+\tau S,2}(k), \ s = 1, 2, \ldots, S,$$

and

$$\kappa_\epsilon^{(s)} := \frac{\mathbb{E}\left(\epsilon_{s+\tau S}^4\right)}{\left[\mathbb{E}\left(\epsilon_{s+\tau S}^2\right)\right]^2} = \frac{\sum_{k=1}^{d} N_{s+\tau S,4}(k)}{\left[\sum_{k=1}^{d} N_{s+\tau S,2}(k)\right]^2}, \ s = 1, 2, \ldots, S.$$

Remark 5.2.

1. When $S = 1$, *then the condition* (21) can be written as follows

   $$\rho\left(\mathbb{P}_{\Phi_m}\right) < 1,$$

   where $\Phi_m(\cdot) = \mathbb{E}\left[\left(\alpha_{1,1}(\cdot)\eta_t^2 + \beta_{1,1}(\cdot)\right)^m\right]$ and which is the same established by Francq and Zakoïan ([20], Corollary 2) in the case of the non-periodic *MS-GARCH* models.

2. If (Δ_t) is a sequence of *i.i.d.* random variables, then the condition ensuring the existence of moments of order $2m$ of the mixture periodic $GARCH$ model is given by

   $$\rho\left(\prod_{s=0}^{S-1} \mathbb{P}_{\Omega_{S-s,m}}\right) < 1, \tag{22}$$

   where $\Omega_{t,m}(\cdot) = \mathbb{E}\left[\left(\alpha_{t,1}(\cdot)\eta_t^2 + \beta_{t,1}(\cdot)\right)^m\right]$. In this particular case, we can rewrite this last condition in another simpler form. Indeed, we have

$$\prod_{s=0}^{S-1} \mathbb{P}_{\Omega_{S-s,m}} = \begin{pmatrix} \pi(1)\mathbb{E}\left[\left(\alpha_{S,1}^{(1)}\eta_S^2 + \beta_{S,1}^{(1)}\right)^m\right] & \cdots & \pi(1)\mathbb{E}\left[\left(\alpha_{S,1}^{(1)}\eta_S^2 + \beta_{S,1}^{(1)}\right)^m\right] \\ \vdots & \ddots & \vdots \\ \pi(d)\mathbb{E}\left[\left(\alpha_{S,1}^{(d)}\eta_S^2 + \beta_{S,1}^{(d)}\right)^m\right] & \cdots & \pi(d)\mathbb{E}\left[\left(\alpha_{S,1}^{(d)}\eta_S^2 + \beta_{S,1}^{(d)}\right)^m\right] \end{pmatrix}$$
$$\times \left\{\prod_{s=1}^{S-1}\left(\sum_{k=1}^{d} \pi(k)\mathbb{E}\left[\left(\alpha_{S-s,1}^{(k)}\eta_{S-s}^2 + \beta_{S-s,1}^{(k)}\right)^m\right]\right)\right\}.$$

Note that this last matrix is positive such that the sum of the terms of each column is constant and equal to

$$\prod_{s=0}^{S-1}\left(\sum_{k=1}^{d}\pi\left(k\right)\mathbb{E}\left[\left(\alpha_{S-s,1}^{(k)}\eta_{S-s}^{2}+\beta_{S-s,1}^{(k)}\right)^{m}\right]\right),$$

which is then an eigenvalue of $\prod_{s=0}^{S-1}\mathbb{P}_{\Omega_{S-s,m}}$ and

$$\rho\left(\prod_{s=0}^{S-1}\mathbb{P}_{\Omega_{S-s,m}}\right)=\prod_{s=0}^{S-1}\left(\sum_{k=1}^{d}\pi\left(k\right)\mathbb{E}\left[\left(\alpha_{S-s,1}^{(k)}\eta_{S-s}^{2}+\beta_{S-s,1}^{(k)}\right)^{m}\right]\right),$$

(see Horn and Johnson [32], Lemma 8.1.21, p. 521).

For $m=1$, the condition (22) coincides with that obtained by Hamdi et Souam ([28], Corollary 1) for the second order periodic stationarity of the mixture $PGARCH$ models.

