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SUFFICIENT CONDITIONS FOR A T-PARTIAL ORDER OBTAINED FROM TRIANGULAR NORMS TO BE A LATTICE

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For a t-norm T on a bounded lattice $(L, \leq)$, a partial order $\leq_T$ was recently defined and studied. In [11], it was pointed out that the binary relation $\leq_T$ is a partial order on L, but $(L, \leq_T)$ may not be a lattice in general. In this paper, several sufficient conditions under which $(L, \leq_T)$ is a lattice are given, as an answer to an open problem posed by the authors of [11]. Furthermore, some examples of t-norms on L such that $(L, \leq_T)$ is a lattice are presented.

Keywords: bounded lattice, triangular norm, T-partial order

Classification: 03E72, 03B52

1. INTRODUCTION

In [7], a partial order generated by a commutative semigroup was introduced by Clifford. There have been numerous attempts to extend this ordering to other semigroups (such as [10, 15]). Especially Mitsch [17] succeeded introducing a natural partial order of semigroups. This order was extended to t-norms on a bounded lattice $(L, \leq, 0, 1)$ by Karaçal and Kesicioğlu in [11], and named T-order. Let $(L, \leq, 0, 1)$ be a bounded lattice, the T-order is defined as follows:

$$x \leq_T y :\Leftrightarrow T(l, y) = x \quad \text{for some} \quad l \in L \tag{1}$$

for any elements x, y of L and T is a t-norm on L. In addition, in [11] it was given the relationship between T-order and partial order "$\leq$" of L:

$$\text{If} \quad x \leq_T y \quad \text{then} \quad x \leq y \tag{2}$$

T-order is a pretty interesting partial order, in resent years, many scholars focus on T-order and other partial orders on $(L, \leq, 0, 1)$. An equivalence relation on the class of t-norms on a bounded lattice was introduced by Kesicioğlu, Karaçal and Mesiar (see [13]) based on T-partial orders, and they also characterized the equivalence classes linked to some special t-norms. Later on, in [3, 9], V and U-partial orders, respectively induced by nullnorms and uninorms were introduced and some basic properties of them

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were investigated. In [16], it was introduced an equivalence relations induced by the U-partial order.

And the authors in [11] pointed out that the binary relation $\leq_T$ is a partial order on L, but $(L, \leq_T)$ may not be a lattice in general. One of the open problems posed in the study [11] is: let $(L, \leq, 0, 1)$ be a bounded lattice and T be a t-norm on L, give examples that $(L, \leq_T)$ is a lattice, where $T \neq T_W$.

If $L = [0, 1]$, in [13] the following example was given to answer this open problem:

Example 1.1. (Kesicioğlu et al. [13]) Consider the function $T^\diamond : [0,1]^2 \to [0,1]$ defined by

$$T^\diamond(x,y) = \begin{cases} 0, & (x,y) \in (0,k)^2; \\ \min(x,y), & \text{otherwise,} \end{cases} \quad 0 < k < 1. \tag{3}$$

Then $([0,1], \leq_{T^\diamond})$ is a lattice.

In [2], the following theorem was given to answer this open problem:

Theorem 1.2. (Aşıcıand Karaçal [2]) Let T be a t-norm on $[0,1]$ and the family $(T_\lambda)_{\lambda \in (0,1)}$ be given by

$$T_\lambda(x,y) = \begin{cases} 0, & T(x,y) \leq \lambda \ \text{ and } \ x, y \neq 1; \\ T(x,y), & \text{otherwise.} \end{cases} \tag{4}$$

Then

(i) $(T_\lambda)_{\lambda \in (0,1)}$ is a t-norm.

(ii) If T is divisible on $[0,1]$, then $(L, \leq_{T_\lambda})$ is complete lattice.

Followed by [2] and [13], in the present paper, we continue to answer this open problem on the condition of L is a complete lattice. The rest of this paper is organized as follows. It reviews fundamental notions and properties of t-norm in Sect.2. In Sect.3, some kinds of t-norms such that $(L, \leq_T)$ is a lattice are given. The paper is concluded with a brief summary and an outlook for further research in Sect.4.

2. PRELIMINARIES

Definition 2.1. (Birkhoff [1], Drygaś [8]) A bounded lattice $(L, \leq, 0, 1)$ is a lattice which has the top and bottom elements, which are written as 1 and 0, respectively, that is, there exist two elements $1, 0 \in L$ such that $0 \leq x \leq 1$, for all $x \in L$.

