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DYNAMIC MODEL OF MARKET WITH UNINFORMED MARKET MAKER

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We model a market with multiple liquidity takers and a single market maker maximizing his discounted consumption while keeping a prescribed probability of bankruptcy. We show that, given this setting, spread and price bias (a difference between the midpoint- and the expected fair price) depend solely on the MM's inventory and his uncertainty concerning the fair price. Tested on ten-second data from ten US electronic markets, our model gives significant results with the price bias decreasing in the inventory and increasing in the uncertainty and with the spread mostly increasing in the uncertainty.

Keywords: market maker, optimal decision, price and inventory, high frequency data, dynamic model

Classification: 91G80, 62P05

1. INTRODUCTION

One of the greatest challenges of mathematical finance is the problem of optimal trading on a continuous-time limit order market. In all its generality, decision problems faced by the traders exhibit infinitely dimensional state spaces (it is possible to put arbitrary numbers of limit orders), so they are intractable. Therefore, researches resort to simplifications; they either use simplified settings (e. g., one-period models or models with restricted strategy space, see e. g., [4] or [18]) or – in extreme – they do not assume any rationality at all, constructing so called zero intelligence models (see e. g. [5] or [22]). In the present paper, we go the former route: we assume that there is only one (possibly representative) rational agent – a market maker[1] – trading against many irrational and many averagely rational liquidity traders by posting only two limit orders at a time.

There is a large number of papers studying behaviour of market makers by means of tractable simplified rational models. One of the earliest paper of that kind was [8], followed by many others, e. g., [17, 25], or [9]. For more references, see surveys [4] or [26]. It follows from those works that functioning of a market maker (MM) is associated with two main risks, for which the MM demands compensations in the form of spread: the risk

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[1] In fact, our model can be applied to *any* agent posting at most two limit orders of different type – even though our MM is *obliged* to put exactly two limit orders, posting only zero or one order can be emulated by setting quotes with unrealistic prices.

of running out of the traded instrument (the inventory risk) and the risk associated with the uncertainty about the "true" value of the instrument (the adverse selection risk).[2] There are many empirical studies confirming the following implications of these theoretical works:

(M1) market makers try to sell (buy) extra (missing) inventory,

(M2) large trades have a permanent impact to price (speaking for the presence of adverse selection)

(M3) spreads widen at the times of uncertainty.

For details, see [4], especially Table 1 therein.

In addition to these classical papers, there are several works dealing with truly dynamical decision problems, e. g. [2, 7, 10, 12], all being similar to (or directly based on) the seminal work [11]. In this paper, the MM maximizes his expected utility from the terminal wealth by continuously setting his quotes under the assumption of decreasing (increasing) demand (supply) of the liquidity traders. It is demonstrated that the resulting optimal quotes depend solely on time, the amount of cash held by the MM, and his inventory. Interestingly, the spread itself does not depend on the inventory.

In the present paper, we build a (discrete time) dynamical model in which we let the MM maximize the discounted running consumption plus the terminal wealth. Even if we in fact assume a linear utility function, our MM cannot be viewed as risk neutral as he keeps the probability of his bankruptcy at a prescribed level. In line with [11], we assume that the market orders' arrival intensities increase linearly with distances of the quotes from the "fair" (log)price; contrary to [11], however, our fair price is random rather than constant, following a normal random walk. Similarly to [7], our MM tries to avoid running out of the cash or the instrument traded; contrary to this paper, however, our constraints are probabilistic.

There are four main original contributions of our paper: Firstly, the value of the fair price might not be known exactly to the MM in our setting, which makes the adverse selection implicit to our model. Secondly, the time horizon may be infinite, so we may avoid problematic final valuation of the inventory. Thirdly, we (approximately) describe a joint distribution of the quotes - hence of the market price - and the MM's inventory. Finally, we show how the price increments and their conditional volatility may be decomposed into three components: the first one "caused" by the fair price, the second one by the inventory and the third one by the uncertainty.

We proceed as follows: after the definitions (Section 2), we derive an approximate joint conditional distribution of the fair price and the inventory given the MM's quotes (Section 3). Further, our main result is formulated, saying that the MM's quotes depend solely on the expected fair price, the inventory level and the MM's uncertainty (Section 4). Further, we discuss some implications of our results, namely the price decomposition resulting from the model and the joint dynamics of the midpoint price, spread and the inventory (Section 5). Consequently we estimate the parameters of the resulting

[2] [26], in addition, mentions an "option" risk stemming from the fact that, by putting a limit order, the MM in fact underwrites an option with the strike price equal to the limit price; this risk, however, may be minimized by frequent adjustments of the quotations.

equations in the presence of noise limit orders using 10-second high frequency data of three US stocks from ten US electronic markets, and we demonstrate that results at least do not contradict our model and that they mostly confirm stylized facts (M1) and (M3) (Section 6). In addition, we show that three simple benchmark models assuming irrationality of the MM's and/or of the liquidity takers may be rejected in favour of our model (Section 7). Finally, we conclude the paper (Section 8). Two longer proofs, a discussion of consistency and asymptotic normality of our estimates, and the detailed results of the estimation are presented in the Appendix.

2. THE SETTING

In the present Section, we formally define our model and do some approximations necessary for the model to be at least partially tractable.

2.1. The agents

We assume that there is a (single representative) market maker, continuum of irrational liquidity traders and continuum of averagely rational (informed) liquidity traders.

At each $t \in \mathbb{N}$,[3] the market maker sets the log-quotes a_t and b_t (the actual best ask and best bid, i. e. the prices for which the liquidity traders can buy, sell, respectively, are then $A_t = e^{a_t}$, $B_t = e^{b_t}$, respectively) in order to maximize his overall discounted consumption and terminal wealth (see Section 2.2).

In reaction to the quotes, averagely rational liquidity traders post buy and sell market orders, i. e. requests to buy or to sell certain amount of the instrument for the ask price, bid price, respectively. The numbers of those buy- and sell-market orders having arrived between times $t-1$ to t, are Poisson distributed with intensities

$$\lambda(a_{t-1} - \pi_{t-1}), \qquad \lambda(\pi_{t-1} - b_{t-1}),$$

respectively, where

$$\lambda(z) = \begin{cases} r\left(1 - \frac{z}{D}\right) & z \le D \\ 0 & z > D \end{cases} \qquad \text{for some } r > 0 \text{ and } D > 0$$

and $\pi_t \in \mathbb{R}$ is a log-fair price.

The numbers of irrational liquidity traders' market orders, on the other hand, follow a Poisson distribution with constant intensity κ for both buy and sell orders.

The sizes of the market orders are random, with a common distribution $\mathcal{D}$ having mean μ and the non-central second moment s (i. e. the variance of $\mathcal{D}$ is $s - \mu^2$), dependent neither on each other nor on the number of the orders arrived.

The log-fair price π follows a normal random walk with $\mathbb{E}\Delta\pi_t = 0$. We distinguish three possible degrees of information, available to the MM:

(I) The MM is fully informed, i. e., the values of π are observable to him.

[3]Different trading frequencies may be modelled by scaling of the time.

(P) The MM is partially informed, i. e., he observes a proxy

$$e_t = \pi_t + \gamma_t$$

instead of π_t at each $t \geq 0$, where γ_t is normal with $\mathbb{E}(\gamma_t) = 0$.

(U) The MM is uninformed.

As cases (I) and (U) may be approximated by (P) with $\mathrm{var}(\gamma_t)$ very small, very large, respectively, we shall assume only (P) in the rest of the paper.

Denoting X_t and Y_t the total volume of the buy market orders, sell market orders, respectively, arriving from $t-1$ to t, and denoting

$$\Xi_\tau = (\pi_0, e_0, X_1, Y_1, e_1, \pi_1, \ldots, X_\tau, Y_\tau, e_\tau, \pi_\tau), \qquad \tau \in \mathbb{N}_0, \tag{1}$$

all the (historical) information relevant for the market, our setting may be formally defined as follows: For each $t > 0$,

(D1) $X_t|\Xi_{t-1} \sim \mathrm{CP}\left(\kappa + \lambda(a_{t-1} - \pi_{t-1}), \mathcal{D}\right)$,

(D2) $Y_t|\Xi_{t-1} \sim \mathrm{CP}\left(\kappa + \lambda(\pi_{t-1} - b_{t-1}), \mathcal{D}\right)$,

(D3) $\Delta\pi_t \sim \mathcal{N}(0, v_{\Delta\pi})$, for some constant $v_{\Delta\pi}$,

(D4) $\gamma_t \sim \mathcal{N}(0, v_\gamma)$ for some constant v_γ,

(I1) $\Delta\pi_t$, γ_t and (Ξ_{t-1}, X_t, Y_t) are mutually independent,

(I2) Ξ_{t-1}, X_t and Y_t are mutually conditionally independent given $(a_{t-1} - \pi_{t-1}, \pi_{t-1} - b_{t-1})$.

Here, $\mathrm{CP}(\iota, \mathcal{Q})$ denotes the compound Poisson distribution with intensity ι and summands' distribution $\mathcal{Q}$.

Remark 2.1. If $a_{t-1} = b_{t-1} = p$ for some p (i. e. if the market price were p), then the expected[4] total volume of the buy market orders, sell market orders, respectively, between $t-1$ and t would be equal to $\bar{D}_t(p) := \mu[\kappa+\lambda(p-\pi_{t-1})]$, $\bar{S}_t(p) := \mu[\kappa+\lambda(\pi_{t-1}-p)]$, respectively. Thus, we may interpret functions $\bar{D}_t$ and $\bar{S}_t$ as the (expected) demand curve, supply curve, respectively. Moreover, since $\pi_{t-1} = \arg\max_p[\bar{D}_t(p) \wedge \bar{S}_t(p)]$ we may regard π_{t-1} as the equilibrium price. Further, if the market price was π_{t-1}, then the expected overall traded volume between $t-1$ and t would be $\bar{D}_t(\pi_{t-1}) = \bar{S}_t(\pi_{t-1}) = \mu(\kappa + r)$ (note that this amount does not depend on π_t).

[4]The expectation is a conditional one, given $(\pi_{t-1}, a_{t-1}, b_{t-1})$

2.2. The MM's decision problem

Let us turn our attention to the decision problem. Assume that MM holds M_0 units of cash and N_0 units of the traded instrument at the time 0. Given that he sets quotes to a_t and b_t at each time $t = 0, 1, \dots$ and consumes C_t at each time $t = 1, 2, \dots,$ the increments of his cash holding, instrument holding, respectively, are

$$\Delta M_t = \Delta m_t - C_t, \qquad \Delta m_t = e^{a_{t-1}} X_t - e^{b_{t-1}} Y_t, \qquad t > 0, \tag{2}$$

$$\Delta N_t = Y_t - X_t, \qquad t > 0. \tag{3}$$

We assume that the consumption C_t may be also negative, i.e. it is allowed to MM to "put money into the business" if needed. Moreover, we allow the MM to borrow stocks for a single period.

As the MM observes values of the fair price only through the proxy, his information set at the time t consists of

$$\xi_t = (e_0, X_1, Y_1, e_1, X_2, Y_2, \dots X_t, Y_t, e_t)$$

(recall that $e_t = \pi_t$ in case of (I)).

As it was premised, we assume the MM to maximize his discounted consumption at time t plus his discounted wealth at a time horizon so that both the probability of running out of the money (i.e. of $M_{t+1} < 0$) and the probability of depleting the instruments (i.e. of $N_{t+1} < 0$) at the next step are less than a prescribed level.

Definition 2.2. The decision problem, solved by the MM at each $t \in \mathbb{N} \cup \{0\}$, is given by

$$\begin{aligned} V_t(\xi_t) \quad &= \sup_{a_\tau, b_\tau, C_\tau \in \mathbb{R}, t \le \tau < T} \quad \mathbb{E}\left[\sum_{\tau=t}^{T-1} e^{-\rho(\tau-t)} C_\tau + e^{-\rho(T-t)}(M_T + e^{\pi_T} N_T) | \xi_t\right] \qquad (4) \\ & \text{subject to} \\ & \mathcal{E}(\tau) \qquad (a_\tau, b_\tau, C_\tau) \text{ is } \sigma(\xi_\tau)\text{measurable,} \\ & \mathcal{A}(\tau) \qquad a_\tau \ge b_\tau \\ & \mathcal{M}(\tau) \qquad \mathbb{P}\left[M_{\tau+1} < 0 | \xi_\tau\right] \le \gamma, \\ & \mathcal{N}(\tau) \qquad \mathbb{P}\left[N_{\tau+1} < 0 | \xi_\tau\right] \le \gamma, \\ & t \le \tau < T. \end{aligned}$$

Here, T is a time horizon fulfilling $t \le T \le \infty$, ρ is a discount factor and γ is a pre-chosen probability level.

2.3. Approximation

For any τ, denote

$$h_\tau = \mathbb{E}(\pi_\tau | \xi_\tau),$$

the (conditionally) expected log-fair price. Our next aim is to describe the dynamics of h_τ and the distribution of the fair price observation error

$$\eta_\tau = h_\tau - \pi_\tau.$$

Denote

$$v_{\eta,\tau} = \operatorname{var}(\eta_\tau|\xi_\tau)$$

its conditional variance.

Because, except of an (unrealistic) case (I), the computation of h_τ and a distribution of η_τ is non-trivial, we have to approximate our model. To this end, note that, once r and κ are high enough, we may approximate the Compound Poisson conditional distribution of X_t given Ξ_{t-1} by

$$\mathcal{N}(\mu[\kappa + \lambda(a_{t-1} - \pi_{t-1})], s[\kappa + \lambda(a_{t-1} - \pi_{t-1})])$$

(see [3], Sec. 3.9). Further, if

$$a_{t-1} - h_{t-1} \ll D$$

and

$$\sqrt{v_{\eta,t-1}} \ll D$$

then, by Chebyshev inequality, the conditional probability of event

$$[a_{t-1} - \pi_{t-1} > D] = [a_{t-1} - h_{t-1} + \eta_{t-1} > D]$$

is small, so we may assume

$$\lambda(a_{t-1} - \pi_{t-1}) = r\left(1 - \frac{a_{t-1} - \pi_{t-1}}{D}\right),$$

(i. e., that λ is linear in b_{t-1}). Both this and an analogous approximation of $\lambda(\bullet - b_{t-1})$ may be formally expressed by keeping *(D3)*, *(D4)*, *(I1)* and *(I2)* and assuming

(A1) $X_t|\Xi_{t-1} \sim \mathcal{N}\left(\mu\kappa + \mu r\left(1 - \frac{a_{t-1}-\pi_{t-1}}{D}\right), s\kappa + sr\left[\left(1 - \frac{a_{t-1}-\pi_{t-1}}{D}\right) \vee 0\right]\right)$,

(A2) $Y_t|\Xi_{t-1}, \sim \mathcal{N}\left(\mu\kappa + \mu r\left(1 - \frac{\pi_{t-1}-b_{t-1}}{D}\right), s\kappa + sr\left[\left(1 - \frac{\pi_{t-1}-b_{t-1}}{D}\right) \vee 0\right]\right)$.