5.2. Autocovariance structure of the powers of the MS-$PGARCH_S\left(1,1\right)$ process

From (14), we clearly see that if $n\in\{0,1,\ldots,m\}$ and $m\in\mathbb{N}^{*}$, we have

$$\epsilon_{t}^{2n}\epsilon_{t-1}^{2m}=\eta_{t}^{2n}\left[\omega_{t}\left(\Delta_{t}\right)+\left\{\alpha_{t}\left(\Delta_{t}\right)\eta_{t-1}^{2}+\beta_{t}\left(\Delta_{t}\right)\right\}h_{t-1}\right]^{n}\epsilon_{t-1}^{2m},$$

and under the assumption that $\mathbb{E}\left(\eta_{t}^{2m}\right)<\infty$ and $\rho\left(\prod_{s=0}^{S-1}\mathbb{P}_{a_{S-s,2m}}\right)<1$, we obtain

$$\begin{aligned}
&\mathbb{E}\left(\epsilon_{t}^{2n}\epsilon_{t-1}^{2m}\right|\left.\Delta_{t}=k\right)\\
&=\mathbb{E}\left(\eta_{t}^{2n}\left[\omega_{t}\left(\Delta_{t}\right)+\left\{\alpha_{t}\left(\Delta_{t}\right)\eta_{t-1}^{2}+\beta_{t}\left(\Delta_{t}\right)\right\}h_{t-1}\right]^{n}\epsilon_{t-1}^{2m}\right|\left.\Delta_{t}=k\right)\\
&=\mathbb{E}\left(\eta_{t}^{2n}\left\{\sum_{i=0}^{n}\tbinom{n}{i}\left\{\omega_{t}\left(\Delta_{t}\right)\right\}^{n-i}\left[\alpha_{t}\left(\Delta_{t}\right)\eta_{t-1}^{2}+\beta_{t}\left(\Delta_{t}\right)\right]^{i}h_{t-1}^{m+i}\eta_{t-1}^{2m}\right\}\right|\left.\Delta_{t}=k\right)\\
&=\sum_{i=0}^{n}\sum_{j=0}^{i}\tbinom{n}{i}\tbinom{i}{j}\mathbb{E}\left(\eta_{t}^{2n}\left\{\omega_{t}\left(\Delta_{t}\right)\right\}^{n-i}\left\{\alpha_{t}\left(\Delta_{t}\right)\right\}^{j}\left\{\beta_{t}\left(\Delta_{t}\right)\right\}^{i-j}\eta_{t-1}^{2(j+m)}h_{t-1}^{m+i}\right|\left.\Delta_{t}=k\right)\\
&=\sum_{i=0}^{n}\xi_{n,i,m}^{(t)}\left(k\right)\mathbb{E}\left(\epsilon_{t-1}^{2(m+i)}\right|\left.\Delta_{t}=k\right)
\end{aligned}$$

where the S-periodic functions $\xi_{n,i,m}^{(t)}\left(\cdot\right)$ are given by

$$\xi_{n,i,m}^{(t)}\left(\cdot\right)=\tbinom{n}{i}\left\{\omega_{t}\left(\cdot\right)\right\}^{n-i}\tfrac{\mu_{2n}}{\mu_{2(m+i)}}\sum_{j=0}^{i}\tbinom{i}{j}\left\{\alpha_{t}\left(\cdot\right)\right\}^{j}\left\{\beta_{t}\left(\cdot\right)\right\}^{i-j}\mu_{2(m+j)},$$

$$\text{for all } i\in\{0,1,\ldots,n\},\ n\in\{0,1,\ldots,m\}\text{ and } m\in\mathbb{N}^{*}.$$

Thus, for all $k=1,\ldots,d$, we have

$$\pi\left(k\right)\mathbb{E}\left(\epsilon_{t}^{2n}\epsilon_{t-1}^{2m}\right|\left.\Delta_{t}=k\right)=\sum_{i=0}^{n}\sum_{l=1}^{d}\xi_{n,i,m}^{(t)}\left(k\right)\mathbb{E}\left(\epsilon_{t-1}^{2(i+m)}\right|\left.\Delta_{t-1}=l\right)p\left(l,k\right)\pi\left(l\right),$$

from which we obtain the following system

$$W_{1,n,m}^{(t)} = \sum_{i=0}^{n} \mathbb{P}_{\xi_{n,i,m}^{(t)}} N_{t-1,2(i+m)}, \tag{23}$$

where $W_{h,n,m}^{(t)} = \left(\pi(1)\mathbb{E}\left(\epsilon_t^{2n}\epsilon_{t-h}^{2m}\middle|\,\Delta_t = 1\right),\ldots,\pi(d)\mathbb{E}\left(\epsilon_t^{2n}\epsilon_{t-h}^{2m}\middle|\,\Delta_t = d\right)\right)'$, $t,h \in \mathbb{Z}$.