Definition 2.2. (Birkhoff [1]) Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, $a \leq b$, a subinterval $[a, b]$ of L is defined as

$$[a,b] = \{x \in L, a \leq x \leq b\}$$

Similarly, $[a, b) = \{x \in L, a \leq x < b\}$, $(a, b] = \{x \in L, a < x \leq b\}$ and $(a, b) = \{x \in L, a < x < b\}$.

Definition 2.3. (Çaylı et al. [5], Karaçal [12]) Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $T : L^2 \to L$ is called a triangular norm (t-norm) if it is commutative, associative, increasing with respect to both variables and it satisfies

$$T(x, 1) = x, \ \forall x \in L.$$

Definition 2.4. (Casasnovas and Mayor [4]) A t-norm T on L is divisible if the following condition holds: $\forall x, y \in L$ with $x \leq y$ there is a $z \in L$ such that $x = T(y, z)$.

Example 2.5. The following are two basic t-norms T_M and T_W which are the strongest and the weakest t-norms, respectively, on a bounded lattice L

$$T_M(x, y) = x \wedge y,$$

$$T_W(x, y) = \begin{cases} x \wedge y, & x, y \in \{1\}; \\ 0, & \text{otherwise.} \end{cases}$$

If $L = [0, 1]$, the following are the four basic t-norms T_M, T_P, T_L, T_D given by, respectively:

$$\begin{aligned} T_M(x, y) &= \min\{x, y\}, \\ T_P(x, y) &= xy, \\ T_L(x, y) &= \max\{x + y - 1, 0\}, \\ T_D(x, y) &= \begin{cases} 0, & (x, y) \in [0, 1)^2, \\ \min\{x, y\}, & \text{otherwise.} \end{cases} \end{aligned}$$

Theorem 2.6. (Casasnovas and Mayor [4]) Let $(L, \leq, 0, 1)$ be a bounded lattice, T be a t-norm on L. Then the binary relation $\leq_T$ is a partial order on L.

3. SOME KINDS OF T-NORMS SUCH THAT $(L, \leq_T)$ IS A LATTICE

Let $(L, \leq, 0, 1)$ be a bounded lattice. Consider a t-norm T on L. For $X \subseteq L$, we denote the set of the upper bounds of X and lower bounds of X with respect to "$\leq_T$" on L by $\overline{X}_T$ and $\underline{X}_T$ respectively. We generalize Theorem 1.1 from the unit interval $[0, 1]$ to an arbitrary complete lattice.

Theorem 3.1. Let T be a t-norm on a complete lattice L and the family $(T_a)_{a \in L}$ be given by

$$T_a(x, y) = \begin{cases} 0, & T(x, y) \leq a \text{ and } 1 \notin \{x, y\}; \\ T(x, y), & \text{otherwise.} \end{cases} \tag{5}$$

Then the following statements hold:

(i) $(T_a)_{a \in L}$ is a t-norm.

(ii) If T is divisible on L, then $(L, \leq_{T_a})$ is complete lattice.

Proof. (i) (a) Since T is a t-norm on L, then T is commutative. It leads to T_a is commutative.

(b) We will show that T_a is associative. For any $x, y, z \in L$, if one of x, y, z is 1, then $T_a(T_a(x, y), z) = T_a(x, T_a(y, z))$. For any $x, y, z \in L \setminus \{1\}$.

(b1) Let $T(x, y) \leq a$, then $T_a(T_a(x, y), z) = T_a(0, z) = 0$ and $T(x, T(y, z)) = T(T(x, y), z) \leq T(a, z) \leq T(a, 1) = a$. Therefore

$$T_a(x, T_a(y, z)) = \begin{cases} T_a(x, 0), & T(y, z) \leq a; \\ T_a(x, T(y, z)), & \text{otherwise.} \end{cases} = 0.$$

Then $T_a(T_a(x, y), z) = T_a(x, T_a(y, z))$.

(b2) Let $T(x, y) \not\leq a$. If $T(T(x, y), z) \leq a$, then $T_a(T_a(x, y), z) = T_a(T(x, y), z) = 0$ and $T(x, T(y, z)) = T(T(x, y), z) \leq a$. Therefore

$$T_a(x, T_a(y, z)) = \begin{cases} T_a(x, 0), & T(y, z) \leq a; \\ T_a(x, T(y, z)), & \text{otherwise.} \end{cases} = 0.$$

If $T(T(x, y), z) \not\leq a$, then $T_a(T_a(x, y), z) = T_a(T(x, y), z) = T(T(x, y), z)$ and $T(T(x, y), z) = T(x, T(y, z)) \leq T(1, T(y, z)) = T(y, z)$. Therefore $T_a(x, T_a(y, z)) = T_a(x, T(y, z)) = T(x, T(y, z))$.