Unfortunately, even given this approximation, we would not get analytical formulas for the conditional distribution of η_t given ξ_t which we need to describe the dynamics of the price-volume process (the reason being dependence of conditional variances on both X_t and Y_t on π_{t-1}). One way to overcome this would be to approximate the conditional density; however, since the formulas resulting from that approach would still be quite complex, we rather assume that, instead of both the volumes X_t and Y_t, the MM takes into account only their difference (the increase of the inventory)

$$\Delta N_t = Y_t - X_t,$$

whose conditional variance does not depend on π_{t-1} given that $\kappa + r[1 - \frac{a_{t-1}-\pi_{t-1}}{D}] > 0$, $\kappa + r[1 - \frac{\pi_{t-1}-b_{t-1}}{D}] > 0$, probability of which, however, we assumed to be negligible. This simplification could be partially justified by the fact that when we do it, the loss of information would not be large.[5] Hence, we assume that the information available to the MM is the proxy and the inventory, i. e.

(A3) $\xi_t = (e_0, \Delta N_1, e_1, \dots, \Delta N_t, e_t)$

until the end of the paper.

Before going on, note that, given (A1) and (A2),

$$\Delta N_t | \Xi_{t-1} \sim \mathcal{N}\left(\frac{\mu r}{D}(a_{t-1} + b_{t-1} - 2\pi_{t-1}), 2s\kappa + sr\left[2 - \frac{a_{t-1} - b_{t-1}}{D}\right]\right)$$

and

$$\Delta m_t | \Xi_{t-1} \sim \mathcal{N}(e^{a_{t-1}}\mathbb{E}(X_t|\Xi_{t-1}) - e^{b_{t-1}}\mathbb{E}(Y_t|\Xi_{t-1}), \\ e^{2a_{t-1}}\operatorname{var}(X_t|\Xi_{t-1}) + e^{2b_{t-1}}\operatorname{var}(Y_t|\Xi_{t-1}))$$

and, specially, that $\operatorname{var}(\Delta m_t|\Xi_t) > 0$ if $\kappa > 0$.

3. CONDITIONAL DISTRIBUTIONS

The goal of the present Section is to determine the conditional distributions, important for the rest of the paper, and to describe the evolution of the fair price estimator h_t. To this end, denote

$$P_\tau = \frac{a_\tau + b_\tau}{2}$$

the (log)midpoint price and define, for any $t > 0$,

$$\delta_t = P_t - h_t, \qquad \sigma_t = \frac{a_t - b_t}{2}, \tag{5}$$

the price bias, half-spread, respectively.

Proposition 3.1. Let $t \in \mathbb{N}$. Assume (D3), (D4), (I1), (I2), (A1)-(A3). If

$$\eta_{t-1}|\xi_{t-1} \sim \mathcal{N}(0, v_{\eta,t-1}) \tag{6}$$

for some ξ_{t-1}-measurable variable $v_{\eta,t-1}$, then

(i)

$$\left.\begin{bmatrix} \Delta\pi_t \\ \gamma_t \\ \Delta N_t \\ \eta_{t-1} \end{bmatrix}\right| \xi_{t-1} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \\ k\delta_{t-1} \\ 0 \end{bmatrix}, \begin{bmatrix} v_{\Delta\pi} & 0 & 0 & 0 \\ 0 & v_\gamma & 0 & 0 \\ 0 & 0 & v_{N,t} & kv_{\eta,t-1} \\ 0 & 0 & kv_{\eta,t-1} & v_{\eta,t-1} \end{bmatrix}\right)$$

[5]Say that we want to construct a linear estimate of π_t based on (e_t, X_t, Y_t). As, due to (2.3), the variances of Y_t and X_t are similar, being around $s\kappa + sr$, and as their relation to e_t is symmetric. the absolute values of the corresponding coefficients in the estimate should be close to being the same. Further, as Y_t depends on π_t reverse way than X_t does, the signs of the coefficients should be opposite. Thus, any linear estimator or π_t based on (X_t, Y_t, e_t) should not be much different from a that based on $(\Delta N_t, e_t)$.

where $k = 2\frac{\mu r}{D}$ and $v_{N,t} = v_N(v_{\eta,t-1}, \sigma_{t-1}) = 2s(\kappa + r - \frac{r\sigma_{t-1}}{D}) + k^2 v_{\eta,t-1}$.

(ii)

$$\Delta h_t = c_{N,t}(k^{-1}\Delta N_t - \delta_{t-1}) + c_{e,t}(e_t - h_{t-1})$$

where

$$c_{N,t} = c_N(v_{\eta,t-1}, \sigma_{t-1}), \qquad c_{e,t} = c_e(v_{\eta,t-1}, \sigma_{t-1}),$$

for some differentiable functions c_N and c_e,

(iii)

$$\eta_t | \xi_t \sim \mathcal{N}(0, v_{\eta,t})$$

where

$$v_{\eta,t} = v_\eta(v_{\eta,t-1}, \sigma_{t-1}) \tag{7}$$

for some differentiable function v_η.

Proof. See Appendix A. □

Remark 3.2. (ii) of the Proposition in fact says that the MM may obtain information about the fair price not only from the proxy but also from the results of trading.

Corollary 3.3. Let $t \in \mathbb{N}$. Given the assumptions of Proposition 3.1, it holds that

$$\left[\begin{array}{c} \Delta\pi_t \\ \gamma_t \\ \eta_{t-1} \end{array}\right] \Bigg| \left[\begin{array}{c} \xi_{t-1} \\ \Delta N_t \end{array}\right] = \mathcal{N}\left(\left[\begin{array}{c} 0 \\ 0 \\ c_{\eta,t}\left(k^{-1}\Delta N_t - \delta_{t-1}\right) \end{array}\right], \left[\begin{array}{ccc} v_{\Delta\pi} & 0 & 0 \\ 0 & v_\gamma & 0 \\ 0 & 0 & w_{\eta,t-1} \end{array}\right]\right)$$

where $w_{\eta,t-1} = v_{\eta,t-1} - \frac{k v_{\eta,t-1}^2}{v_{N,t}}$, $c_{\eta,t} = \frac{v_{\eta,t-1}}{v_{N,t}}$.

Proof. The corollary follows from the well known formula for the conditional distribution of a Gaussian sub-vector ([6], Proposition 3.13) applied to (i) of Proposition 3.1. □

Corollary 3.4. If η_0 is centred normal, then (i)-(iii) hold for any t, and, moreover, there exists a function $\tilde{v}_\eta$ such that $v_{\eta,t} = \tilde{v}_\eta(\sigma_{t-1}, \sigma_{t-2}, \ldots, \sigma_0)$, and similarly for $v_{N,t}$, $c_{N,t}$, $c_{e,t}$ and $c_{\eta,t}$.

Proof. The assertion follows from a recursive application of Proposition 3.1. □

Throughout the rest of the paper, assume that η_0 is normal with known unconditional variance $v_{\eta,0}$.

4. THE OPTIMAL DECISION

Before proceeding, note that, as h_t is (by definition) ξ_t-measurable, any feasible strategy of (4) may be alternatively expressed by $(\delta_\tau, \sigma_\tau, C_\tau)_{\tau<T}$ where δ_τ and σ_τ are defined by (5).

The following Proposition shows that, out of the whole MM's past information ξ_t, the optimal decision at t depends solely on M_t, N_t, h_t and $v_{\eta,t}$ and that δ_t and σ_t alone depend solely on N_t and $v_{\eta,t}$.

Proposition 4.1. Let $\kappa > 0$ and let either

(F) T be finite

or

(N) $T = \infty$, $\rho > \frac{1}{2}(v_\gamma + v_{\Delta\pi})$ and the problem (4) has additional constraints

$$C(\tau) \qquad \delta_\tau \in [-D_0, D_0], \qquad \sigma_\tau \leq S_0, \qquad \tau < T, \quad D_0, S_0 \in \mathbb{R}^+.$$

Let $0 \leq t < T$. Then

$$V_t(\xi_t) = M_t + e^{h_t} W(N_t, v_{\eta,t}, T - t) \tag{8}$$

where $W(N, v, 0) = N e^{v^2/2}$ and, for any $\tau < T$,

$$\begin{aligned} W(N.v, T - \tau) \quad &= \quad \sup_{\delta,\sigma} F(\delta, \sigma, N, v, T - \tau) \\ &\quad\ \tilde{\mathcal{A}} \qquad \sigma \geq 0 \\ &\quad\ \tilde{\mathcal{N}} \qquad \varphi\left(-\frac{N + k\delta}{\sqrt{v_N(v,\sigma)}}\right) \leq \gamma \end{aligned} \tag{9}$$

for some function F. Here φ is the standard normal c.d.f.

Further, for any optimal solution $(\delta_\tau, \sigma_\tau, C_\tau)_{t \leq \tau \leq T}$, of (4) it holds that

$$\delta_t = \delta(N_t, v_{\eta,t}, T - t) \text{ for some function } \delta, \tag{10}$$

$$\sigma_\tau = \sigma(N_t, v_{\eta,t}, T - t) \text{ for some function } \sigma, \tag{11}$$

$$C_\tau = M_\tau + e^{h_\tau} \Psi(\delta_\tau, \sigma_\tau, v_{\eta,\tau}) \text{ for some function } \Psi. \tag{12}$$

Here, $\infty - t = \infty$ by definition; in particular, $f(n, v, \infty - t)$ denotes a function dependent only on its first two arguments.

Proof. See Appendix B. □

Remark 4.2. An analytical formula for Ψ exists and is given in the proof.

As it is clear from the proof, there is probably no chance for any analytical expression of the optimal policies δ and σ; thus, δ and σ have to be computed numerically. The important fact for us is, however, that, given the infinite horizon, both δ and σ depend solely on the inventory and the conditional variance of η.

Until the end of the paper, assume (N), i. e. that the horizon is infinite.

5. IMPLICATIONS

In the present Section, we show how the price and its volatility may be naturally decomposed into the parts associated with the fair price, the inventory and the uncertainty, and we formulate equations approximately describing the dynamics of the quotes and the inventory stemming from the optimal behaviour of the MM.

5.1. Price decomposition

If we approximate (10) by

$$\delta(n, v) \doteq d_0 + d_N n + d_v v, \tag{13}$$

where d_0, d_N and d_v are some real constants, then the log-midpoint price may be naturally decomposed as

$$P_t = h_t + \delta(N_t, v_{\eta,t}) = \pi_t + \eta_t + \delta(N_t, v_{\eta,t}) \doteq \underbrace{\pi_t}_{\text{fair price}} + \underbrace{\eta_t + d_v v_{\eta,t}}_{\text{uncertainty}} + \underbrace{d_0 + d_N N_t}_{\text{inventory}},$$

for any $t > 0$. This is, to a certain extent, analogous to a well known decomposition of spread into order processing, inventory and adverse selection components, widely discussed in market macro-structure (see e. g., [13]). Moreover, as $\mathbb{E}\eta_t = 0$, the price bias δ_t may be understood as a correction of the price for the inventory and uncertainty.

Before we give a decomposition of the mean price increments and their volatility, note that, given (I), we would have $h \equiv \pi$ and $v_\eta \equiv 0$ so it would be

$$\Delta P_t = \underbrace{\Delta\pi_t}_{\text{fair price}} + \underbrace{\Delta\delta(N_t)}_{\text{inventory}} \doteq \underbrace{\Delta\pi_t}_{\text{fair price}} + \underbrace{d_N \Delta N_t}_{\text{inventory}}$$

with

$$\Delta P_t|\xi_{t-1} \dot\sim \mathcal{N}(\underbrace{d_N\delta_{t-1}}_{\text{inventory}}, \underbrace{v_{\Delta\pi}}_{\text{fair price}} + \underbrace{d_N v^\star_{N,t}}_{\text{inventory}}) \tag{14}$$

where $v^\star_{N,t} = 2s(\kappa + r - \frac{r\sigma_{t-1}}{D})$. The components of the mean and variance associated with the uncertainty thus naturally emerge by a comparison of (14) with the distribution of price increments given (P):

$$\Delta P_t \doteq \Delta\pi_t + d_N \Delta N_t + c_{e,t}(\gamma_t - \eta_{t-1}) + c_{N,t}(k^{-1}\Delta N_t - \delta_{t-1}) + d_v \Delta v_{\eta,t},$$

with

$$\Delta P_t|\xi_{t-1} \dot\sim \mathcal{N}(\mu_t, v_{\Delta P,t}) \qquad \mu_t = \underbrace{d_N\delta_{t-1}}_{\text{inventory}} + \underbrace{c_{N,t}\delta_{t-1} + d_v\Delta v_{\eta,t}}_{\text{uncertainty}}$$

$$\begin{aligned}
v_{\Delta P,t} &= \operatorname{var}(\Delta\pi_t) + (d_N + c_{N,t}k^{-1})^2 \operatorname{var}(\Delta N_t|\xi_{t-1}) + c^2_{e,t}\operatorname{var}(\gamma_t) \\
&\quad + c^2_{e,t}\operatorname{var}(\eta_{t-1}|\xi_{t-1}) + 2c_{e,t}(d_N + c_{N,t}k^{-1})\operatorname{cov}(\eta_{t-1}, \Delta N_t|\xi_{t-1}) \\
&= v_{\Delta\pi} + (d_N + c_{N,t}k^{-1})^2(v^\star_{N,t} + k^2 v_{\eta,t-1}) + c^2_{e,t}v_\gamma + c^2_{e,t}v_{\eta,t-1} - 2c_{e,t}(d_N + c_{N,t}k^{-1})k v_{\eta,t-1} \\
&= \underbrace{v_{\Delta\pi}}_{\text{fair price}} + \underbrace{d_N^2 v^\star_{N,t}}_{\text{inventory}} \\
&\quad + \underbrace{(2d_N c_{N,t}k^{-1} + c^2_{N,t}k^{-2})v^\star_{N,t}d_N + c^2_{e,t}v_\gamma + [(kd_N + c_{N,t})^2 + c_{e,t}(kc_{e,t} - 2kd_N - 2c_{N,t})]v_{\eta,t-1}}_{\text{uncertainty}}
\end{aligned}$$

(To understand the calculation, see Proposition 3.1 (i) and note that $v_{N,t} = v^\star_{N,t} + k^2 v_{\eta,t-1}$).