Moreover, we have for all $k = 1,\ldots,d$ and all $h > 1$, $m \in \mathbb{N}^*$ and $n \in \{0,1,\ldots,m\}$,

$$\begin{aligned}
&\pi(k)\mathbb{E}\left(\epsilon_t^{2n}\epsilon_{t-h}^{2m}\middle|\,\Delta_t = k\right)\\
&\quad= \pi(k)\mu_{2n}\mathbb{E}\left(h_t^{n}\epsilon_{t-h}^{2m}\middle|\,\Delta_t = k\right)\\
&\quad= \pi(k)\mu_{2n}\sum_{i=0}^{n}\tbinom{n}{i}\{\omega_t(k)\}^{n-i}\,\mathbb{E}\left(\left[\alpha_t(k)\eta_{t-1}^2+\beta_t(k)\right]^i h_{t-1}^{i}\epsilon_{t-h}^{2m}\middle|\,\Delta_t = k\right)\\
&\quad= \pi(k)\mu_{2n}\sum_{i=0}^{n}\sum_{j=0}^{i}\tbinom{n}{i}\tbinom{i}{j}\{\omega_t(k)\}^{n-i}\{\alpha_t(k)\}^{j}\{\beta_t(k)\}^{i-j}\,\mathbb{E}\left(\eta_{t-1}^{2j}h_{t-1}^{i}\epsilon_{t-h}^{2m}\middle|\,\Delta_t = k\right)\\
&\quad= \pi(k)\sum_{i=0}^{n}\sum_{j=0}^{i}\tbinom{n}{i}\tbinom{i}{j}\{\omega_t(k)\}^{n-i}\{\alpha_t(k)\}^{j}\{\beta_t(k)\}^{i-j}\tfrac{\mu_{2j}\mu_{2n}}{\mu_{2i}}\mathbb{E}\left(\epsilon_{t-1}^{2i}\epsilon_{t-h}^{2m}\middle|\,\Delta_t = k\right)\\
&\quad= \pi(k)\sum_{i=0}^{n}\psi_{n,i}^{(t)}(k)\,\mathbb{E}\left(\epsilon_{t-1}^{2i}\epsilon_{t-h}^{2m}\middle|\,\Delta_t = k\right)\\
&\quad= \sum_{i=0}^{n}\sum_{l=1}^{d}\psi_{n,i}^{(t)}(k)\,\mathbb{E}\left(\epsilon_{t-1}^{2i}\epsilon_{t-h}^{2m}\middle|\,\Delta_{t-1} = l\right)p(l,k)\,\pi(l),
\end{aligned}$$

where

$$\psi_{n,i}^{(t)}(\cdot) = \binom{n}{i}\{\omega_t(\cdot)\}^{n-i}\frac{\mu_{2n}}{\mu_{2i}}\sum_{j=0}^{i}\binom{i}{j}\{\alpha_t(\cdot)\}^{j}\{\beta_t(\cdot)\}^{i-j}\mu_{2j}.$$

Therefore, we obtain the following system

$$W_{h,n,m}^{(t)} = \sum_{i=0}^{n}\mathbb{P}_{\psi_{n,i}^{(t)}} W_{h-1,i,m}^{(t-1)}. \tag{24}$$

Consequently, we get for all $h \in \mathbb{N}$

$$\gamma_{\epsilon^{2m},h}^{(t)} := cov\left(\epsilon_t^{2m},\epsilon_{t-h}^{2m}\right) = \sum_{k=1}^{d} W_{h,m,m}^{(t)}(k) - \left(\sum_{k=1}^{d} N_{t,2m}(k)\right)\left(\sum_{k=1}^{d} N_{t-h,2m}(k)\right). \tag{25}$$

The following algorithm summarizes the calculation of the autocovariances of the powers of an $MS\text{-}PGARCH_S(1,1)$ process.

Algorithm 5.3.

1. For $s = 1,2,\ldots,S$ compute $N_{t,2i}$ for $i = m, m+1,\ldots,2m$, using (20).
2. For $s = 1,2,\ldots,S$, $m \in \mathbb{N}^*$ and $n = 0,1,\ldots,m$ calculate $W_{1,n,m}^{(t)}$ using (23).

3. For $s = 1, 2, \ldots, S$, $m \in \mathbb{N}^*$, $h > 1$ and $n = 0, 1, \ldots, m$ compute $W_{h,n,m}^{(t)}$ from (24).
4. For $s = 1, 2, \ldots, S$, $m \in \mathbb{N}^*$ and $h \in \mathbb{N}^*$, compute the autocovariances of $\left\{\epsilon_t^{2m}, t \in \mathbb{Z}\right\}$ using (25).

6. ESTIMATION OF THE MS-$PGARCH_S\,(1,1)$ MODEL USING GMM METHOD

6.1. GMM estimation

The estimation of the Markov-switching $PGARCH$ model is so hard, and this is due to the recursiveness of the conditional variance equation. Indeed, through equation (14), one can clearly see that the conditional variance depend at time t on the whole history of regimes generated by the markov chain (Δ_t) up to time t. So, since the regimes are unobservable, one should to integrate over all possible regime paths to evaluate the likelihood function. Whereas, the number of possible paths grows exponentially with t and becomes quickly unmanageable. For instance, the computation of the likelihood function over a sample of length T, require summation over all d^T unobserved states. Hence, the path dependence property of this class of models renders the exact calculation of the likelihood very cumbersome numerically. This prompted many authors (e. g. Gray [24]; Klaassen [33] to propose estimation procedures based on modified versions of the non-periodic MS-$GARCH$ model that circumvent the path dependence problem by maximum likelihood. Other authors suggested alternative estimation methods such as generalized method of moments (GMM) (Francq and Zakoïan, [21]), Bayesian $MCMC$ methods (e. g. Bauwens et al. [7]; Henneke et al. [31]; Billio et al. [14] and Collapsing procedure (e. g. Augustyniak et al. [6]).