Then $T_a(T_a(x, y), z) = T_a(x, T_a(y, z))$.

(c) We show that T_a satisfies the monotonicity. Let $x_1 \leq x_2$, if $T(x_1, y) \leq a$ and $x_1 \neq 1, y \neq 1$ (The case $x_1 = 1$ or $y = 1$ is trivial), then $0 = T_a(x_1, y) \leq T_a(x_2, y)$. If $T(x_1, y) \not\leq a$, from $T(x_1, y) \leq T(x_2, y)$, we have $T(x_2, y) \not\leq a$. Thus $T_a(x_1, y) = T(x_1, y) \leq T(x_2, y) = T_a(x_2, y)$.

(d) Since $T_a(x, 1) = T(x, 1) = x$ for all $x \in L$, we have that 1 is neutral element. Thus, T_a is a t-norm on L.

(ii) Since $0 = T_a(0, x)$ and $x = T_a(x, 1)$, then $0 \leq_{T_a} x \leq_{T_a} 1$ for any $x \in L$. Thus,

$$\bigvee_{T_a}\{x_\tau \mid \tau \in \Phi\} = \bigvee_{T_a}\{x_\tau \mid \tau \in \Phi, x_\tau \neq 0\}$$

$$\bigwedge_{T_a}\{x_\tau \mid \tau \in \Phi\} = \bigwedge_{T_a}\{x_\tau \mid \tau \in \Phi, x_\tau \neq 1\}.$$

Let T be a divisible t-norm on complete lattice L and $\{x_\tau \mid \tau \in \Phi\} \subseteq L \backslash \{0, 1\}$ be arbitrary.

(a) We will show existence of $\bigvee_{T_a}\{x_\tau \mid \tau \in \Phi\}$.

(a1) Suppose that there exists $x_{\tau_0} \in \{x_\tau \mid \tau \in \Phi\}$ such that $x_{\tau_0} \leq a$. Since $T_a(x_{\tau_0}, 1) = T(x_{\tau_0}, 1) = x_{\tau_0}$, then $\{x_{\tau_0}, 1\} \subseteq \overline{\{x_{\tau_0}\}}_{T_a}$. Suppose $k \in \overline{\{x_{\tau_0}\}}_{T_a}$, then there exists $z \in L$ such that $x_{\tau_0} = T_a(z, k)$. Because of $0 \neq x_{\tau_0} \leq a$, it leads to $0 \neq T_a(z, k) = T(z, k) \leq a$. From the definition of T_a, we have $z = 1$ or $k = 1$. If $z = 1$, $x_{\tau_0} = T_a(z, k) = T(1, k) = k$.

Thus $\overline{\{x_{\tau_0}\}}_{T_a} = \{x_{\tau_0}, 1\}$. Therefore $\bigvee_{T_a}\{x_\tau \mid \tau \in \Phi\}$ exists and $\bigvee_{T_a}\{x_\tau \mid \tau \in \Phi\} = 1$.

(a2) Suppose $x_\tau \nleq a$ for all $\tau \in \Phi$. Let $k = \bigvee\{x_\tau \mid \tau \in \Phi\}$, then $x_\tau \leq k$. Since T is a divisible t-norm, then there exist $z_\tau \in L$ such that $x_\tau = T(z_\tau, k)$. Because of $x_\tau \nleq a$ and from the definition of T_a, we have $x_\tau = T(z_\tau, k) = T_a(z_\tau, k)$, therefore $x_\tau \leq_{T_a} k$, i. e. $k \in \overline{\{x_\tau \mid \tau \in \Phi\}}_{T_a}$. Suppose $s \in \overline{\{x_\tau \mid \tau \in \Phi\}}_{T_a}$ and $s \neq k$, then $x_\tau \leq_{T_a} s$, it leads to $x_\tau \leq s$. Therefore $k = \bigvee\{x_\tau \mid \tau \in \Phi\} \leq s$. Because of $x_\tau \nleq a$ for all $\tau \in \Phi$ and T is a divisible t-norm on complete lattice L, it leads to $k \nleq a$ and there exists $z \in L$ such that $k = T(z, s)$, therefore $k \leq_{T_a} s$. Thus $\bigvee_{T_a}\{x_\tau \mid \tau \in \Phi\} = \bigvee\{x_\tau \mid \tau \in \Phi\}$.

(b) We will show existence of $\bigwedge_{T_a}\{x_\tau \mid \tau \in \Phi\}$.