5.2. Approximate dynamics

In the present Subsection, we describe the dynamics of (a_t, b_t, N_t) expressed equivalently by (P_t, σ_t, N_t). As, by Proposition 4.1, both P_t and σ_t are computed by means of functions of δ, σ, v_η, whose analytic description is unknown to us, we have to use an approximation - in particular, we use a linearisation (13) for δ, and we approximate the non-negative functions σ and v_η geometrically, i. e.

$$\sigma(N, v) \doteq s_0 \exp(N)^{s_N} v^{s_v}, \qquad s_0 > 0, \quad s_N, s_v \in \mathbb{R} \tag{15}$$

and

$$v_\eta(\sigma, v) \doteq \omega_0 \sigma^{\omega_\sigma} v^{\omega_v}, \qquad \omega_0 > 0, \quad \omega_\sigma, \omega_v \in \mathbb{R}. \tag{16}$$

Denote

$$S_{t,i} = S(\sigma_{t-i}, \sigma_{t-i-1}, \ldots, \sigma_{t-i-j_0}) = \sum_{j=0}^{j_0} \omega_v^j \ln(\sigma_{t-i-j}), \qquad i \in \{1, 2\},$$

where j_0 is large enough. Given that $|\omega_v| < 1$, it follows from (16), applied iteratively, that

$$\ln v_{\eta,t} \doteq \frac{\ln \omega_0}{1 - \omega_v} + \omega_\sigma S_{t,1},$$

or equivalently,

$$v_{\eta,t} \doteq \omega_0^{\frac{1}{1-\omega_v}} \exp(\omega_\sigma S_{t,1}) = \omega_0^{\frac{1}{1-\omega_v}} \prod_{j=0}^{j_0} \sigma_{t-1-j}^{\omega_\sigma \omega_v^j}. \tag{17}$$

A combination of (17) and (13) then yields

$$\delta_t \doteq d_0 + d_N N_t + d_S \exp(\omega_\sigma S_{t,1}), \qquad d_S = d_v \omega_0^{\frac{1}{1-\omega_v}}. \tag{18}$$

Similarly, we may express (15) by

$$\ln \sigma_t \doteq \ln s_0 + s_v \frac{\ln \omega_0}{1 - \omega_v} + s_N N_t + s_v \omega_\sigma S_{t,1}. \tag{19}$$

According to Proposition 3.1 (i), we then have

$$\Delta N_t = k\delta(N_{t-1}, v_{\eta,t-1}) + \sqrt{v_{N,t}} F_t \doteq k\left(d_0 + d_N N_{t-1} + d_S \exp(\omega_\sigma S_{t,2})\right) + \sqrt{v_{N,t}} F_t \tag{20}$$

where F_t is standard normal independent of ξ_{t-1}, and where, by Proposition 3.1 (i) and (17),

$$v_{N,t} \doteq \nu_0 + \nu_\sigma \sigma_{t-1} + \nu_S \exp(\omega_\sigma S_{t,2}), \qquad v_0 = 2s(\kappa + r), \quad v_\sigma = \frac{2sr}{D}, \quad v_S = k^2 \omega_0^{\frac{1}{1-\omega_v}}. \tag{21}$$

Analogously, by Proposition 3.1 (ii),

$$\Delta P_t = \Delta\delta_t + \Delta h_t \doteq \Delta\delta_t + c_{N,t}(k^{-1}\Delta N_t - \delta_{t-1}) + c_{e,t}(e_t - h_{t-1}).$$

As

$$e_t - h_{t-1} = (\pi_t + \gamma_t) - (\pi_{t-1} + \eta_{t-1}) = \Delta\pi_t + \gamma_t - \eta_{t-1}$$

ans as, by Corollary 3.3,

$$e_t - h_{t-1}|\xi_{t-1}, \Delta N_t = \mathcal{N}\left(-c_{\eta,t}(k^{-1}\Delta N_t - \delta_{t-1}), w_{\eta,t-1} + v_\gamma + v_{\Delta\pi}\right),$$

we have

$$\Delta P_t = \Delta\delta_t + (c_{N,t} - c_{e,t}c_{\eta,t})(k^{-1}\Delta N_t - \delta_{t-1}) + c_{e,t}\sqrt{w_{\eta,t-1} + v_\gamma + v_{\Delta\pi}}G_t$$

where G_t is standard normal, independent on $(\xi_{t-1}, \Delta N_t)$, hence on $(\xi_{t-1}, \Delta N_t, F_t)$.[6] Consequently, by (18),

$$\begin{aligned}\Delta P_t &\doteq d_N \Delta N_t + d_S[\exp(\omega_\sigma S_{t,1}) - \exp(\omega_\sigma S_{t,2})]\\ &+(c_{N,t}-c_{e,t}c_{\eta,t})[k^{-1}\Delta N_t - d_0 - d_N N_{t-1} - d_S \exp(\omega_\sigma S_{t,2})] + c_{e,t}\sqrt{w_{\eta,t-1} + v_\gamma + v_{\Delta\pi}}G_t.\end{aligned} \tag{22}$$

Thus, the approximate dynamics of the "trade and quote" data is given by (19), (20) and (22).

6. ECONOMETRIC EVIDENCE

The goal of the present Section is to get an econometric evidence supporting our model, namely whether and how σ_t and δ_t depend on the inventory and/or the uncertainty, as it is predicted by our model.

6.1. Dynamics with noise limit orders

Having "trade and quote" data (a, b, N) at our disposal, we may observe the quotations and the inventory increments. In real-life, however, not all the agents putting limit orders are MM's. Therefore, we assume that there are some "noise" limit orders being put into the spread by the "non-MM" agents[7] so the information observed by an econometrician at time t is

$$\tilde{\xi}_t = (\tilde{a}_1, \tilde{b}_1, \Delta N_1, \ldots, \tilde{a}_t, \tilde{b}_t, \Delta N_t)$$

where

$$b_t \le \tilde{b}_t < \tilde{a}_t \le a_t.$$

Denote

$$\epsilon_{\sigma,t} = \frac{\tilde{\sigma}_t}{\sigma_t}, \qquad \tilde{\sigma}_t = \frac{\tilde{a}_t - \tilde{b}_t}{2}$$

[6] To see it, note that $\Delta N_t = f(\xi_{t-1}, F_t)$ for some function f, bijective in its second argument.

[7] Consequently, not all the trades are made through the MM's. Moreover, because the trade data we have at our disposal are not matched with the quote changes so we had to use our own algorithm succeeding to match only about approximately 70% cases, the inventory data themselves are known only subject to an error, too. We, however, do not take this "matching" noise into account in the present paper because this would prevent the parameters or the model to be identifiable.

the relative spread reduction and

$$\epsilon_{P,t} = \frac{(\tilde{a}_t - a_t) + (\tilde{b}_t - b_t)}{2}$$

the imbalance of the one-sided spread reductions. It is quite natural to assume that the intensity of noise trading does not depend on the side of trading, i. e.

$$\mathbb{E}(\epsilon_{P,t}|\tilde{\sigma}_{t-1}, \Xi_{t-1}) = 0.$$

It is also likely (and evident from the data) that the rate of in-spread order placement increases with widening of the spread, which can be expressed as

$$\mathbb{E}(\epsilon_{\sigma,t}|\tilde{\sigma}_{t-1}, \epsilon_{P,t}, \Xi_t) = \exp(\alpha_1\tilde{\sigma}_{t-1} + \alpha_2\tilde{\sigma}^2_{t-1}) \tag{23}$$

for some constants α_1 and α_2 such that $\alpha_1 s + \alpha_2 s^2 \leq 0$ for each $s \geq 0$. Strengthening (23) a little, we assume

$$\epsilon_{\sigma,t} = \exp(\alpha_1\tilde{\sigma}_{t-1} + \alpha_2\tilde{\sigma}^2_{t-1})\epsilon^\star_t \tag{24}$$

where $(\epsilon^\star_t)_{\tau\in\mathbb{N}}$ are i.i.d. positive independent of $(\Xi_\tau, \epsilon_{P,\tau})_{\tau\in\mathbb{N}}$ with $\mathbb{E}e^\star_\tau = 1$. Note that then, for any τ,

$$\ln\tilde{\sigma}_\tau = \ln\sigma_\tau + \alpha_1\tilde{\sigma}_{\tau-1} + \alpha_2\tilde{\sigma}^2_{\tau-1} + \ln\epsilon^\star_t. \tag{25}$$

Further, denote

$$\begin{aligned} Z_{t,i} &= Z(\tilde{\sigma}_{t-i}, \dots, \tilde{\sigma}_{t-i-1-j_0}; \omega_v, \alpha_1, \alpha_2) \\ &= \sum_{j=0}^{j_0} \omega_v^j \ln\tilde{\sigma}_{t-i-j} - \sum_{j=0}^{j_0} \omega_v^j (\alpha_1\tilde{\sigma}_{t-i-1-j} + \alpha_2\tilde{\sigma}^2_{t-i-1-j}), \qquad i \in \{1,2\} \end{aligned}$$

and note that

$$S_{\tau,i} = Z_{\tau,i} - \sum_{j=0}^{j_0} \omega_v^j \ln\epsilon^\star_{\tau-i-j}$$

for any τ. Using this and (25), we are getting

$$\begin{aligned} \ln\tilde{\sigma}_t &\doteq \tilde{s}_0 + \alpha_1\tilde{\sigma}_{t-1} + \alpha_2\tilde{\sigma}^2_{t-1} + s_N\sum_{\tau=1}^{t}\Delta N_\tau + s_S Z_{t,1} + \mathcal{E}_t \qquad (26) \\ & \quad \tilde{s}_0 = \ln s_0 + s_v\frac{\ln\omega_0}{1-\omega_0} + s_N N_0 + \left(1 - s_S\sum_{j=0}^{j_0}\omega_v^j\right)\mathbb{E}\ln\epsilon^\star_1 \\ & \quad s_S = s_v\omega_\sigma \\ & \quad \mathcal{E}_t = \ln\epsilon^\star_t - \mathbb{E}\ln\epsilon^\star_1 - s_S\sum_{j=0}^{j_0}\omega_v^j(\ln\epsilon^\star_{t-1-j} - \mathbb{E}(\ln\epsilon^\star_1)). \end{aligned}$$

Further, if we assume that the initial inventory, denoted by N_0, is constant unknown, then we get from (20) that

$$
\begin{aligned}
\Delta N_t \quad = \quad & n_0 + n_N \sum_{\tau=1}^{t-1} \Delta N_\tau + n_S \exp\left(\omega_\sigma Z_{t,2}\right) + \mathcal{F}_t \qquad (27)\\
& n_0 = kd_0 + kd_N N_0 \\
& n_N = kd_N \\
& n_S = kd_S \mathbb{E}(\theta_1) \\
& \mathcal{F}_t = kd_S \exp\left(\omega_\sigma Z_{t,2}\right)\left(\theta_{t-2} - \mathbb{E}(\theta_1)\right) + \sqrt{v_{N,t}} F_t, \\
& \theta_t = \exp(-\omega_\sigma \sum_{j=0}^{j_0} \omega_v^j \ln \epsilon^\star_{t-j}).
\end{aligned}
$$

Finally, if we take $c_{N,t} \doteq c_N$, $c_{e,t} \doteq c_e$, $c_{\eta,t} \doteq c_\eta$ for some constants c_N, c_e, c_η, [8] and denote

$$c_P = c_N - c_e c_\eta.$$

relation (22) becomes

$$
\begin{aligned}
\Delta P_t \doteq & -c_P(d_0 + d_N N_0) + (d_N + c_P k^{-1})\Delta N_t - c_P d_N \sum_{\tau=1}^{t-1} \Delta N_\tau \\
& d_S \exp(\omega_\sigma S_{t,1}) - d_S(1 + c_P)\exp(\omega_\sigma S_{t,2}) + c_e \sqrt{w_{\eta,t-1} + v_\gamma + v_{\Delta\pi}} G_t.
\end{aligned}
$$

Consequently, the dynamics of the increments of the noised price

$$\tilde{P}_t = \frac{\tilde{a}_t + \tilde{b}_t}{2} = P_t + \epsilon_{P,t}$$

will be

$$
\begin{aligned}
\Delta \tilde{P}_t \quad = \quad & \phi_0 + \phi_{\Delta N} \Delta N_t + \phi_N \sum_{\tau=1}^{t-1} \Delta N_\tau + \phi_{S,1} \exp(\omega_\sigma Z_{t,1}) + \phi_{S,2} \exp(\omega_\sigma Z_{t,2}) + \mathcal{G}_t \quad (28)\\
& \phi_0 = -c_P(d_0 + d_N N_0) \\
& \phi_{\Delta N} = d_N + k^{-1}(c_N - c_e c_\eta) \\
& \phi_N = -c_P d_N \\
& \phi_{S,1} = d_S \mathbb{E}\theta_1 \\
& \phi_{S,2} = -d_S(1 + c_P)\mathbb{E}\theta_1 \\
& \mathcal{G}_t = \epsilon_{P,t} - \epsilon_{P,t-1} + d_S \exp(\omega_0 Z_{t,1})\left(\theta_{t-1} - \mathbb{E}(\theta_1)\right) \\
& \qquad -d_S(1 + c_P)\exp(\omega_0 Z_{t,2})\left(\theta_{t-2} - \mathbb{E}(\theta_1)\right) + c_e \sqrt{w_{\eta,t-1} + v_\gamma} G_t.
\end{aligned}
$$

The noised version of the approximate dynamics (19), (20) and (22) is thus given by (26), (27) and (28).