Since we gave the explicit expressions of the moments of the squared process and the autocovariances of its powers, we propose in this section, a generalization of the GMM procedure, proposed by Francq and Zakoïan [21], into the periodic case. Suppose that the number of regimes d and the period S are known. The unknown parameters are gathered in a vector $\theta = \left(vec\,(\mathbb{P})', \theta_1^{(1)}, \ldots, \theta_S^{(1)}, \theta_1^{(2)}, \ldots, \theta_S^{(2)}, \ldots, \theta_1^{(d)}, \ldots, \theta_S^{(d)}\right)'$ belonging in to a parameter space Θ, where $\theta_s^{(k)} = \left(\omega_s^{(k)}, \alpha_{s,1}^{(k)}, \beta_{s,1}^{(k)}\right)$, for $s = 1, \ldots, S$, $k = 1, \ldots, d$ and vec is the usual column stacking. The true parameter value denoted by θ_0 is unknown and should to be estimated. For this purpose, let $\{\epsilon_1, \ldots, \epsilon_T \mid T = NS,\ N \in \mathbb{N}^*\}$ be a realization of length $T \geq l$, where l is a given lag. To estimate the unknown parameters of the model via a GMM procedure, we use the orthogonality conditions given by

$$\mathbb{E}_{\theta_0}\left[H_\tau\left(\theta_0, \underline{\epsilon}_\tau\right)\right] = 0, \tag{26}$$

where

$$\begin{aligned}
\underline{\epsilon}_\tau &= \left(\epsilon_{1+S\tau}, \ldots, \epsilon_{S+S\tau}, \epsilon_{S\tau}, \ldots, \epsilon_{S+S\tau-1}, \ldots, \epsilon_{1+S\tau-l}, \ldots, \epsilon_{S+S\tau-l}\right), \\
H_\tau\left(\theta, \underline{\epsilon}_\tau\right) &= \left(h_{1,\tau}^{(0)}, h_{1,\tau}^{(1)}, \ldots, h_{1,\tau}^{(m)}, h_{2,\tau}^{(0)}, h_{2,\tau}^{(1)}, \ldots, h_{2,\tau}^{(m)}, \ldots, h_{S,\tau}^{(0)}, h_{S,\tau}^{(1)}, \ldots, h_{S,\tau}^{(m)}\right)',
\end{aligned}$$

with $h_{s,\tau}^{(0)} = \epsilon_{s+S\tau}^2 - \mathbb{E}_\theta\left(\epsilon_{s+S\tau}^2\right)$ and $h_{s,\tau}^{(i)} = \left(\epsilon_{s+S\tau}^{4i} - \mathbb{E}_\theta\left(\epsilon_{s+S\tau}^{4i}\right),\ \epsilon_{s+S\tau}^{2i}\epsilon_{s+S\tau-1}^{2i} - \mathbb{E}_\theta\left(\epsilon_{s+S\tau}^{2i}\epsilon_{s+S\tau-1}^{2i}\right), \ldots, \epsilon_{s+S\tau}^{2i}\epsilon_{s+S\tau-l}^{2i} - \mathbb{E}_\theta\left(\epsilon_{s+S\tau}^{2i}\epsilon_{s+S\tau-l}^{2i}\right)\right)$, for $i = 1, \ldots, m$, and $s =$

$1, \ldots, S$. Note that m and l are chosen such that the system (26) be identifiable and the conditions of existence of moments, given by Proposition 5.1, are satisfied.

The idea behind GMM is to choose θ so as to make the sample moment $H_N(\theta)$ as close as possible to the population moment which equal to zero; that is, the GMM estimator $\widehat{\theta}$ is the value of θ that minimizes

$$Q_N(\theta) = H_N'(\theta) W_N H_N(\theta),$$

where $H_N(\theta)$ denote the sample average of $H_\tau(\theta, \underline{\epsilon}_\tau)$, i.e.,

$$H_N(\theta) = \frac{1}{N - \tau_0} \sum_{\tau=\tau_0+1}^{N} H_\tau(\theta, \underline{\epsilon}_\tau),$$

and (W_N) is a sequence of positive definite weighting matrices. Here τ_0 denotes the largest integer less than or equal to l/S. Finally, to estimate the unknown parameters of the model, one should compute the optimal weighting matrix $\widehat{W}_N = \widehat{V}^{-1}$, using the estimator of Newey and West [36], which is defined by