(b1) Suppose that there exists $x_{\tau_0} \in \{x_\tau \mid \tau \in \Phi\}$ such that $x_{\tau_0} \leq a$. Since $T_a(x_{\tau_0}, 0) = 0$ and $T_a(x_{\tau_0}, 1) = T(x_{\tau_0}, 1) = x_{\tau_0}$, then $\{0, x_{\tau_0}\} \subseteq \underline{\{x_{\tau_0}\}}_{T_a}$. Suppose $k \in \underline{\{x_{\tau_0}\}}_{T_a}$, then there exists $z \in L$ such that $k = T_a(z, x_{\tau_0})$. If $k \neq 0$, from the definition of T_a we have $T_a(z, x_{\tau_0}) = T(z, x_{\tau_0})$. Combing with $T(z, x_{\tau_0}) \leq T(1, x_{\tau_0}) = x_{\tau_0} \leq a$, we have $z = 1$. Therefore $k = T_a(z, x_{\tau_0}) = T(1, x_{\tau_0}) = x_{\tau_0}$. Thus, $\underline{\{x_{\tau_0}\}}_{T_a} = \{0, x_{\tau_0}\}$. Therefore $\bigwedge_{T_a}\{x_\tau \mid \tau \in \Phi\}$ exists and $\bigwedge_{T_a}\{x_\tau \mid \tau \in \Phi\} = 0$.

(b2) Suppose $x_\tau \nleq a$ for all $\tau \in \Phi$. Let $k = \bigwedge\{x_\tau \mid \tau \in \Phi\}$, then $k \nless a$, i. e. $k \nleq a$ or $k = a$.

(b21) In the case of $k \nleq a$. Since $k = \bigwedge\{x_\tau \mid \tau \in \Phi\}$, then $k \leq x_\tau$, therefore $k = T(l_\tau, x_\tau)$ for some $l_\tau \in L$. Because of $k \nleq a$, then $k \nleq 0$. Therefore $k = T(l_\tau, x_\tau) = T_a(l_\tau, x_\tau)$ for some $l_\tau \in L$, i. e.$k \leq_{T_a} x_\tau$. Thus $k \in \underline{\{x_\tau \mid \tau \in \Phi\}}_{T_a}$. Suppose $0 \neq s \in \underline{\{x_\tau \mid \tau \in \Phi\}}_{T_a}$, then $s \leq_{T_a} x_\tau$, therefore $s \leq x_\tau$, i. e. $s \leq \bigwedge\{x_\tau \mid \tau \in \Phi\} = k$. Thus, $s = T(l, k) = T_a(l, k)$ for some $l \in L$. Therefore $s \leq_{T_a} k$. That is to say $\bigwedge_{T_a}\{x_\tau \mid \tau \in \Phi\} = \bigwedge\{x_\tau \mid \tau \in \Phi\}$.

(b22) In the case of $k = a$. Obviously $0 \in \underline{\{x_\tau \mid \tau \in \Phi\}}_{T_a}$. Suppose $0 \neq s \in \underline{\{x_\tau \mid \tau \in \Phi\}}_{T_a}$, then $s \leq_{T_a} x_\tau$. Thus $s \leq x_\tau$, i. e. $s \leq \bigwedge\{x_\tau \mid \tau \in \Phi\} = a$ on the one hand. On the other hand, $0 \neq s \leq x_\tau$ implies $s = T(l, x_\tau) = T_a(l, x_\tau)$ for some $l \in L$. From the definition of T_a, we have $s = T(l, x_\tau) \nleq a$, which is a contradiction. Thus $0 = \bigwedge_{T_a}\{x_\tau \mid \tau \in \Phi\}$. □

Example 3.2. Consider the lattice $(L = \{0, a, b, c, d, 1\}, \leq, 0, 1)$ given in Figure 1, and the function T_b on L defined by

$$T_b(x, y) = \begin{cases} 0, & x \wedge y \leq b \text{ and } 1 \notin \{x, y\}; \\ x \wedge y, & \text{otherwise.} \end{cases}$$

then T_b is a t-norm and T_b can also been described in Table 1. The order $\leq_{T_b}$ on L is given in Figure 2.

T_b	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	0	b
c	0	0	0	c	0	c
d	0	0	0	0	d	d
1	0	a	b	c	d	1

Tab. 1. T-norm T_b.

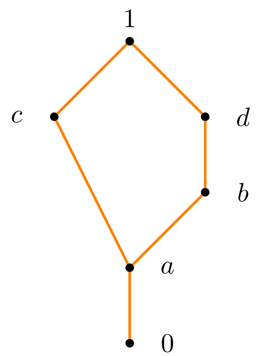

Fig 1. The order $\leq$ on L.