[8] An alternative would be to approximated $c_{N,t}$ and $c_{e,t}$, too, which would lead to an equation with additional independent variables (e. g. $\exp(2\omega_\sigma S_{t,2})$) – we, however, do not go this way for simplicity.

6.2. Estimation

Because the equations defining the noised dynamics are non-linear and, moreover, some of the coefficients are shared among equations, we estimated the equations jointly by a non-linear least squares weighted by standard errors of the individual equations (see Appendix C for details and discussion about consistency and asymptotic normality of the estimators).

The actual estimation was performed by a C++ program, developed by the authors, employing the MMA (Method of Moving Asymptotes) minimization algorithm from the NLOPT package (see [15]).

As a dataset, we used 10 seconds high frequency trade and quote data from March 2009[9] supplied by Tickdata Inc., of three stocks:

GE General Electric

MSFT Microsoft

XOM Exxon Mobile

from ten electronic markets: ISE, NASDAQ OMX BX,NSE,NASD ADF, Chicago, NYSE, ARCA, NASDAQ T, CBOE, BATS, which were actually all the markets, covered by the data, where the particular stocks were traded. The results of the estimation of parameters from (26), (27) and (28) can be seen in Appendix D.

6.3. Testing for implications of the decision model

The aim of the present Subsection is to examine a dependence of δ and σ on the inventory and the uncertainty, predicted by our model. Given our approximations (13) and (15), this dependence would manifest itself by non-zero linear coefficients in (20), (19), respectively. Even though, except for s_N, these coefficients cannot be estimated directly from (26), (27) and (28), their signs and significances may be deduced from these equations. In particular, the significance and the sign of...

- ... s_v may be taken from this of s_S (note that $\omega_\sigma > 0$ by assumption).
- ... d_N may be taken from this of n_N (its significance and sign is inherited by d_N).
- ... d_v may be got from those of n_S and $\phi_{S,1}$ (as $\mathbb{E}\theta_1 > 0$, their signs and significances are inherited by d_S and consequently by d_v).

The signs of dependence resulting from our estimation together with significances of the corresponding coefficients may be seen in Table 1 (the blank fields in the table mean that the given stock was not traded on the given market).

Immediately we see that the most obvious dependence is the largely prevailing positive dependence of the spread on the uncertainty, speaking in favour of *M3* (see the Introduction). Then there is a mostly significant dependence of the spread on the inventory which, however, changes sign.

[9] We chose this month arbitrarily in advance.

		ISE	NAS. QB	NSE	NAS. ADF	Chicago
XOM	σ in N	$\downarrow^{*}$	$\downarrow^{***}$	$\uparrow$	$\downarrow^{***}$	
	σ in v	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$	
	δ in N	$\downarrow^{*}$	$\downarrow$	$\uparrow^{**}$	$\downarrow^{***}$	
	δ in v (via ΔN)	$\downarrow$	$\uparrow$	$\downarrow^{*}$	$\uparrow^{***}$	
	δ in v (via ΔP)	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$	
GE	σ in N	$\downarrow^{***}$	$\downarrow^{***}$	$\uparrow^{***}$	$\downarrow$	$\downarrow^{***}$
	σ in v	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$
	δ in N	$\downarrow$	$\downarrow^{***}$	$\downarrow$	$\downarrow^{***}$	$\downarrow^{*}$
	δ in v (via ΔN)	$\uparrow$	$\downarrow^{*}$	$\uparrow$	$\uparrow^{***}$	$\downarrow$
	δ in v (via ΔP)	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$
MSFT	σ in N	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$	$\downarrow^{***}$
	σ in v	$\uparrow^{***}$	$\uparrow^{***}$	$\downarrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$
	δ in N	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow^{***}$	$\uparrow$
	δ in v (via ΔN)	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow^{**}$
	δ in v (via ΔP)	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$
		NYSE	ARCA	NAS. T	COEB	BATS
XOM	σ in N	$\downarrow^{***}$	$\downarrow^{***}$	$\downarrow^{***}$	$\downarrow^{***}$	$\uparrow^{**}$
	σ in v	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$
	δ in N	$\downarrow^{**}$	$\downarrow$	$\downarrow^{*}$	$\uparrow$	$\downarrow^{***}$
	δ in v (via ΔN)	$\downarrow^{***}$	$\downarrow^{*}$	$\uparrow$	$\downarrow^{***}$	$\downarrow^{***}$
	δ in v (via ΔP)	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow^{***}$	$\downarrow^{**}$
GE	σ in N	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$	$\downarrow^{***}$	$\uparrow^{***}$
	σ in v	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$	$\uparrow^{***}$
	δ in N	$\downarrow^{*}$	$\downarrow$	$\downarrow$	$\downarrow^{**}$	$\downarrow$
	δ in v (via ΔN)	$\uparrow^{*}$	$\downarrow$	$\uparrow$	$\downarrow$	$\downarrow$
	δ in v (via ΔP)	$\uparrow$	$\downarrow$	$\uparrow$	$\uparrow^{***}$	$\uparrow^{*}$
MSFT	σ in N		$\uparrow$		$\downarrow^{***}$	$\uparrow^{***}$
	σ in v		$\uparrow^{***}$		$\uparrow^{***}$	$\uparrow^{***}$
	δ in N		$\uparrow$		$\downarrow$	$\downarrow^{*}$
	δ in v (via ΔN)		$\uparrow$		$\downarrow$	$\uparrow$
	δ in v (via ΔP)		$\uparrow^{***}$		$\uparrow^{***}$	$\uparrow^{***}$

Tab. 1. Dependence of δ and σ on the inventory and uncertainty. The number of stars denote a significance on levels 0.05, 0.01 and 0.001, respectively.

The dependence of the price bias δ on the inventory is almost always negative (but not always significant), which again speaks for the findings of the market micro-structure theory, namely for *M1* from the Introduction. Further, we can observe mostly positive dependence of δ on the uncertainty.

7. COMPARISON WITH BENCHMARK MODELS

An objection may arise whether our empirical findings are not spurious in some way. Even if it is impossible to refute completely such an objection (one can never exclude a possibility that another, perhaps richer, model explains the data better), we try to meet such objection by examining three simple benchmark hypotheses.

7.1. All agents irrational

Assume first that neither the MM nor the liquidity traders are rational and the limit and market orders are put in the zero-intelligence way in the sense of [24]. Then, however, the flow of the market orders would be independent of the past, which is proved to be false by the significant results in (27).

7.2. Irrational liquidity takers

Now, say that the MM's are possibly rational but the liquidity traders are not, meaning that they do not consider (their estimate of) the fair price but buy and sell the stocks randomly, i. e., it is $\lambda(\bullet) \equiv 0$. This, however, would mean that

$$X_t|\Xi_{t-1} \sim \mathrm{CP}(\kappa, \mathcal{D}), \qquad Y_t|\Xi_{t-1}, X_t \sim \mathrm{CP}(\kappa, \mathcal{D})$$

saying again that ΔN_t are independent of the past, which is falsified by the significant results from (27).

7.3. Irrational market makers

Another case could be that the LT's would be possibly rational and the actions of the MM would not depend on his uncertainty. Instead, the dependence would be caused only by the price movements. In particular, once a market order arrived and caused the movement of a corresponding quote, the MM would set the new quote proportionally to the size of the market order, which could be mathematically described as

$$\Delta\sigma_t = f(|\Delta N_t|) + \epsilon_t$$

for some f and white noise $\epsilon_\bullet$. Then, however, σ_t would depend only on σ_{t-1} and ΔN_t but not on the earlier values of σ, which is evidently not true whenever s_S comes out significant in (26).

8. CONCLUSION

A model describing the behaviour of a market with a rational partially informed market maker has been proposed and partially verified.

Even though, as it is common in finance, the model did not succeed to explain major parts of corresponding variances, it may contribute to understanding of such phenomenons as micro-structure noise or the relation between price and inventory (hence of the traded volume).

The model might be further refined in many possible ways.

A step towards its realism, suggesting itself the most, would be to assume multiple rational agents; then, however, the model would become a dynamical stochastic game, analysis of which would require methods of analysis completely different from those used in the present paper.

Less radical enhancements of the model could be done, too, such as considering other than linear utility function, or modelling the fair price by a process different from the random walk, such as the GARCH. In both these cases, our main theoretical results could be preserved for the price of increasing of the state space of the dynamics. The econometric verification, however, would be probably much more complicated than this performed in the present paper.

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A. PROOF OF PROPOSITION 3.1

As, by (I1), $\Delta\pi_t \perp\!\!\!\perp \xi_{t-1}, \eta_{t-1}$, we have, by [16] Proposition 6.8, that $\Delta\pi_t \perp\!\!\!\perp_{\xi_{t-1}} \eta_{t-1}$. Therefore and because, by [16] Proposition 6.6, $\mathcal{L}(\Delta\pi_t|\xi_{t-1}) = \mathcal{L}(\Delta\pi_t)$, we get. using also (6), that $\Delta\pi_t, \eta_{t-1}|\xi_{t-1} \sim \mathcal{N}(0, \operatorname{diag}(v_{\Delta\pi}, v_{\eta,t-1}))$.

By an analogous procedure, we get

$$\gamma_t, \Delta\pi_t, \eta_{t-1}|\xi_{t-1} \sim \mathcal{N}(0, \operatorname{diag}(v_\gamma, v_{\Delta\pi}, v_{\eta,t-1})). \tag{29}$$

Further, denote

$$\vartheta_t = \Delta N_t - m_t, \qquad m_t = \mathbb{E}(\Delta N_t|\Xi_{t-1}) = k\left(\frac{a_{t-1}+b_{t-1}}{2} - \pi_{t-1}\right) = k(\delta_{t-1} + \eta_{t-1}).$$

By (I1), $X_t, Y_t \perp\!\!\!\perp_{\Xi_{t-1}} \gamma_t, \Delta\pi_t$ and, as $m_t \in \Xi_{t-1}$ and $(\xi_{t-1}, \eta_{t-1}) \in \Xi_{t-1}$, we get, by [16] Corollary 6.7 used twice, that

$$X_t, Y_t, m_t \perp\!\!\!\perp_{\Xi_{t-1}} \xi_{t-1}, \eta_{t-1}, \gamma_t, \Delta\pi_t,$$

which is, by [16], Proposition 6.6., equivalent to

$$\mathcal{L}(X_t, Y_t, m_t|\Xi_{t-1}, \xi_{t-1}, \eta_{t-1}, \gamma_t, \Delta\pi_t) = \mathcal{L}(X_t, Y_t, m_t|\Xi_{t-1}).$$

trivially implying

$$\mathcal{L}(\vartheta_t|\Xi_{t-1}, \xi_{t-1}, \eta_{t-1}, \gamma_t, \Delta\pi_t) = \mathcal{L}(\vartheta_t|\Xi_{t-1}). \tag{30}$$

Further, by (I2) and the fact that m_t is conditionally constant given Ξ_{t-1} hence conditionally independent on any random variable given Ξ_{t-1}, we have

$$\mathcal{L}(\vartheta_t|\Xi_{t-1}) = \mathcal{L}(X_t|\Xi_{t-1}) \circ \mathcal{L}(-Y_t|\Xi_{t-1}) \circ \delta_{-m_t} \overset{A1,A2}{=} \mathcal{N}(0, v_{\vartheta,t}) \tag{31}$$

where

$$v_{\vartheta,t} = 2s\kappa + sr\left(2 - \frac{a_{t-1}-b_{t-1}}{D}\right) = 2s\left(\kappa + r - \frac{r}{D}\sigma_{t-1}\right).$$

As $v_{\vartheta,t}$ is a function of σ_{t-1}, $\mathcal{L}(\vartheta_t|\Xi_{t-1})$ is σ_{t-1}-measurable, hence ξ_{t-1}-measurable, so it may serve as a conditional probability of ϑ_t given ξ_{t-1} which, in the combination with (30) and (31), gives

$$\mathcal{L}(\vartheta_t|\Xi_{t-1}, \xi_{t-1}, \eta_{t-1}, \gamma_t, \Delta\pi_t) = \mathcal{L}(\vartheta_t|\xi_{t-1}) = \mathcal{N}(0, v_{\vartheta,t})$$

i. e. $\vartheta_t \perp\!\!\!\perp_{\xi_{t-1}} \Xi_{t-1}, \eta_{t-1}, \gamma_t, \Delta\pi_t$, trivially implying

$$\vartheta_t \perp\!\!\!\perp_{\xi_{t-1}} \eta_{t-1}, \gamma_t, \Delta\pi_t.$$

By combining this and (29), we are getting

$$\vartheta_t, \gamma_t, \Delta\pi_t, \eta_{t-1}|\xi_{t-1} \sim \mathcal{N}(0, \operatorname{diag}(v_{\vartheta,t}, v_\gamma, v_{\Delta\pi}, v_{\eta,t-1})).$$

Point (i) of the Proposition now follows from the fact that

$$\begin{bmatrix} \Delta\pi_t \\ \gamma_t \\ \Delta N_t \\ \eta_{t-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ k\delta_{t-1} \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\pi_t \\ \gamma_t \\ \vartheta_t \\ \eta_{t-1} \end{bmatrix}.$$