$$\widehat{V} = \Omega_N(0) + \sum_{i=1}^{\nu} K\left(\frac{i}{\nu}\right)\{\Omega_N(i) + \Omega_N'(i)\}$$

where

$$\begin{aligned} \Omega_N(i) &= \frac{1}{N - \tau_0} \sum_{\tau=\tau_0+1+i}^{N} \overline{\mathbf{H}}_\tau\left(\widehat{\theta}, \underline{\epsilon}_\tau\right) \overline{\mathbf{H}}_{\tau-i}'\left(\widehat{\theta}, \underline{\epsilon}_\tau\right), \\ \overline{\mathbf{H}}_\tau\left(\widehat{\theta}, \underline{\epsilon}_\tau\right) &= H_\tau\left(\widehat{\theta}, \underline{\epsilon}_\tau\right) - \frac{1}{N - \tau_0} \sum_{\tau=\tau_0+1}^{N} H_\tau\left(\widehat{\theta}, \underline{\epsilon}_\tau\right), \end{aligned}$$

and ν is a truncation parameter which is a function of N allowed to grow slowly enough with the sample size and required to grow slower than $\sqrt[4]{N}$ (see Newey and West [36] for more details about the choice of ν). For the simulation study reported in Section 6.2, we used the Bartlett kernel weight $K(x)$ defined by $(1 - |x|)\,\mathbf{1}_{(|x| \le 1)}$.

Hence, a GMM estimator $\widehat{\theta}$ is obtained as

$$\widehat{\theta} = \arg\min_{\theta \in \Theta} H_N'(\theta) \widehat{W}_N H_N(\theta).$$

6.2. Simulation study

In order to investigate the performance of the GMM method for parameters estimation, we carried out a simulation study based on two $MS\text{-}PGARCH_S(1,1)$ models ($S = 2, 4$, $d = 2$ and $\eta_t \sim \mathcal{N}(0,1)$). We simulated 1000 data samples with different lengths. The sample sizes to be examined in this simulation study are 1000, 2000, 5000 and 10000. The corresponding parameter values are chosen to satisfy the condition (21) for $m = 4$, i.e. the existence of moments of order 8. The GMM criterion was constructed on the basis of the expectations of the following $(19 \times S)$ variables $\epsilon^2_{s+S\tau}$ and $\epsilon^{2i}_{s+S\tau}\epsilon^{2i}_{s+S\tau-h}$, $h = 1, \ldots, 8$, and $i = 1, 2$.

In Tables 1-2 are reported the true values (TV) of the parameters of each of the considered MS-$PGARCH_S\,(1,1)$ data-generating process, the mean and the root-mean-square error $(RMSE)$ of their estimates for the 1000 replications.

		$T = 1000$		$T = 2000$		$T = 5000$		$T = 10000$	
	TV	Mean	$RMSE$	Mean	$RMSE$	Mean	$RMSE$	Mean	$RMSE$
$p\,(1,1)$	0.85	0.5747	0.3210	0.6727	0.2714	0.8100	0.1655	0.8350	0.1567
$p\,(2,1)$	0.25	0.2718	0.1195	0.2424	0.1391	0.2454	0.1173	0.2551	0.1068
$\omega_1^{(1)}$	0.30	0.4750	0.2673	0.4741	0.2089	0.2735	0.1715	0,2752	0.1579
$\alpha_{1,1}^{(1)}$	0.10	0.1316	0.0984	0.1312	0.0938	0.1055	0.0765	0.1036	0.0679
$\beta_{1,1}^{(1)}$	0.10	0.2322	0.1720	0.2089	0.1817	0.0567	0.1018	0.0864	0.1011
$\omega_2^{(1)}$	0.50	0.6563	0.2571	0.6545	0.2226	0.3364	0.2533	0.3570	0.2089
$\alpha_{2,1}^{(1)}$	0.15	0.1956	0.1179	0.1795	0.1166	0.1862	0.1048	0.1772	0.0841
$\beta_{2,1}^{(1)}$	0.20	0.3088	0.2122	0.2925	0.1924	0.2493	0.1816	0.2481	0.1453
$\omega_1^{(2)}$	0.90	0.4463	0.5138	0.5590	0.5013	0.8444	0.4368	0.8591	0.3566
$\alpha_{1,1}^{(2)}$	0.13	0.1168	0.0771	0.1097	0.0671	0.1115	0.0542	0.1203	0.0518
$\beta_{1,1}^{(2)}$	0.70	0.2513	0.4741	0.2307	0.5019	0.5115	0.3854	0.6282	0.3500
$\omega_2^{(2)}$	1.10	0.5892	0.5823	0.7007	0.5722	1.2022	0.4104	1.1365	0.4087
$\alpha_{2,1}^{(2)}$	0.18	0.2262	0.1518	0.1917	0.1080	0.1440	0.0652	0.1869	0.0443
$\beta_{2,1}^{(2)}$	0.50	0.3357	0.2358	0.3229	0.2138	0.2130	0.2006	0.4087	0.1981

Tab. 1. Results of a simulation study for an MS-$PGARCH_2\,(1,1)$ model with different values of the sample size T and 1000 replications.