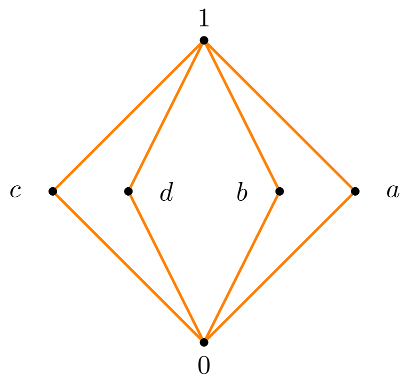

Fig. 2. The order $\leq_{T_b}$ on L.

Suppose $H = (0, k) \subseteq [0, 1)$, Let $* : H^2 \to H$ be an operation on H which is commutative, associative, increasing with respect to both variables and

$$x * y \leq \min\{x, y\}$$

the function $T : [0, 1]^2 \to [0, 1]$ is defined by

$$T(x,y) = \begin{cases} x * y, & (x,y) \in H^2; \\ \min\{x,y\} & \text{otherwise.} \end{cases} \quad . \tag{6}$$

Then, T is a t-norm (Proposition 3.60 in [14]).

If $x * y = 0$, then $T(x, y) = T^{\diamond}(x, y)$ (Example 1.1 of this paper or Example 7 in [13]), and authors in [13] proved that $([0, 1], \leq_{T^{\diamond}})$ is a lattice, and

$$x \vee_{T^{\diamond}} y = \begin{cases} k, & (x,y) \in H^2; \\ \max\{x,y\}, & \text{otherwise.} \end{cases} \quad ,$$

$$x \wedge_{T^{\diamond}} y = \begin{cases} 0, & (x,y) \in H^2; \\ \min\{x,y\}, & \text{otherwise.} \end{cases} \quad .$$

Example 3.3. Let $H = (0, k) \subseteq [0, 1)$, consider the t-norm on $[0, 1]$ defined as follows:

$$T(x,y) = \begin{cases} \max\{x+y-k, 0\}, & (x,y) \in H^2; \\ \min\{x,y\}, & \text{otherwise}; \end{cases}$$

then $([0, 1], \leq_T)$ is lattice, and for all $x, y \in [0, 1]$,

$$\begin{aligned} x \vee_T y &= \max\{x, y\}, \\ x \wedge_T y &= \min\{x, y\}. \end{aligned}$$

Suppose $x, y \in (0, 1)$ and $x < y$.

(a) We will show existence of $x \vee_T y$.

(a1) If $y \geq k$. Since $x = \min\{x, y\} = T(x, y)$, then $x \leq_T y$, i. e. $x \vee_T y = y$.

(a2) If $y < k$. Let $z = x + k - y < k$, then $x = \max\{y + z - k, 0\} = T(z, y)$. Therefore, $x \leq_T y$, i. e. $x \vee_T y = y$.

(b) We shall show existence of $x \wedge_T y$.

(b1) If $y \geq k$. Since $x = \min\{x, y\} = T(x, y)$, then $x \leq_T y$, i. e. $x \wedge_T y = x$.

(b2) If $y < k$. Let $z = x + k - y < k$, then $x = x = \max\{y + z - k, 0\} = T(z, y)$. Therefore, $x \leq_T y$, i. e. $x \wedge_T y = x$.

In general, $([0, 1], \leq_T)$ is not a lattice, which illustrated by the following example.

Example 3.4. Let $H = (0, \frac{1}{2})$, consider the t-norm on $[0, 1]$ defined as follows:

$$T(x,y) = \begin{cases} xy, & (x,y) \in H^2, \\ \min\{x,y\}, & \text{otherwise.} \end{cases}$$

For any $x \in (0, 1)$, since $x = \min\{x, 1\} = T(x, 1)$, then $x \in \overline{\{x\}}_T$ and $x \in \underline{\{x\}}_T$.

(a) Let $y \in \overline{\{x\}}_T$, i. e. $x \leq_T y$, then $x \leq y$, thus $\overline{\{x\}}_T \subseteq [x, 1]$.