Further, since

$$\pi_t = h_{t-1} - \eta_{t-1} + \Delta\pi_t \tag{32}$$

$$\Delta N_t = k\delta_{t-1} + k\eta_{t-1} + \vartheta_t, \tag{33}$$

$$e_t = \pi_t + \gamma_t = h_{t-1} - \eta_{t-1} + \Delta\pi_t + \gamma_t, \tag{34}$$

we are getting that

$$\begin{aligned} &\left[\begin{matrix} \pi_t \\ \Delta N_t \\ e_t \end{matrix} \right] \Bigg| \xi_{t-1} \\ &\sim \mathcal{N}\left(\begin{bmatrix} h_{t-1} \\ k\delta_{t-1} \\ h_{t-1} \end{bmatrix}, \begin{bmatrix} v_{\eta,t-1} + v_{\Delta\pi} & -kv_{\eta,t-1} & v_{\eta,t-1} + v_{\Delta\pi} \\ -kv_{\eta,t-1} & v_{\vartheta,t} + k^2 v_{\eta,t-1} & -kv_{\eta,t-1} \\ v_{\eta,t-1} + v_{\Delta\pi} & -kv_{\eta,t-1} & v_{\Delta\pi} + v_\gamma + v_{\eta,t-1} \end{bmatrix} \right), \end{aligned} \tag{35}$$

from which we get by the well known formula for conditional normal distribution ([6], Proposition 3.13), that $\mathcal{L}(\pi_t|\xi_t) = \mathcal{L}(\pi_t|\Delta N_t, e_t, \xi_{t-1})$ is normal with mean

$$\begin{aligned} &h_{t-1} + \begin{bmatrix} -kv_{\eta,t-1} \\ v_{\eta,t-1} + v_{\Delta\pi} \end{bmatrix} \\ &\qquad \times \begin{pmatrix} v_{\vartheta,t} + k^2 v_{\eta,t-1} & -kv_{\eta,t-1} \\ -kv_{\eta,t-1} & v_{\Delta\pi} + v_\gamma + v_{\eta,t-1} \end{pmatrix}^{-1} \begin{bmatrix} k^{-1}\Delta N_t - \delta_{t-1} \\ e_t \end{bmatrix} \end{aligned} \tag{36}$$

and variance

$$\begin{aligned} v_{\eta,t} &= v_{\eta,t-1} + v_{\Delta\pi} - \begin{bmatrix} -kv_{\eta,t-1} \\ v_{\eta,t-1} + v_{\Delta\pi} \end{bmatrix} \\ &\qquad \times \begin{pmatrix} v_{\vartheta,t} + k^2 v_{\eta,t-1} & -kv_{\eta,t-1} \\ -kv_{\eta,t-1} & v_{\Delta\pi} + v_\gamma + v_{\eta,t-1} \end{pmatrix}^{-1} \begin{bmatrix} -kv_{\eta,t-1} \\ v_{\eta,t-1} + v_{\Delta\pi} \end{bmatrix}. \end{aligned} \tag{37}$$

As all the variables on the RHS's in (36) and (37) are constants except for $v_{\eta,t-1}$ and $v_{\vartheta,t}$, which is a function of σ_{t-1} and $v_{\eta,t-1}$, (36) proves (ii) (note that $h_t = \mathbb{E}[\pi_t|\xi_t]$) and (37) proves (iii) (note that $\eta_t = \mathbb{E}(\pi_t|\xi_t) - \pi_t$ and that the conditional variance of $\mathbb{E}(\pi_t|\xi_t)$, which is ξ_t-measurable with respect to ξ_t, is zero).

B. PROOF OF PROPOSITION 4.1

Assume (F) first. Let us prove (8) by induction: If $T = t$ then (8) holds because

$$V_T(\xi_T) = M_T + N_T \mathbb{E}(e^{\pi_T}|\xi_T) = M_T + N_T e^{h_T} e^{v_{\eta,T}/2}$$

(to see it, note that $\mathbb{E}(e^{\pi_T}|\xi_T) = \mathbb{E}(e^{-\eta_T}e^{h_T}|\xi_T) = e^{h_T}\mathbb{E}(e^{-\eta_T}|\xi_T))$

Now let $t < T$ and assume (8) to hold for $t+1$. Applying the Bellman principle, we get that the value function V_t and the optimal solution $(C_t, \delta_t, \sigma_t)$ must satisfy

$$V_t(\xi_t) = \sup_{\substack{\delta_t,\sigma_t,C_t \\ \mathcal{A}(t),\mathcal{M}(t),\mathcal{N}(t)}} \left[C_t + e^{-\rho}\mathbb{E}\left(V_{t+1}(\xi_{t+1})|\xi_t\right)\right]. \tag{38}$$

which, by (8), may be written as

$$V_t(\xi_t) = \sup_{\substack{\delta_t,\sigma_t,C_t \\ \mathcal{A}(t),\mathcal{M}(t),\mathcal{N}(t)}} \left[C_t + e^{-\rho}\mathbb{E}\left(M_{t+1} + e^{h_{t+1}}W(N_{t+1}, v_{\eta,t+1}, T-t-1)|\xi_t\right)\right]. \tag{39}$$

Using trivial identity $\mathbb{E}(M_{t+1}|\xi_t) = M_t + \mathbb{E}(\Delta M_t|\xi_t)$, imposing into (2) and noting that $a_t = h_t + \delta_t + \sigma_t$, $b_t = h_t + \delta_t - \sigma_t$, we further rewrite (39) as

$$\begin{aligned} V_t(\xi_t) = e^{-\rho}M_t + \sup_{\delta_t,\sigma_t,C_t,\ldots} [C_t + e^{-\rho}(\mathbb{E}(\Delta m_{t+1}|\xi_t) - C_t) \\ + e^{-\rho}e^{h_t}\mathbb{E}(e^{\Delta h_{t+1}}W(N_{t+1}, v_{\eta,t+1}, T-t-1)|\xi_t)]. \end{aligned} \tag{40}$$

Now, as

$$\begin{aligned} \mathbb{E}(\Delta X_{t+1}|\xi_t) &= \mathbb{E}(\mathbb{E}(\Delta X_{t+1}|\Xi_t)|\xi_t) = \mathbb{E}(\mu(\kappa + r(1 - \frac{a_t - \pi_t}{D})|\xi_t) \\ &= \mathbb{E}(\mu(\kappa + r(1 - \frac{\eta_t + \delta_t + \sigma_t}{D})|\xi_t) = \mu(\kappa + r(1 - \frac{\delta_t + \sigma_t}{D})) \end{aligned}$$

and as an analogous procedure may be applied to $\mathbb{E}(\Delta X_{t+1}|\xi_t)$, it holds that

$$\mathbb{E}(\Delta m_{t+1}|\xi_t) = e^{h_t+\delta_t+\sigma_t}\mathbb{E}(\Delta X_{t+1}|\xi_t) - e^{h_t+\delta_t-\sigma_t}\mathbb{E}(\Delta Y_{t+1}|\xi_t) = e^{h_t}f(\delta_t,\sigma_t) \tag{41}$$

where

$$f(\delta,\sigma) = e^{\delta+\sigma}\mu(\kappa + r(1 - \frac{\delta+\sigma}{D})) - e^{\delta-\sigma}\mu(\kappa + r(1 - \frac{\sigma-\delta}{D})).$$

Further, as both $v_{\eta,t+1}$ and Δh_{t+1} are functions of $(\Delta N_{t+1}, e_{t+1} - h_t, v_{\eta,t}, \delta_t, \sigma_t)$ (see Proposition 3.1), we have

$$\begin{aligned} &\mathbb{E}(e^{\Delta h_{t+1}}W(N_{t+1}, v_{\eta,t+1}, T-t-1)|\xi_t)] \\ &\quad = \mathbb{E}(g(N_t, \Delta N_{t+1}, e_{t+1} - h_t, v_{\eta,t}, \delta_t, \sigma_t, T-t)|\xi_t) = G(N_t, v_{\eta,t}, \delta_t, \sigma_t, T-t) \end{aligned} \tag{42}$$

for some g and G (the last "=" is due to the fact that the conditional distribution of $(\Delta N_{t+1}, e_{t+1} - h_t)$ given ξ_t depends only on $(\delta_t, \sigma_t, v_{\eta,t})$, see Proposition 3.1 (i) and note that $e_{t+1} - h_t = \pi_{t+1} + \gamma_{t+1} - h_t = \Delta\pi_{t+1} - \eta_t + \gamma_{t+1})$.

Using (41) and (42), we may now write

$$\begin{aligned} V_t(\xi_t) = e^{-\rho}M_t + \sup_{\delta_t,\sigma_t,C_t,\ldots} [(1 - e^{-\rho})C_t + e^{-\rho}e^{h_t}f(\delta_t,\sigma_t) \\ + e^{-\rho}e^{h_t}G(N_t, v_{\eta,t}, \delta_t, \sigma_t, T-t)]. \end{aligned} \tag{43}$$

Because neither f nor G depend on C_t, $\mathcal{M}(t)$ will be fulfilled with "=" given that C_t is optimal (if not then a greater C would strictly increase the objective function) which determines optimal C_t uniquely as

$$C_t = \sup\{c : \phi(c - M_t|\xi_t) \le \gamma) \tag{44}$$

where ϕ is a conditional c.d.f. of $\Delta m_{t+1}|\xi_t$, which is given by

$$\begin{aligned}\phi(x|\xi_t) &= \mathbb{E}(\mathbb{P}(\Delta m_{t+1} \le x|\Xi_t)|\xi_t)\\ &= \mathbb{E}\left(\varphi\left(\frac{x - \mathbb{E}(\Delta m_{t+1}|\Xi_t)}{\sqrt{\mathrm{var}(\Delta m_{t+1}|\Xi_t)}}\right)\middle|\,\xi_t\right)\\ &= \mathbb{E}\left(\varphi\left(\frac{x - e^{h_t}\left(e^{\delta+\sigma}\mu[\kappa + (1 - \frac{\delta+\sigma+\eta_t}{D})] - e^{\delta-\sigma}\mu[\kappa + (1 - \frac{\sigma-\delta-\eta_t}{D})]\right)}{e^{h_t}\sqrt{e^{2(\delta+\sigma)}s(\kappa + (1 - \frac{\delta+\sigma+\eta_t}{D})) + e^{2(\delta-\sigma)}s(\kappa + r(1 - \frac{\sigma-\delta-\eta_t}{D}))}}\right)\middle|\,\xi_t\right)\\ &= \mathbb{E}(\theta(e^{-h_t}x; \delta, \sigma, \eta_t)|\xi_t)\end{aligned}$$

where

$$\theta(y; \delta, \sigma, \eta) = \varphi\left(\frac{y - \left(e^{\delta+\sigma}\mu[\kappa + (1 - \frac{\delta+\sigma+\eta}{D})] - e^{\delta-\sigma}\mu[\kappa + (1 - \frac{\sigma-\delta-\eta}{D})]\right)}{\sqrt{e^{2(\delta+\sigma)}s(\kappa + (1 - \frac{\delta+\sigma+\eta}{D})) + e^{2(\delta-\sigma)}s(\kappa + r(1 - \frac{\sigma-\delta-\eta}{D}))}}\right).$$

As the only random element inside the expectation is η_t, we may write

$$\phi(x|\xi_t) = \Phi(e^{-h_t}x; \delta_t, \sigma_t, v_{\eta,t}), \qquad \Phi(y; \delta, \sigma, v) = \mathbb{E}_{\eta\sim\mathcal{N}(0,v)}\theta(y; \delta, \sigma, \eta).$$

(recall that the conditional distribution of $\eta_t|\xi_t$ depends only on $v_{\eta,t}$ by Proposition 3.1). Therefore, (44) may be rewritten as

$$\begin{aligned}C_t &= \sup\{c : \Phi(e^{-h_t}(c - M_t)|\delta_t, \sigma_t, v_{\eta,t}) \le \gamma)\\ &= M_t + \sup\{c - M_t : \Phi(e^{-h_t}(c - M_t); \delta_t, \sigma_t, v_{\eta,t}) \le \gamma)\\ &= M_t + e^{h_t}\sup\{e^{-h_t}(c - M_t) : \Phi(e^{-h_t}(c - M_t); \delta_t, \sigma_t, v_{\eta,t}) \le \gamma)\\ &= M_t + e^{h_t}\sup\{x : \Phi(x; \delta_t, \sigma_t, v_{\eta,t}) \le \gamma)\\ &= M_t + e^{h_t}\Psi(\delta_t, \sigma_t, v_{\eta,t}), \qquad \Psi(\delta, \sigma, v) = \Phi^{-1}(\gamma; \delta, \sigma, v)\end{aligned} \tag{45}$$

which proves (12) (the last "=" holds because $\Phi(\bullet; \delta, \sigma, v)$ is continuous and strictly monotonous for each δ, σ, v).

Now, by imposing (45) into (43) and dropping the constraint $\tilde{\mathcal{M}}$, we are getting

$$\begin{aligned}V_t(\xi_t) &= e^{-\rho}M_t + \sup_{\delta_t,\sigma_t,\tilde{\mathcal{A}},\tilde{\mathcal{N}}}\{(1 - e^{-\rho})[M_t + e^{h_t}\Psi(\delta_t, \sigma_t, v_{\eta,t})]\\ &\quad + e^{-\rho}e^{h_t}f(\delta_t, \sigma_t) + e^{-\rho}e^{h_t}G(N_t, v_{\eta,t}, \delta_t, \sigma_t, T - t)\}\\ &= M_t + e^{h_t}W(N_t, v_{\eta,t}, T - t)\end{aligned}$$

where

$$W(N.v,T-t) = \sup_{\delta,\sigma,\tilde{\mathcal{A}},\tilde{\mathcal{N}}} [(1-e^{-\rho})\Phi^{-1}(\gamma;\delta,\sigma,v)+e^{-\rho}f(\delta,\sigma)+e^{-\rho}G(N,v,\sigma,\delta,T-t)] \quad (46)$$

i. e., (8) is proved, because W is of the form of (9) and ($\tilde{\mathcal{N}}$) may be rewritten as in (8). Relations (11) and (10) follow from (8) and the Bellman principle.[10]

Before dealing with the infinite horizon, let us prove a lemma.

Lemma B.1. (i) $\mathrm{var}(\eta_\tau|\xi_\tau) \le v_\gamma$,

(ii) $\mathrm{var}(\Delta h_\tau|\xi_{\tau-1}) \le v_\gamma + v_{\Delta\pi}$,

(iii) $\mathrm{var}(\Delta h_\tau) \le v_\gamma + v_{\Delta\pi}$

(iv) if $\delta_{\tau-1}$ and $\sigma_{\tau-1}$ are bounded then $|X_\tau| + |Y_\tau| \le d_1 + d_2 V$ for some constants d_1, d_2 and random variable V with $E(V|\xi_{\tau-1}) = 0$ and bounded $\mathrm{var}(V|\xi_{\tau-1})$.