From Tables 1 and 2, we can observe that the estimates of the MS-$PGARCH$ coefficients based on GMM model display significant biases which decrease as the sample size is increased. Unlike the non-periodic case, the transition probabilities are estimated with low accuracy. It can also be seen that the $RMSE$s of some parameters are relatively large but they often gradually decrease as the sample size is increased.

CONCLUSION

This article proposes a new Markov-switching periodic $GARCH$ model, which captures both the periodicity and the regime change phenomenon of time series in conditional variance. We studied some probabilistic properties of this class of models as well as the autocovariance structure of the squared MS-$PGARCH$ process.

As in the non-periodic model, the estimation can be done by using the GMM method. Simulation examples showed that the proposed GMM procedure does not perform well when the sample size is small to moderate. However, it does provide a good starting value for others methods. In order to improve the estimation quality, it could be interesting in a future research to develop a new approach, for example by adopting the simulated methods (Bauwens et al. [7]; Henneke et al. [31]; Augustyniak [5]; Billio et al. [14]; Augustyniak et al. [6]). It could be also interesting to see what happens if one fits a non-periodic MS-$GARCH$ model to a periodic data.

		$T = 1000$		$T = 2000$		$T = 5000$		$T = 10000$	
	TV	Mean	$RMSE$	Mean	$RMSE$	Mean	$RMSE$	Mean	$RMSE$
$p(1,1)$	0.95	0.7838	0.2482	0.8338	0.1619	0.8786	0.1369	0.9371	0.1244
$p(2,1)$	0.10	0.3960	0.3438	0.1132	0.3521	0.1298	0.2223	0.1712	0.2002
$\omega_1^{(1)}$	0.60	0.6169	0.1970	0.6306	0.1741	0.6349	0.1631	0.6264	0.1560
$\alpha_{1,1}^{(1)}$	0.35	0.1772	0.2295	0.1826	0.1893	0.2393	0.1838	0.3398	0.1693
$\beta_{1,1}^{(1)}$	0.30	0.2845	0.1589	0.3023	0.1513	0.3164	0.1447	0.3156	0.1320
$\omega_2^{(1)}$	0.40	0.4438	0.1536	0.4915	0.1866	0.5080	0.1633	0.4481	0.1579
$\alpha_{2,1}^{(1)}$	0.20	0.1323	0.1886	0.1716	0.1019	0.1861	0.1403	0.1885	0.0962
$\beta_{2,1}^{(1)}$	0.25	0.2130	0.1462	0.2225	0.1348	0.1793	0.1334	0.2474	0.1221
$\omega_3^{(1)}$	0.30	0.3337	0.2137	0.3861	0.1499	0.3779	0.1506	0.3121	0.1376
$\alpha_{3,1}^{(1)}$	0.15	0.1196	0.1006	0.1282	0.0724	0.1346	0.0717	0.1396	0.0668
$\beta_{3,1}^{(1)}$	0.20	0.1849	0.1270	0.2032	0.1158	0.1687	0.1110	0.2013	0.1025
$\omega_4^{(1)}$	0.50	0.4812	0.2515	0.5278	0.2219	0.5346	0.2064	0.5028	0.1988
$\alpha_{4,1}^{(1)}$	0.22	0.1688	0.1573	0.2175	0.1481	0.2079	0.1403	0.2139	0.0785
$\beta_{4,1}^{(1)}$	0.35	0.2578	0.1743	0.3147	0.1393	0.3423	0.1221	0.3402	0.1054
$\omega_1^{(2)}$	1.50	0.8108	0.7806	1.2268	0.4268	1.3627	0.3756	1.4403	0.2605
$\alpha_{1,1}^{(2)}$	0.30	0.0914	0.2191	0.0903	0.2121	0.1115	0.1926	0.2270	0.1808
$\beta_{1,1}^{(2)}$	0.42	0.2983	0.2127	0.3046	0.1891	0.3339	0.1821	0.3807	0.1544
$\omega_2^{(2)}$	1.20	0.7466	0.5442	1.0454	0.3227	1.1345	0.3160	1.2269	0.2927
$\alpha_{2,1}^{(2)}$	0.25	0.0756	0.1811	0.0892	0.1658	0.0933	0.1604	0.2254	0.1500
$\beta_{2,1}^{(2)}$	0.40	0.2482	0.2247	0.2599	0.1865	0.2763	0.1953	0.3575	0.1728
$\omega_3^{(2)}$	0.90	0.5679	0.4316	0.8009	0.2919	0.8292	0.2863	0.8371	0.2672
$\alpha_{3,1}^{(2)}$	0.20	0.0659	0.1405	0.0795	0.1250	0.0829	0.1221	0.1696	0.1104
$\beta_{3,1}^{(2)}$	0.35	0.2304	0.1963	0.2123	0.1632	0.2497	0.1623	0.3178	0.1331
$\omega_4^{(2)}$	1.00	0.7074	0.4087	0.9409	0.2805	0.9600	0.2731	1.0216	0.2147
$\alpha_{4,1}^{(2)}$	0.27	0.0917	0.1880	0.0969	0.1779	0.1132	0.1632	0.2229	0.1550
$\beta_{4,1}^{(2)}$	0.45	0.2910	0.2317	0.3120	0.2120	0.3422	0.2183	0.3877	0.1536