(a1) Suppose $x \geq \frac{1}{2}$. $x \leq y$ implies $x = \min\{x, y\} = T(x, y)$ by definition of T, then $x \leq_T y$, therefore, $[x, 1] \subseteq \overline{\{x\}}_T$, thus $\overline{\{x\}}_T = [x, 1]$;

(a2) Suppose $\frac{1}{4} \leq x < \frac{1}{2}$. If $y \geq \frac{1}{2} > x$, then $x = \min\{x, y\} = T(x, y)$, we have $x \leq_T y$, therefore $[\frac{1}{2}, 1] \subseteq \overline{\{x\}}_T$. If $x < y < \frac{1}{2}$ and $y \in \overline{\{x\}}_T$, then there exists $0 \leq l < \frac{1}{2}$ such that $x = T(l, y) = ly$. $(l, y) \in H^2$ implies $ly < \frac{1}{4}$, it is contradict with $\frac{1}{4} \leq x = T(l, y) = ly < \frac{1}{2}$. Therefore, $\overline{\{x\}}_T = \{x\} \cup [\frac{1}{2}, 1]$;

(a3) Suppose $x < \frac{1}{4}$. If $y \geq \frac{1}{2}$, then $x = \min\{x, y\} = T(x, y)$, we have $x \leq_T y$, therefore, $[\frac{1}{2}, 1] \subseteq \overline{\{x\}}_T$. If $2x < y < \frac{1}{2}$, then there exists $0 \leq l < \frac{1}{2}$ such that $x = ly = T(l, y)$. $0 \leq l < \frac{1}{2}$ implies $x = ly < \frac{1}{2}y$, i. e., $2x < y$, therefore, $(2x, \frac{1}{2}) \subseteq \overline{\{x\}}_T$. If $x < y \leq 2x < \frac{1}{2}$, then there is no exists l such that $x = T(l, y)$. Therefore, $\overline{\{x\}}_T = \{x\} \cup (2x, 1]$.

Then we have:

$$\overline{\{x\}}_T = \begin{cases} [x,1], & \frac{1}{2} \le x \\ \{x\} \cup [\frac{1}{2},1], & \frac{1}{4} \le x < \frac{1}{2} \\ \{x\} \cup (2x,1], & x < \frac{1}{4} \end{cases}$$

(b) Let $z \in \underline{\{x\}}_T$, i. e. $z \le_T x$, then $z \le x$, thus $\underline{\{x\}}_T \subseteq [0,x]$.

(b1) Suppose $x \ge \frac{1}{2}$. $z \le x$ implies $z = \min\{z,x\} = T(z,x)$ by definition of T, then $z \le_T x$, therefore, $[0,x] \subseteq \underline{\{x\}}_T$, thus $\underline{\{x\}}_T = [0,x]$.

(b2) Suppose $x < \frac{1}{2}$. If $\frac{1}{2}x \le z < x$, then there is no exists l such that $z = T(l,x)$. If $z < \frac{1}{2}x$, then there exists $0 \le l < \frac{1}{2}$ such that $z = lx = T(l,x)$, therefore, $[0, \frac{1}{2}x) \subseteq \underline{\{x\}}_T$. Thus, $\underline{\{x\}}_T = \{x\} \cup [0, \frac{1}{2}x)$.

Then we have:

$$\underline{\{x\}}_T = \begin{cases} [0,x], & \frac{1}{2} \le x \\ \{x\} \cup [0, \frac{1}{2}x), & x < \frac{1}{2} \end{cases}.$$

Taking $x = \frac{1}{8}$ and $y = \frac{1}{6}$, $\frac{1}{8} \vee_T \frac{1}{6}$ and $\frac{1}{8} \wedge_T \frac{1}{6}$, however, does not exist, since $\overline{\{\frac{1}{8}\}}_T = \{\frac{1}{8}\} \cup (\frac{1}{4}, 1]$ and $\overline{\{\frac{1}{6}\}}_T = \{\frac{1}{6}\} \cup (\frac{1}{3}, 1]$, $\underline{\{\frac{1}{8}\}}_T = \{\frac{1}{8}\} \cup [0, \frac{1}{16})$ and $\underline{\{\frac{1}{6}\}}_T = \{\frac{1}{6}\} \cup [0, \frac{1}{12})$. Follows from Example 3.3, we have that $([0,1], \le_T)$ is neither a join-semilattice nor a meet-semilattice.

Remark 3.5. Example 1.1 can not be generalized from the unit interval [0,1] to arbitrary complete lattice. For arbitrary bounded lattice $(L, \le, 0, 1)$, the function T defined by the formula (3) in Example 1.1 needs not generate a t-norm on L. For example, consider the lattice $(L = \{0, a, b, c, d, e, 1\}, \le, 0, 1)$ given in Figure 3. $H = (0, e)$, the function T be given by

$$T(x,y) = \begin{cases} 0, & (x,y) \in H^2, \\ x \wedge y, & \text{otherwise.} \end{cases}$$

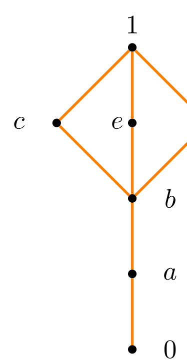

Fig. 3. The order $\le$ on L.