P r o o f. (i) follows from the fact that $\mathrm{var}(\eta_\tau|\xi_\tau) = \mathrm{var}(\gamma_\tau|\xi_\tau) = v_\gamma -$ [positive term], which we obtain by a procedure similar to the computation of $v_{\eta,t}$ in the previous proof (see (37)) with γ_t instead of π_t and with the covariances changed appropriately in (35).
(ii) Noting that $\mathrm{var}(\Delta h_\tau|\xi_{\tau-1}) = \mathrm{var}(h_\tau|\xi_{\tau-1}) = \mathrm{var}(\mathbb{E}(\pi_\tau|\xi_\tau)|\xi_{\tau-1})$ we get, by the Law of Iterated Variance, that

$$\begin{aligned}\mathrm{var}(\Delta h_\tau|\xi_{\tau-1}) &= \mathrm{var}(\pi_\tau|\xi_{\tau-1}) - \mathbb{E}(\mathrm{var}(\pi_\tau|\xi_\tau)|\xi_{\tau-1}) \\ &\le \mathrm{var}(\pi_\tau|\xi_{\tau-1}) = v_{\Delta\pi_\tau} + v_{\eta,\tau-1} \overset{(i)}{\le} v_{\Delta\pi} + v_\gamma.\end{aligned}$$

(iii) By the same Law,

$$\mathrm{var}(\Delta h_\tau) = \mathbb{E}(\mathrm{var}(\Delta h_\tau|\xi_{\tau-1}) + \mathrm{var}(\mathbb{E}(\Delta h_\tau|\xi_{\tau-1})) = \mathbb{E}(\mathrm{var}(\Delta h_\tau|\xi_{\tau-1})) \overset{(ii)}{\le} v_{\Delta\pi_\tau} + v_\gamma.$$

(iv) There exist constants $c_{1,2,3.4}$ such that the distributions would not change when we put $X_\tau = c_1 + c_2(a_{\tau-1} - \pi_{\tau-1}) + \sqrt{(c_3 + c_4(a_{\tau-1} - \pi_{\tau-1})) \vee 0}\, U_1$ with $U_1 \sim \mathcal{N}(0,1)$ independent of $\xi_{\tau-1}$. Further, because $a_\tau - \pi_\tau = \eta_\tau + \delta_\tau + \sigma_\tau$ and as $\eta_{\tau-1} = \sqrt{v_{\eta,\tau-1}}U_2$ for some $U_2 \sim \mathcal{N}(0,1)$ such that (U_1, U_2) is independent of $\xi_{\tau-1}$, we have

$$|X_\tau| \le k_1 + k_2|U_2| + \sqrt{k_3 + k_4|U_2|}|U_1| \le k_1 + k_2|U_2| + (\sqrt{k_3} + \sqrt{k_4}\sqrt{|U_2|})|U_1|$$

for some $k_{1,2,3,4}$. As the term has finite expectation and, by the Schwarz inequality, also finite second moment, and as $|Y_\tau|$ may be bounded analogously, we are getting (iv). □

[10]Strictly speaking, (11) and (10) hold only under additional assumption that the choice between possible multiple solutions does not depend on random variables which are not arguments of the value function.

Now assume (N). Recall that then

$$V_t(\xi_t) = \sup_{\substack{C_\tau,\delta_\tau,\sigma_\tau \text{ fulfilling} \\ \mathcal{E}(\tau),\mathcal{C}(\tau),\mathcal{A}(\tau),\mathcal{N}(\tau),\mathcal{M}(\tau),\tau\geq t}} \mathbb{E}\left[\sum_{\tau=t}^{\infty} e^{-\rho(\tau-t)}C_\tau \middle| \xi_t\right]. \tag{47}$$

First we prove that, for any $t \geq 0$,

$$V_t(\xi_t) = \lim_{T\to\infty} \tilde{V}_{t,T}(\xi_t), \qquad \tilde{V}_{t,T}(\xi_t) = \sup_{\substack{C_\tau,\delta_\tau,\sigma_\tau \text{ fulfilling} \\ \mathcal{E}(\tau),\mathcal{C}(\tau),\mathcal{A}(\tau),\mathcal{N}(\tau),\mathcal{M}(\tau),\tau\geq t}} \mathbb{E}\left[\sum_{\tau=t}^{T-1} e^{-\rho(\tau-t)}C_\tau \middle| \xi_t\right]. \tag{48}$$

Before doing so, note that that the Proposition holds for the problem underlying $\tilde{V}_{t,T}$ (indeed, our proof would be valid even if the constraint $\mathcal{C}(\tau)$ is added and the zero terminal criterion $\tilde{V}_{t,T}(\xi_T) = M_T = M_T + e^{h_{T-1}} \cdot 0$ is assumed).

Now let us show the limit and the expectation may be exchanged in (48). In order to do so, realize that, by (12), $\underline{\psi}e^{h_\tau} \leq C_\tau - M_\tau \leq \overline{\psi}e^{h_\tau}$ where $\underline{\psi} = \min_{\mathcal{C}(\tau),v\leq v_\gamma} \Psi(\delta_\tau,\sigma_\tau,v)$, $\overline{\psi} = \max_{\mathcal{C}(\tau),v\leq v_\gamma} \Psi(\delta_\tau,\sigma_\tau,v)$ whenever C_τ is optimal.[11] Consequently

$$|C_\tau - M_\tau| \leq e^{h_\tau}\psi, \qquad \psi = |\underline{\psi} + \overline{\psi}|,$$

and, further,

$$\begin{aligned} |C_\tau| \leq |C_\tau - M_\tau| + |M_\tau| &= |C_\tau - M_\tau| + |M_{\tau-1} + \Delta m_\tau - C_{\tau-1}| \\ &\leq \psi(e^{h_\tau} + e^{h_{\tau-1}}) + |\Delta m_\tau| \end{aligned}$$

with

$$\begin{aligned} |\Delta m_\tau| = |e^{a_{\tau-1}}X_\tau - e^{b_{\tau-1}}Y_\tau| &\leq e^{h_{\tau-1}}(e^{\delta_{\tau-1}+\sigma_{\tau-1}}|X_\tau| + e^{\delta_{\tau-1}-\sigma_{\tau-1}}|Y_\tau|) \\ &\leq e^{h_{\tau-1}}c(|X_\tau| + |Y_\tau|) \end{aligned}$$

for some c (thanks to $\mathcal{C}(\tau-1)$).

Now, let $\rho_1, \rho_2 > 0$ be such that $\rho_1 + \rho_2 = \rho$ and $\rho_2 \geq \frac{v_\gamma + v_{\Delta\pi}}{2}$. Assume $t = 0$ w.l.o.g. and denote $\epsilon_\tau = \exp\{-\rho\tau\}C_\tau$. We have

$$\begin{aligned} |\epsilon_\tau| &\leq \exp\{-\rho\tau\}\left(\exp\{h_\tau\}\psi + \exp\{h_{\tau-1}\}(\psi + c(|X_\tau| + |Y_\tau|))\right) \\ &= \psi\exp\{-\rho_1\tau + u_\tau\} + \psi\exp\{-\rho_1\tau + v_\tau\} + c\exp\{-\rho_1\tau + w_\tau\}, \end{aligned}$$

$$u_\tau = -\rho_2\tau + h_\tau, \qquad v_\tau = -\rho_2\tau + h_{\tau-1}, \qquad w_\tau = -\rho_2\tau + h_{\tau-1} + \log(|X_\tau| + |Y_\tau|).$$

As, by Proposition (3.1) (ii), h_τ is a ξ_τ-martingale with differences having bounded variance (by Lemma B.1 (iii)), we have $\frac{1}{\tau}h_\tau \to 0$ a.s. by the Strong LLN for Martingale Differences ([20] Ch. 15 par 4.1. Theorem 8). As the probability space may be chosen so that $\frac{1}{\tau}(w_\tau - h_\tau) \to -\rho_1$ a.s.,[12] we have $\frac{1}{\tau}w_\tau \to -\rho_1$ a.s. implying $w_\tau \to -\infty$ a.s.

[11] The minimum and maximum of Ψ exist as Ψ is continuous on a compact, the variable v may be restricted thanks to Lemma B.1 (i).

[12] To see this, note that $\frac{1}{\tau}|X_\tau| \to 0$ in distribution thanks to Lemma B.1 (iv) and that, by the Almost Sure Representation, we cam choose the probability space space such that the convergence is a.s., similarly for $|Y_\tau|$.

Consequently, there exists a r.v. K, finite almost sure, such that $w_\tau \le 0, \tau \ge K$, which further gives

$$\sum_{\tau=0}^{\infty} \exp\{-\rho_1\tau + w_\tau\} = \sum_{\tau=0}^{K-1} \exp\{-\rho_1\tau + w_\tau\} + \sum_{\tau=K}^{\infty} \exp\{-\rho_1\tau\} < \infty. \qquad \text{a.s.}$$

After analogous procedures with u_τ and v_τ we get that that $\sum_\tau^t |\epsilon_\tau|$ converges a.s., which implies a.s. convergence of $\sum_\tau^t \epsilon_\tau$.

Further, as $\mathbb{E}(\exp\{\Delta h_\tau - \rho_2\}|\xi_{\tau-1}) = \exp\{\frac{1}{2}\mathrm{var}(\Delta h_\tau|\xi_{\tau-1}) - \rho_2\} \le \exp\{\frac{1}{2}(v_{\Delta\pi} + v_\gamma) - \rho_2\} \le 1$ (by Lemma B.1 (ii) and by (N)) and $\mathbb{E}(|X_\tau| + |Y_\tau||\xi_{\tau-1}) \le d_1$ (by Lemma B.1 (iv)), we have

$$\mathbb{E}e^{w_\tau} = e^{-\rho_2}\mathbb{E}(e^{\Delta h_1 - \rho_2}\mathbb{E}(\dots\mathbb{E}(e^{\Delta h_{\tau-1}-\rho_2}\mathbb{E}(\psi + c\mathbb{E}(|X_\tau| + |Y_\tau||\xi_{\tau-1})\xi_{t-2}))\dots)\xi_1) \\ \le \phi + cd_1,$$

which gives, together with similar inequalities for e^{u_τ} and e^{v_τ}, that $\mathbb{E}|\epsilon_\tau|) \le \exp\{-\rho_1\tau\}k$ for some constant k. Therefore and by the Monotone Convergence Theorem, $\mathbb{E}(\sum_\tau^\infty |\epsilon_\tau|) = \sum_\tau^\infty \mathbb{E}(|\epsilon_\tau|) < \infty$. i.e. $\sum_\tau^\infty |\epsilon_\tau|$ may serve as an integrable majorant for $\sum_\tau^t \epsilon_\tau$; this, together with the convergence of $\sum_\tau^t \epsilon_\tau$, proves the desired interchangeability by the Fatou-Lesbeque Theorem.

The fact that the sup and the lim may be interchanged follows from variational analysis theory: First, let us take

$$p_\tau = \varphi(C_\tau), \qquad \tau > 0,$$

as a decision variable rather than C_τ to get a compact feasible set. Further, since $\varphi^{-1}(\bullet)$ is monotone and differentiable, the objective functions $f_T = \mathbb{E}(\sum_{\tau=t}^{T-1} e^{-\rho(\tau-t)}\varphi^{-1}(p_\tau)|\xi_t)$, $t \in \mathbb{N}$, are monotone and differentiable, too. Monotonicity, differentiability and compactness imply the epi-convergence of f_T to $f_\infty = \mathbb{E}(\sum_{\tau=t}^{\infty} \varphi^{-1}(p_\tau)|\xi_t)$. Finally, having compact set of feasible solutions, the epi-convergence of objective functions allows for interchange of supremum and limit. See [1] (Theorem 1.10, Theorem 2.11) or [19](Th. 7.33) for more details.

Thus, we have proved (49).

Further, by (8) with $\tilde{V}_{t,T}$ instead of V_t (the fact that (8) holds for $\tilde{V}_{t,T}$ follows from the discussion above),

$$V_t(\xi_t) = M_t + e^{h_t}W(N_t, v_{\eta,t}, \infty),$$

where $W(N, v, \infty) = \lim_{T\to\infty} W(N, v, T-t)$ where W is computed from the problem underlying $\tilde{V}_{t,T}$. Using this and the Bellman equation

$$V_t(\xi_t) = \sup_{\delta_t,\sigma_t,C_t,\ \text{fulfilling}\ \mathcal{C}(t),\mathcal{A}(t),\mathcal{M}(t),\mathcal{N}(t)} \left[C_t + e^{-\rho}\mathbb{E}(V_{t+1}(\xi_{t+1})|\xi_t)\right]. \qquad (49)$$

easily following from (47), we further get the Proposition the same way as in case (F).

C. ASYMPTOTIC PROPERTIES OF THE LS ESTIMATOR

Denote

$$\theta = (n_0, \tilde{s}_0, \phi_0, n_N, s_N, \phi_{\Delta N}, \phi_N, n_S, s_S, \phi_{S,1}, \phi_{S.2}, \omega_\sigma, \omega_v, \alpha_1, \alpha_2)$$

the vector of true values of the parameters and assume that

A1 the parameter space $\mathcal{S}$ is bounded (the bound may be arbitrarily large);

A2 vector θ is such that there exist (possibly very large) K, L, such that, with probability 1, there exists an infinite increasing sequence $(k_1, k_2, \dots)$ fulfilling

$$(N_{k_i - j}, \ln \sigma_{k_i - j}) \in [-K, K] \times [-L, L], \qquad 0 < j \le j_0, \qquad i \in \mathbb{N}, \tag{50}$$

(note that then necessarily $n_N < 0$ because otherwise N_t would explode).