Tab. 2. Results of a simulation study for an $MS\text{-}PGARCH_4\,(1,1)$ model with different values of the sample size T and 1000 replications.

ACKNOWLEDGEMENTS

The authors are deeply grateful to the Associate Editor and to the referee for their careful reading, useful remarks and constructive suggestions that substantially improved the quality of the paper.

(Received February 5, 2019)

REFERENCES

[1] A. Aknouche and M. Bentarzi: On the existence of higher-order moments for periodic *GARCH* models. Statist. Probab. Lett. *78* (2008), 3262–3268. DOI:10.1016/j.spl.2008.06.010

[2] A. Aknouche and A. Bibi: Quasi-maximum likelihood estimation of periodic *GARCH* and periodic *ARMA-GARCH* processes. J. Time Series Anal. *28* (2009), 19–46. DOI:10.1016/j.spl.2008.06.010

[3] A. Aknouche and H. Guerbyenne: Periodic stationarity of random coefficient periodic autoregressions. Statist. Probab. Lett. *79* (2009), 990–996. DOI:10.1016/j.spl.2008.12.012

[4] B. Aliat and F. Hamdi: On markov-switching periodic *ARMA* models. Comm. Statist. Theory Methods *47* (2018), 344–364. DOI:10.1080/03610926.2017.1303734

[5] M. Augustyniak: Maximum likelihood estimation of the Markov-switching *GARCH* model. Comput. Statist. Data Anal. *76* (2014), 61–75. DOI:10.1016/j.csda.2013.01.026

[6] M. Augustyniak, M. Boudreault, and M. Morales: Maximum likelihood estimation of the Markov-switching GARCH model based on a general collapsing procedure. Methodol. Comput. Appl. Probab. *20* (2018), 165–188. DOI:10.1007/s11009-016-9541-4

[7] L. Bauwens, A. Preminger, and J. V. K. Rombouts: Theory and inference for Markov switching *GARCH* model. Econometr. J. *13* (2010), 218–244. DOI:10.1111/j.1368-423x.2009.00307.x

[8] M. Bentarzi and F. Hamdi: Mixture periodic autoregressive conditional heteroskedastic models. Comput. Statist. Data Anal. *53* (2008), 1–16. DOI:10.1016/j.csda.2008.06.019

[9] M. Bentarzi and F. Hamdi: Mixture periodic autoregression with aeriodic *ARCH* errors. Adv. Appl. Statist. *8* (2008), 219–46.

[10] A. Bibi and A. Aknouche: On periodic *GARCH* processes: Stationarity Statist. *17* (2008), 305–316. DOI:10.3103/s1066530708040029

[11] A. Bibi and I. Lescheb: Strong consistency and asymptotic normality of least squares estimators for *PGARCH* and *PARMA-PGARCH* models. Statist. Probab. Lett. *80* (2010), 1532–1542. DOI:10.1016/j.spl.2010.06.007

[12] A. Bibi and I. Lescheb: A conditional least squares approach to *PGARCH* and *PARMA-PGARCH* time series estimation. Comptes Rendus Math. *348* (2010), 1211–1216. DOI:10.1016/j.crma.2010.10.019

[13] A. Bibi and I. Lescheb: Estimation and asymptotic properties in periodic $GARCH(1,1)$ models. Comm. Statist. Theory Methods *42* (2013), 3497-3513. DOI:10.1080/03610926.2011.633201

[14] M. Billio, R. Casarin, and A. Osuntuyi: Efficient Gibbs sampling for Markov switching *GARCH* models. Comput. Statist. Data Anal. *100* (2016), 37–57. DOI:10.1016/j.csda.2014.04.011

[15] T. Bollerslev: Generalized autoregressive conditional heteroskedasticity. J. Econometr. *31* (1986), 307–327. DOI:10.1016/0304-4076(86)90063-1