Then $T(T(a,c),d) = T(a,d) = a$ and $T(a,T(c,d)) = T(a,b) = 0$. Hence, T is not a t-norm on L depicted in Figure 3, since the associativity is violated.

Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L \setminus \{0,1\}$. Çaylı in [6] gave a new t-norm $T_V : L^2 \to L$ on L, where V is a t-norm on $[a,1]$, and

$$T_V(x,y) = \begin{cases} V(x,y), & (x,y) \in [a,1)^2; \\ x \wedge y, & 1 \in \{x,y\}; \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Theorem 3.6. If L is a complete lattice, $a \in L \setminus \{0,1\}$ and V is a divisible t-norm on $[a,1]$, then $(L, \leq_{T_V})$ is a complete lattice.

P r o o f. Let V be divisible on $[a,1]$, and $\{x_\tau \mid \tau \in \Phi\}$ be an arbitrary subset of $L \setminus \{0,1\}$.

(a) We will show existence of $\bigvee_{T_V} \{x_\tau \mid \tau \in \Phi\}$.

(a1) Suppose that there exists $x_{\tau_0} \in \{x_\tau \mid \tau \in \Phi\}$ such that $x_{\tau_0} \notin [a,1)$. Since $T_V(x_{\tau_0}, 1) = x_{\tau_0} \wedge 1 = x_{\tau_0}$, then $\{x_{\tau_0}, 1\} \subseteq \overline{\{x_{\tau_0}\}}_{T_V}$. Assume $k \in \overline{\{x_{\tau_0}\}}_{T_V}$, then there exists $z \in L$ such that $x_{\tau_0} = T_V(z,k)$. Because of $x_{\tau_0} \neq 0$, it leads to $T_V(z,k) = \begin{cases} V(z,k), & (z,k) \in [a,1)^2; \\ z \wedge k, & 1 \in \{z,k\}. \end{cases}$. If $(z,k) \in [a,1)^2$, since V is a t-norm on $[a,1]$, then $x_{\tau_0} = T_V(z,k) = V(z,k) \in [a,1)$, which contradicts with $x_{\tau_0} \notin [a,1)$, and thus $z = 1$ or $k = 1$. If $z = 1$, $x_{\tau_0} = T_V(z,k) = 1 \wedge k = k$. Thus $\overline{\{x_{\tau_0}\}}_{T_V} = \{x_{\tau_0}, 1\}$. Therefore $\bigvee_{T_V} \{x_\tau \mid \tau \in \Phi\}$ exists and $\bigvee_{T_V} \{x_\tau \mid \tau \in \Phi\} = 1$.

(a2) Suppose $x_\tau \in [a,1)$ for all $\tau \in \Phi$. Let $k = \bigvee \{x_\tau \mid \tau \in \Phi\}$, then $x_\tau \leq k$. Since V is a divisible t-norm on $[a,1]$, then there exist $z_\tau \in [a,1]$ such that $x_\tau = V(z_\tau, k) = T_V(z_\tau, k)$. Therefore $x_\tau \leq_{T_V} k$, i.e. $k \in \overline{\{x_\tau \mid \tau \in \Phi\}}_{T_V}$. Suppose $s \in \overline{\{x_\tau \mid \tau \in \Phi\}}_{T_V}$, then $x_\tau \leq_{T_V} s$, it leads to $x_\tau \leq s$. Therefore $k \leq s$. Because of $x_\tau \in [a,1)$ for all $\tau \in \Phi$ and V is a divisible t-norm on complete lattice L, it leads to $k \in [a,1]$ and there exists $z \in L$ such that $k = T_V(z,s)$, therefore $k \leq_{T_V} s$. Thus $\bigvee_{T_V} \{x_\tau \mid \tau \in \Phi\} = \bigvee \{x_\tau \mid \tau \in \Phi\}$.

(b) We will show existence of $\bigwedge_{T_V} \{x_\tau \mid \tau \in \Phi\}$.