First, let us renumber the observations so that observations violating (50) (i. e., observations with indices other than k_i) are excluded[13] and rewrite the estimated equations (26), (27) and (28), weighted by corresponding standard errors, as

$$Y_n = \lambda_n g_n(\theta) + h_n, \qquad n \in \mathbb{N},$$

where, for any $k \in \mathbb{N} \cup \{0\}$,

$$Y_{3k+1} = \lambda_{k,1} \Delta N_k, \qquad Y_{3k+2} = \lambda_{k,2} \ln \tilde{\sigma}_k, \qquad Y_{3k+3} = \lambda_{k,3} \Delta \tilde{P}_k$$

$$h_{3k+1} = \lambda_{k,1} \mathcal{F}_k, \qquad h_{3k+2} = \lambda_{k,2} \mathcal{E}_k, \qquad Y_{3k+3} = \lambda_{k,3} \mathcal{G}_k,$$

$$\lambda_{k,1} = \frac{1}{\hat{s}_{k,N}}, \qquad \lambda_{k,2} = \frac{1}{\hat{s}_{k,\sigma}}, \qquad \lambda_{k,3} = \frac{1}{\hat{s}_{k,P}},$$

where $\hat{s}_{k,\sigma}, \hat{s}_{k,N}$ and $\hat{s}_{k,P,}$ are standard errors of residuals from (26), (27) and (28), respectively, estimated individually, and

$$g_{3k+j} = g_{k,j}, \qquad k \in \mathbb{N}, \quad j \in \{1, 2, 3\},$$

where

$$g_{k,1}(n_0, n_N, n_S, \omega_\sigma, \omega_v, \alpha_1, \alpha_2) = n_0 + n_N \sum_{\tau=1}^{k-1} \Delta N_\tau + n_S \exp\left(\omega_\sigma Z(\tilde{\sigma}_{k-2}, \dots, \tilde{\sigma}_{k-j_0-3}; \omega_v, \alpha_1, \alpha_2)\right)$$

$$g_{k,2}(\tilde{s}_0, s_N, s_S, \omega_v, \alpha_1, \alpha_2) = \tilde{s}_0 + \alpha_1 \tilde{\sigma}_{k-1} + \alpha_2 \tilde{\sigma}_{k-1}^2 + s_N \sum_{\tau=1}^{k} \Delta N_\tau + s_S Z(\tilde{\sigma}_{k-1}, \dots, \tilde{\sigma}_{k-j_0-2}; \omega_v, \alpha_1, \alpha_2)$$

[13]In the actual data, both $\tilde{\sigma}_t$ and N_t seem to be mean reverting so, given large enough bounds, no observation should be thrown out.

$$
\begin{aligned}
&g_{k,3}(\phi_0, \phi_{\Delta N}, \phi_N, \phi_{S,1}, \phi_{S,2}, \omega_v, \alpha_1, \alpha_2) \\
&\quad = \phi_0 + \phi_{\Delta N}\Delta N_k + \phi_N \sum_{\tau=1}^{k-1} \Delta N_\tau + \phi_{S,1}\exp(\omega_\sigma Z(\tilde{\sigma}_{k-1}, \dots, \tilde{\sigma}_{k-j_0-2}; \omega_v, \alpha_1, \alpha_2)) \\
&\qquad\qquad + \phi_{S,2}\exp(\omega_\sigma Z(\tilde{\sigma}_{k-2}, \dots, \tilde{\sigma}_{k-j_0-3}; \omega_v, \alpha_1, \alpha_2)).
\end{aligned}
$$

First we show that the residuals of Y's are martingale differences:

Proposition C.1. $\mathbb{E}(h_{k,j}|\mathcal{H}_{3k+j}) = 0$ where $\mathcal{H}_{3k+j} = \begin{cases} \tilde{\xi}_{k-1} & \text{if } j = 1 \\ (\tilde{\xi}_{k-1}, \Delta N_k) & \text{if } j > 1 \end{cases}$.

P r o o f. Let $j > 1$ (the slightly more complicated case) first. The assertion follows from [23], Lemma A.1 (ii); to see it, put $U = (\epsilon^\star_k, \dots, \epsilon^\star_{k-j_0-1}, \mathcal{E}_k, \mathcal{G}_k)$. and $V = (\xi_{k-1}, N_k)$ in the Lemma. For $j = 1$, the proof is similar. □

Further, let us assume

A3 $n_S \neq 0, \omega_\sigma \neq 0$

A4 All moments of $\epsilon^\star_1$ are finite and the fourth moment of $\log \epsilon^\star_1$ is finite.

Remark C.2. Note that (A4) together with (A1) and (A2) implies that (A4) holds with $\epsilon_{\sigma,1}$ instead of $\epsilon^\star_1$.

Consistency

For any $\hat{\theta}$ and k and j, denote

$$\Delta\hat{\theta} = \hat{\theta} - \theta, \qquad d_{k,j} = d_{k,j}(\hat{\theta}) = g_{k,j}(\hat{\theta}) - g_{k,j}(\theta),$$

First, note that, thanks to the fact that the variance of the residuals in (26), (27) and (28) is bounded from below (in the less transparent case (27) it is because $v_{N,t}$ is bounded from below), we have that

$$\liminf_k \hat{s}_{k,\sigma} > 0, \qquad \liminf_k \hat{s}_{k,N} > 0, \qquad \liminf_k \hat{s}_{k,P} > 0. \tag{51}$$

We gradually verify the conditions, sufficient for the consistency by [14]:

LIP($f_k(\theta)$) holds because in for all j the derivatives of $g_{k,j}$ are continuous bounded (which is thanks to A1).

SI($\{D_n(\theta)\}$) Denote $\|\|$ the max norm. Note that the condition is satisfied if, for any $\delta > 0$ and $\hat{\theta}$ fulfilling $\left\|\Delta\hat{\theta}\right\| \geq \delta$,

$$H_j = \infty, \qquad H_j := \sum_{k=1}^{\infty} d^2_{k,j} \tag{52}$$

almost surely for at least one $j \in \{1,2,3\}$; note further that this is satisfied if, for some j, there exists a filtration $\mathcal{I}_i$, a constant ϵ and a (possibly random) subsequence k_i fulfilling

$$\operatorname{var}(d_{k_i,j}|\mathcal{I}_i) \geq \epsilon, \tag{53}$$

$$\operatorname{var}(\delta_{k_i,j}) < \infty, \qquad \delta_{i,j} = d^2_{i,,j} - \mathbb{E}(d^2_{i,,j}|\mathcal{I}_i) \tag{54}$$

and

$$\lim_\tau \sum_{i=1}^{\tau} (\operatorname{var}(\delta_{k_i,j}|\mathcal{I}_i))/\tau^2 < \infty \tag{55}$$

in which case

$$\frac{1}{\tau}\sum_{i=1}^{\tau} \delta_{k_i,j} = 0 \tag{56}$$

a.s. by the Strong LLN for martingale differences ([21] p 487, Theorem 4), implying

$$\lim \frac{1}{\tau}\sum_{i=1}^{\tau} d^2_{k_i,j} = \lim \frac{1}{\tau}\sum_{k=1}^{\tau}[\delta_{k_i,j} + \mathbb{E}(d^2_{k_i,j}|\mathcal{I}_i)] \geq \lim \frac{1}{\tau}\sum_{k=1}^{\tau}[\delta_{k_i,j} + \operatorname{var}(d^2_{k_i,j}|\mathcal{I}_i)] \geq \epsilon$$

by (56), further yielding $\sum d^2_{k,j} = \infty$ which suffices for (52).
In particular, if, for some j,

$$d_{k,j} = c_k + f_k \tag{57}$$

where c_k is $\mathcal{I}_k$ measurable then

$$\delta_{k,j} = 2c_k(f_k - \mathbb{E}(f_k|\mathcal{I}_k)) + (f_k^2 - \mathbb{E}(f_k^2|\mathcal{I}_k))$$

so, thanks to the fact that $(x+y)^2 \leq 3(x^2+y^2)$ and the Hölder inequality,

$$\begin{aligned}\operatorname{var}(\delta_{k,j}|\mathcal{I}_k) = \mathbb{E}(\delta^2_{k,j}|\mathcal{I}_k) &\leq 12c_k^2 \operatorname{var}(f_k|\mathcal{I}_k) + 3\operatorname{var}(f_k^2|\mathcal{I}_k)\\ &\leq 12c_k^2\mathbb{E}(f_k^2|\mathcal{I}_k) + 3\mathbb{E}(f_k^4|\mathcal{I}_k) \leq 12c_k^2\sqrt{\mathbb{E}(f_k^4|\mathcal{I}_k)} + 3(f_k^4|\mathcal{I}_k).\end{aligned}$$

Thus, for (55) to hold, it suffices when the conditional fourth moments of f_k and the conditional second moments of c_k are uniformly bounded by constants F, C, respectively which, in addition, proves (54) because

$$\operatorname{var}(\delta_{k,j}) = \mathbb{E}(\delta^2_{k,j}) \leq 12\mathbb{E}(|c_k|f_k^2) + 3\mathbb{E}(f_k^4) \leq 12\sqrt{\mathbb{E}c_k^2\mathbb{E}(f_k^4)} + 3\mathbb{E}(f_k^4) \leq 12\sqrt{CF} + 3F.$$

Coming to condition SI itself, agree to write Δn_0, Δs_0, etc., for the first, second, etc., component of $\Delta\hat{\theta}$ and assume first that at least one of values $\Delta\alpha_1$,$\Delta\alpha_2$,Δs_S is non-zero. Then

$$d_{k,2} = \Delta\tilde{s}_0 + \Delta\alpha_1\tilde{\sigma}_{k-1} + \Delta\alpha_2\tilde{\sigma}^2_{k-1} + \Delta s_N\sum_i^k \Delta N_i + \Delta s_S Z_{k,1} + s_S\Delta Z_{k,1}$$

which may be expressed in the language of (57) with

$$\mathcal{I}_k = (\xi_\infty, \epsilon_{\sigma,1}, \ldots, \epsilon_{\sigma,k-2}),$$

$$c_k = \Delta\tilde{s}_0 + \Delta s_N \sum \Delta N_\tau + \Delta s_S (\sum_{j=1}^{j_0} \omega_v^j \ln \tilde{\sigma}_{k-1-j} - \sum_{j=0}^{j_0} \omega_v^j (\alpha_1 \tilde{\sigma}_{k-2-j} + \alpha_2 \tilde{\sigma}^2_{k-2-j}))$$
$$+ s_S (\sum_{j=1}^{j_0} (\hat{\omega}_v^j - \omega_v^j) \ln \tilde{\sigma}_{k-1-j} - \sum_{j=0}^{j_0} \omega_v^j (\Delta\alpha_1 \tilde{\sigma}_{k-2-j} + \Delta\alpha_2 \tilde{\sigma}^2_{k-2-j})$$

and

$$f_k = a_k E_k, \qquad a_k = \begin{pmatrix} \Delta\alpha_1 \eta_k \\ \Delta\alpha_2 \eta_k^2 \\ \Delta s_S \end{pmatrix}, \qquad E_k = \begin{pmatrix} \epsilon^\star_{k-1} \\ (\epsilon^\star_{k-1})^2 \\ \ln \epsilon^\star_{k-1} \end{pmatrix}$$
$$\eta_k = \sigma_{k-1} \exp(\alpha_1 \tilde{\sigma}_{k-1} + \alpha_2 \tilde{\sigma}^2_{k-1}).$$

As the residuals in (26) are i.i.d. non-degenerated, we can select a sequence k_i such that $\tilde{\sigma}_{k_i} \geq h$ for some $h > 0$. Thus, there exists $\epsilon_\eta > 0$ fulfilling $\eta_{k_i} \geq \epsilon_\eta$ for each i, which implies existence of $\epsilon_a > 0$ such that $\|a_{k_i}\| = \alpha_1^2 \eta_k^2 + \alpha_2^2 \eta_k^4 + s_S^2 \geq \epsilon_a$ and, consequently, $\mathrm{var}(d_{k,2}|\mathcal{I}_k) = a'_{k_i} \mathrm{var}(E_1) a_{k_i} \geq \epsilon_a^2 \lambda_{\min}$ where $\lambda_{\min}$ is the least eigenvalue of $\mathrm{var}(E_1)$ i. e. (53) is fulfilled. Further, by A2 and A4 and by the fact that $\epsilon^\star_{k_i}$ are i.i.d., $\mathbb{E}c^2_{k_i}$ and $\mathbb{E}f^4_{k_i}$ are uniformly bounded, hence also (54) and (55) are verified. Summarized, (52) is proved if at least one of values $\Delta\alpha_1$,$\Delta\alpha_2$,Δs_S is non-zero.

Now let $\Delta\alpha_1, \Delta\alpha_2, \Delta s_S$ be zero but let $\Delta\omega_v$ be non-zero. Then

$$d_{k,2} = \Delta s_0 + \Delta s_N \sum \Delta N_\tau + s_S (\sum_{j=1}^{j_0} (\hat{\omega}_v^j - \omega_v^j) \tilde{\sigma}_{k-j-1}) = s_S (\sum_{j=1}^{j_0} (\hat{\omega}_v^j - \omega_v^j)(\tilde{\sigma}_{k-j-1} + q_t)),$$

where $q_t = \frac{\Delta s_0 + s_N \sum \Delta N_\tau}{s_S \sum_{j=1}^{j_0} (\omega_v^j - \hat{\omega}_v^j)}$: as neither $\tilde{\sigma}_k$ nor N_k is convergent (note that the variances in the residuals in (27) are bounded from below), a subsequence k_i exists such that $d^2_{k_i,2} \geq \epsilon$ directly implying (52).

If $\Delta\alpha_1, \Delta\alpha_2, \Delta s_S, \Delta\omega_v$ are zero but at least one of the values $\Delta s_N, \Delta s_0$ are non zero then, for similar reasons, ϵ and k_i exist such that $d^2_{k_i,2} \geq \epsilon$ yielding (52).