[16] T. Bollerslev and E. Ghysels: Periodic autoregressive conditional heteroskedasticity. J. Business Econom. Statist. *14* (1996), 139–152. DOI:10.2307/1392425

[17] P. Bougerol and N. Picard: Strict stationarity of generalized autoregressive processes. Ann. Probab. *20* (1992), 1714–1730. DOI:10.1214/aop/1176989526

[18] J. Cai: A Markov model of Switching-regime $ARCH$. J. Business Econom. Statist. *12* (1994), 309–316. DOI:10.2307/1392087

[19] C. Francq, M. Roussignol, and J. M. Zakoïan: Conditional heteroskedasticity driven by hidden Markov chains. J. Time Ser. Anal. *22* (2001), 197–220. DOI:10.1111/1467-9892.00219

[20] C. Francq and J. M. Zakoïan: The L^2-structures of standard and switching-regime $GARCH$ models. Stoch. Process. Appl. *115* (2005), 1557–1582.

[21] C. Francq and J. M. Zakoïan: Deriving the autocovariances of powers of Markov-switching $GARCH$ models with applications to statistical inference. Computat. Statist. Data Anal. *52* (2008), 3027–3046. DOI:10.1016/j.csda.2007.08.003

[22] P. H. Franses and R. Paap: Modeling changing day-of-the-week seasonality in stock returns and volatility. Appl. Financ. Econom. *52* (2000), 3027–3046.

[23] E. G. Gladyshev: Periodically correlated random sequences. Doklady Akademii Nauk SSSR *137* (1961), 1026–1029.

[24] S. F. Gray: Modeling the conditional distribution of interest rates as a regime-switching process. J. Financ. Econom. *42* (1996), 27–62. DOI:10.1016/0304-405x(96)00875-6

[25] M. Haas and M. S. Paolella: Mixture and regime-switching $GARCH$ models. In: Handbook of Volatility Models and their Applications 2012, pp. 71–102. DOI:10.1002/9781118272039.ch3

[26] M. Haas, S. Mittnik, and M. S. Paolella: A new approach to Markov-switching $GARCH$ models. J. Financ. Econometr. *2* (2004), 493–530. DOI:10.1093/jjfinec/nbh020

[27] F. Hamdi and S. Souam: Mixture periodic $GARCH$ models: Applications to exchange rate modeling. In: Modeling, Simulation and Applied Optimization (ICMSAO), 5th International Conference on IEEE, 2013. DOI:10.1109/icmsao.2013.6552570

[28] F. Hamdi and S. Souam: Mixture periodic $GARCH$ models: theory and applications. Empir. Econom. *55* (2018), 1925–1956. DOI:10.1007/s00181-017-1348-9

[29] J. D. Hamilton: A new approach to the economie analysis of nonstationary time series and the business cycle. Econometrica *57* (1989), 357–384. DOI:10.2307/1912559

[30] J. D. Hamilton and R. Susmel: Autoregressive condiational heteroskedasticity and changes in regime. J. Econometr. *64* (1994), 307–333. DOI:10.1016/0304-4076(94)90067-1

[31] J. S. Henneke, S. T. Rachev, F. J. Fabozzi, and M.Nikolov: MCMC-based estimation of Markov switching $ARMA$-$GARCH$ models. Appl. Econom. *43* (2011), 259–271. DOI:10.1080/00036840802552379

[32] R. A. Horn and C. R. Johnson: Matrix Analysis. Second edition. Cambridge University Press, New York 2013.

[33] F. Klaassen: Improving $GARCH$ volatility forecasts. Empir. Econometr. *27* (2002), 363–394. DOI:10.1007/s001810100100

[34] P. Lancaster and M. Tismenetsky: The Theory of Matrices. Academic Press, New York 1985.

[35] O. Lee and D. W. Shin: Geometric ergodicity and moment conditions for a seasonal *GARCH* model with periodic coefficients. Comm. Statist. Theory Methods *39* (2009), 38–51. DOI:10.1080/03610920802715032

[36] W. K. Newey and K. D. West: Hypothesis testing with efficient method of moments estimation. Int. Econom. Rev. *28* (1987), 3, 777–787. DOI:10.2307/2526578

[37] N. Regnard and J. M. Zakoïan: Structure and estimation of a class of nonstationary yet nonexplosive *GARCH* models. J. Time Series Anal. *31* (2010), 348–364. DOI:10.1111/j.1467-9892.2010.00669.x

Billel Aliat, RECITS Laboratory, Faculty of Mathematics, USTHB, Algiers, Algeria and Centre de Recherche en Economie Appliquée pour le Développement (CREAD), Alger. Algeria.
e-mail: aliatbillel@yahoo.fr

Fayçal Hamdi, RECITS Laboratory, Faculty of Mathematics, USTHB, Algiers. Algeria.
e-mail: fhamdi@usthb.dz