(b1) Suppose that there exists $x_{\tau_0} \in \{x_\tau \mid \tau \in \Phi\}$ such that $x_{\tau_0} \notin [a,1)$. Since $T_V(x_\tau, 0) = 0$ and $T_V(x_\tau, 1) = x_\tau$, we have $\{0, x_\tau\} \subseteq \underline{\{x_\tau\}}_{T_V}$. Suppose $k \in \underline{\{x_{\tau_0}\}}_{T_V}$, then there exists $z \in L$ such that $k = T_V(z, x_{\tau_0})$. Since $x_{\tau_0} \notin [a,1)$, then $k = T_V(z, x_{\tau_0}) = \begin{cases} x_{\tau_0}, & z = 1; \\ 0, & z < 1. \end{cases}$. Thus, $\underline{\{x_{\tau_0}\}}_{T_V} = \{0, x_{\tau_0}\}$. Therefore $\bigwedge_{T_V} \{x_\tau \mid \tau \in \Phi\}$ exists and $\bigwedge_{T_V} \{x_\tau \mid \tau \in \Phi\} = 0$.

(b2) Suppose $x_\tau \in [a,1)$ for all $\tau \in \Phi$. Let $k = \bigwedge \{x_\tau \mid \tau \in \Phi\}$, then $a \leq k \leq x_\tau$. Since V is a divisible t-norm on $[a,1]$, we obtain that $\bigwedge_{T_V} \{x_\tau \mid \tau \in \Phi\} = \bigwedge \{x_\tau \mid \tau \in \Phi\}$.

□

Example 3.7. Consider the lattice $(L = \{0,a,b,c,d,1\}, \leq, 0, 1)$ given in Figure 1. The function T_{V1} on L is defined by

$$T_{V1}(x,y) = \begin{cases} x \wedge y, & (x,y) \in [b,1)^2 \text{ or } 1 \in \{x,y\}; \\ 0, & \text{otherwise.} \end{cases}$$

T_{V1}	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	b	0	b	b
c	0	0	0	0	0	c
d	0	0	b	0	d	d
1	0	a	b	c	d	1

Tab. 2. T-norm T_{V1}.

then T_{V1} is a t-norm and T_{V1} can also be described in Table 2. The order $\leq_{T_{V1}}$ on L has its diagram as given in Figure 4.

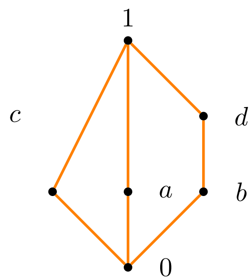

Fig. 4. The order $\leq_{T_{V1}}$ and $\leq_{T_{V2}}$ on L.

The following example shows that converse of Theorem 3.2 is not true in general.

Example 3.8. Consider the lattice $(L = \{0, a, b, c, d, 1\}, \leq, 0, 1)$ given in Figure 1. Taking t-norm V on $[b, 1]$ as

$$V(x, y) = \begin{cases} b, & (x, y) \in [b, 1)^2; \\ x \wedge y, & \text{otherwise.} \end{cases}.$$

Consider the function T_{V2} on L defined by

$$T_{V2}(x, y) = \begin{cases} b, & (x, y) \in [b, 1)^2; \\ x \wedge y, & 1 \in \{x, y\}; \\ 0, & \text{otherwise.} \end{cases}$$

T_{V2} described in Table 3 is a t-norm. The order $\leq_{T_{V2}}$ on L is given in Figure 4. Hence $(L, \leq_{T_{V2}})$ is a complete lattice, but $V(x, y)$ is not a divisible t-norm on $[b, 1]$.

Example 3.9. Consider the t-norm on $[0, 1]$ defined as follows:

$$T_{V3}(x, y) = \begin{cases} \min\{x, y\}, & (x, y) \in [\frac{1}{2}, 1)^2; \\ \min\{x, y\}, & 1 \in \{x, y\}; \\ 0, & \text{otherwise.} \end{cases}$$

T_{V2}	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	b	0	b	b
c	0	0	0	0	0	c
d	0	0	b	0	b	d
1	0	a	b	c	d	1

Tab. 3. T-norm T_{V2}.

Since $V(x,y) = \min\{x,y\}$ is a divisible t-norm on $[\frac{1}{2},1)$, then $([0,1], \leq_{T_{V3}})$ is lattice. And

$$x \vee_{T_{V3}} y = \begin{cases} 1, & (x,y) \notin [\frac{1}{2},1)^2; \\ \max\{x,y\}, & (x,y) \in [\frac{1}{2},1)^2. \end{cases},$$

$$x \wedge_{T_{V3}} y = \begin{cases} 0, & (x,y) \notin [\frac{1}{2},1)^2; \\ \min\{x,y\}, & (x,y) \in [\frac{1}{2},1)^2. \end{cases}.$$

4. CONCLUSION

The objective of this paper is to give some sufficient conditions for a T-partial order obtained from triangular norms to be a lattice. Sufficient conditions for other partial order (for example U-partial order and V-partial order) to be a lattice will be considered in future work.

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