Let now $\Delta\alpha_1, \Delta\alpha_2, \Delta s_S, \Delta\omega_v, \Delta s_N, \Delta s_0$ be zero (implying that $\Delta Z_{t,2} = 0$). Then we can decompose $d_{k,2}$ according to (57) as

$$d_{k,2} = c_k + f_k$$

with $\mathcal{I}_k = (\xi_\infty, \epsilon_{\sigma,1}, \ldots, \epsilon_{\sigma,k-3})$.

$$c_k = \Delta n_0 + \Delta n_N \sum \Delta N_\tau,$$

$$f_k = \Delta n_S \exp(\hat{\omega}_\sigma Z_{t,2}) + n_S [\exp(\hat{\omega}_\sigma Z_{t,2}) - \exp(\omega_\sigma Z_{t,2})]$$
$$= (\Delta n_s + n_S)(\epsilon^\star_{k-2})^{\hat{\omega}_\sigma} \exp(\hat{\omega}_\sigma z_k) - n_S (\epsilon^\star_{k-2})^{\omega_\sigma} \exp(\omega_\sigma z_k) = b_k F_k$$

where

$$b_k = ((\Delta n_s + n_S) \exp(\hat{\omega}_\sigma z_k), -n_S \exp(\omega_\sigma z_k))$$
$$= \exp(\hat{\omega}_\sigma z_k)\, n_S \left((\frac{\Delta n_s}{n_S} + 1), -\exp(-\Delta\omega_\sigma z_k) \right)$$
$$F_k = ((\epsilon^\star_{k-2})^{\hat{\omega}_\sigma}, (\epsilon^\star_{k-2})^{\omega_\sigma}), \qquad z_k = Z_{k,2} - \ln \epsilon^\star_{k-2}$$

so

$$\operatorname{var}(f_k|\mathcal{I}_k) = b'_k V b_k$$

where V is a variance matrix of F_1 (note that it is regular because otherwise $\Delta\omega_\sigma$ would be zero). Note also that

$$\|b_k\|^2 = \exp(2\hat{\omega}_\sigma z_k)\, n_S^2 \left[\left(\frac{\Delta n_s}{n_S} + 1 \right) - \exp(-\Delta\omega_\sigma z_k) \right]^2.$$

If at least one of values $\Delta\omega_\sigma, \Delta n_S$ is non-zero then, thanks to A3 and the non-convergence of $\tilde{\sigma}'$s, there exists k_i and $h > 0$ such that $\|b_{k_i}\| > h$ implying (53) analogously as above. Thanks to A1 and A2, also (54) holds true so (52) is satisfied. Let now $\Delta\alpha_1, \Delta\alpha_2, \Delta s_S, \Delta s_N, \Delta s_0, \Delta\omega_v, \Delta n, \Delta\omega_\sigma, \Delta n_S$ be zero but at least one of the values $\Delta n_0, \Delta n_N$ be non zero. Then

$$d_{k,1} = \Delta n_0 + \Delta n_N \sum_\tau \Delta N_\tau,$$

which diverges thanks to the non-convergence of N_k which again implies (52). Finally, let $\Delta\alpha_1, \Delta\alpha_2, \Delta s_S, \Delta s_N, \Delta s_0, \Delta\omega_v, \Delta n, \Delta\omega_\sigma, \Delta n_S, \Delta n_0, \Delta n_N$ be zero. Then, however,

$$d_{k,3} = \Delta\phi_0 + \Delta\phi_{\Delta N} \Delta N_k + \Delta\phi_N N_k + \Delta\phi_{S,1} \exp(\omega_\sigma Z_{k,1}) + \Delta\phi_{S,2} \exp(\omega_\sigma Z_{k,2})$$

with at least one of the coefficients being non-zero – the proof of (52) follows the non-convergence of σ_k and ΔN_k similarly as above.

VAR(σ_k) Thanks to (51) and the fact that only observations fulfilling A2 are involved, it is easily seen from (26), (27) and (28) that $\limsup_k \operatorname{var}(Y_k|\mathcal{F}_{k-1}) < \infty$ which suffices for VAR by [14], Section 5.

The consistency now follows from Proposition 3.1. of [14].

Asymptotic normality

Finally, we verify the conditions sufficient form the asymptotic normality. In order to do this, we put an additional assumption

A5 Limit of

$$L_k(\theta) = \frac{1}{k} \sum_{i=1}^{k} \mathbf{h}_i(\theta) \mathbf{h}_i^T(\theta)$$

is deterministic where $\mathbf{h}'_{3k+j} = \mathbf{h}'_{k,j}$, $j \in \{1, 2, 3\}$,

$$\mathbf{h}_{k,1}(\theta) = \frac{1}{\lambda_{k,1}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \sum^{k-1} \Delta N_\tau \\ 0 \\ 0 \\ 0 \\ \exp\{\omega_\sigma Z_{k,2}\} \\ 0 \\ 0 \\ 0 \\ n_S Z_{k,2} \exp\{\omega_\sigma Z_{k,2}\} \\ n_S \omega_\sigma \exp\{\omega_\sigma Z_{k,2}\} \frac{\partial}{\partial \omega_v} Z_{k,2} \\ n_S \omega_\sigma \exp\{\omega_\sigma Z_{k,2}\} \frac{\partial}{\partial \alpha_1} Z_{k,2} \\ n_S \omega_\sigma \exp\{\omega_\sigma Z_{k,2}\} \frac{\partial}{\partial \alpha_2} Z_{k,2} \end{bmatrix}, \quad \mathbf{h}_{k,2}(\theta) = \frac{1}{\lambda_{k,2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \sum_\tau^k \Delta N_\tau \\ 0 \\ 0 \\ 0 \\ Z_{k,1} \\ 0 \\ 0 \\ 0 \\ s_S \frac{\partial}{\partial \omega_v} Z_{k,1} \\ \tilde{\sigma}_{k-1} + s_S \frac{\partial}{\partial \alpha_1} Z_{k,1} \\ \tilde{\sigma}^2_{k-1} + s_S \frac{\partial}{\partial \alpha_2} Z_{k,1} \end{bmatrix},$$

$$\mathbf{h}_{k,3}(\theta) = \frac{1}{\lambda_{k,3}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \Delta N_k \\ \sum_\tau^{k-1} \Delta N_\tau \\ 0 \\ 0 \\ \exp\{\omega_\sigma Z_{k,1}\} \\ \exp\{\omega_\sigma Z_{k,2}\} \\ \phi_{S,1} Z_{k,1} \exp\{\omega_\sigma Z_{k,1}\} + \phi_{S,2} Z_{k,2} \exp\{\omega_\sigma Z_{k,2}\} \\ \phi_{S,1} \exp\{\omega_\sigma Z_{k,1}\} \frac{\partial}{\partial \alpha_1} Z_{k,1} + \phi_{S,2} \exp\{\omega_\sigma Z_{k,2}\} \frac{\partial}{\partial \alpha_1} Z_{k,2} \\ \phi_{S,1} \exp\{\omega_\sigma Z_{k,1}\} \frac{\partial}{\partial \alpha_2} Z_{k,1} + \phi_{S,2} \exp\{\omega_\sigma Z_{k,2}\} \frac{\partial}{\partial \alpha_2} Z_{k,2} \\ \phi_{S,1} \exp\{\omega_\sigma Z_{k,1}\} \frac{\partial}{\partial \omega_v} Z_{k,1} + \phi_{S,2} \exp\{\omega_\sigma Z_{k,2}\} \frac{\partial}{\partial \omega_v} Z_{k,2} \end{bmatrix}$$

with $\frac{\partial}{\partial \alpha_1} Z_{k,i} = -\sum_{j=0}^{j_0} \omega_v^j \tilde{\sigma}_{t-i-1-j}$, $\frac{\partial}{\partial \alpha_2} Z_{k,i} = -\sum_{j=0}^{j_0} \omega_v^j \tilde{\sigma}_{t-i-1-j}$, and $\frac{\partial}{\partial \omega_v} Z_{k,i} = \sum_{j=1}^{j_0} j \omega_v^{j-1} (\ln \tilde{\sigma}_{t-i-j} - \alpha_1 \tilde{\sigma}_{t-i-1-j} - \alpha_2 \tilde{\sigma}^2_{t-i-1-j})$

Before verifying the conditions required for the asymptotic normality, note that

$$M_n = \frac{1}{n} L_n^{-1}$$

for matrix M_n defined in Section 6 of [14].

Now let us verify the conditions, required for the normality by [14].

UNC($\{\theta_n\}$) Because the first and the second derivatives of $\mathbf{h}$ are bounded (say by a constant C) and because the parameter space S is compact, we have

$$\begin{aligned}
\left|\frac{1}{k}\sum_{i=1}^{k}\mathbf{h}_i(\theta_n)\mathbf{h}_i^T(\theta_n)L_k^{-1}-I\right| &= \left|\frac{1}{k}\left(\sum_{i=1}^{k}\mathbf{h}_i(\theta_n)\mathbf{h}_i^T(\theta_n)-\frac{1}{k}\sum_{i=1}^{k}\mathbf{h}_i(\theta)\mathbf{h}_i^T(\theta)\right)L_k^{-1}\right| \\
&\le \frac{1}{k}\left\|\sum_{i=1}^{k}\mathbf{h}_i(\theta_n)(\mathbf{h}_i(\theta_n)-\mathbf{h}_i(\theta))^T+\sum_{i=1}^{k}\mathbf{h}_i(\theta)(\mathbf{h}_i(\theta_n)-\mathbf{h}_i(\theta))^T\right\|\|L_k^{-1}\| \\
\le \Bigg(\left\|\frac{1}{k}\sum_{i=1}^{k}\mathbf{h}_i(\theta_n)(\mathbf{h'}_i(\theta)+o(\theta_n-\theta))^T\right\| &+ \left\|\frac{1}{k}\sum_{i=1}^{k}\mathbf{h}_i(\theta)(\mathbf{h}_i'(\theta)+o(\theta_n-\theta))^T\right\|\Bigg) \\
&\times\|\theta_n-\theta\|\left\|L_k^{-1}\right\| \\
&\le \|\theta_n-\theta\|\left(2C+o(\theta_n-\theta)\right)\left\|L_k^{-1}\right\|\to 0
\end{aligned}$$

which proves UNC.

LIM$\{\{\theta_n\}\}$ would be proved similarly

SI$\{(M_n[l,j])^{-1}\}$ For SI to hold, it suffices that there exists n_0 such that

$$|L_k^{-1}[l,j]|\le K, \qquad n\ge n_0,$$

for some deterministic K because then

$$\left|\frac{1}{M_n[l,j]}\right| = \left|\frac{n}{L_n[l,j]}\right| \ge \frac{n}{K}, \qquad n\ge n_0, \tag{58}$$

which verifies SI. However, as the inversion of a matrix is continuous in all the points in which the inversion exists, (58) is verified by A5.

Now, thanks to A5 and the CLT for martingale arrays (as cited e. g. in [14], Sec 8, p. 24), the estimate is asymptotically normal by Proposition 6.2 of [14].

Remark C.3. Note that the seemingly arbitrary assumptions A2 and A5 are satisfied if the process (a_t, b_t, N_t) is stationary ergodic.

D. ESTIMATES OF NOISED APPROXIMATE DYNAMICS

The following table shows a brief descriptive statistics and results of a joint estimation of equations (26), (27) and (28) from Section 6 for each stock-market pair. The number of stars denote significance on levels 0.05, 0.01 and 0.001, respectively. The actual values of parameters σ_N, ϕ_N and n_N are scaled by 10^5, parameters $\phi_{S,1}$ and $\phi_{S,2}$ are scaled by 10^8.

D.1. XOM

XOM / ISE

Volume/s: 7.09537
Trades/s: 0.05105
Avg. spread: 0.05510

ω_σ 0.05344(0.019317) **
ω_v 0.87165(0.0073743) ***
α_1 426.2(7.7373) ***
α_2 $-$ 48407(1882.5) ***

$\tilde{s}_0$ $-$ 3.6899(0.033508) ***
s_N $-$ 0.018084(0.0058002) ***
s_S 0.13473(0.0024467) ***

ϕ_0 $-$ 0(0) *
$\phi_{\Delta N}$ $-$ 0.024115(0.00023285) ***
ϕ_N $8.0031e-06(3.5142e-06)$ *
$\phi_{S,1}$ $5.0649e+05(1.0585e+05)$ ***
$\phi_{S,2}$ $-4.5818e+05(97310)$ ***

n_0 12.469(10.676)
n_N $-$ 9.8757(2.3125) ***
n_S $-$ 93.184(43.857) *

XOM / NASD OB

Volume/s: 4.5922
Trades/s: 0.054805
Avg. spread: 0.022481

ω_σ $0.95(6.099e+12)$
ω_v 1(0.0070012) ***
α_1 893.54(16.052) ***
α_2 $-1.1861e+05(4888.5)$ ***

$\tilde{s}_0$ $-$ 3.4517(0.035459) ***
s_N $-$ 0.16428(0.012957) ***
s_S 0.11421(0.0020317) ***

ϕ_0 $-$ 0(0) ***
$\phi_{\Delta N}$ $-$ 0.073961(0.00092821) ***
ϕ_N $4.3614e-05(1.0476e-05)$ ***
$\phi_{S,1}$ $3.5831e+05(7.1984e+19)$
$\phi_{S,2}$ $-3.8423e+05(8.453e+19)$

n_0 0.35364(0.28444)
n_N $-$ 4.5709(2.2478) *
n_S $7.0504(3.5887e+15)$

XOM / NSE

Volume/s: 6.4547
Trades/s: 0.041187
Avg. spread: 0.37276

ω_σ 0.083754(0.018494) ***
ω_v 0.77437(0.00543) ***
α_1 105.7(1.1647) ***
α_2 $-$ 1271.5(25.103) ***

$\tilde{s}_0$ $-$ 2.4592(0.020582) ***
s_N 0.0016863(0.00079383) *
s_S 0.18717(0.0022769) ***

ϕ_0 $-$ 0(0) *
$\phi_{\Delta N}$ $-$ 0.057546(0.0015561) ***
ϕ_N $9.0614e-07(1.9825e-06)$
$\phi_{S,1}$ $1.0607e+06(1.5185e+05)$ ***
$\phi_{S,2}$ $-1.0353e+06(1.4727e+05)$ ***

n_0 10.449(5.0425) *
n_N 1.3314(0.26952) ***
n_S $-$ 90.467(24.021) ***

XOM / NASD ADF

Volume/s: 26.465
Trades/s: 0.15398
Avg. spread: 0.32196

ω_σ 0.059761(0.0029591) ***
ω_v 0.96131(0.003483) ***
α_1 57.467(2.3121) ***
α_2 $-$ 735.07(155.42) ***

$\tilde{s}_0$ $-$ 2.5948(0.02412) ***
s_N $-$ 0.0030441(0.00010484) ***
s_S 0.12215(0.0011744) ***

ϕ_0 0(0)
$\phi_{\Delta N}$ $-$ 0.066904(0.0010776) ***
ϕ_N $-2.2403e-07(1.5121e-07)$
$\phi_{S,1}$ $2.7007e+06(93368)$ ***
$\phi_{S,2}$ $-2.6154e+06(89575)$ ***

n_0 197.8(5.9637) ***
n_N $-$ 0.72481(0.031148) ***
n_S 376.77(29.54) ***

XOM / NYSE

Volume/s: 118.22
Trades/s: 0.55785
Avg. spread: 0.025279

ω_σ $0.95(1.1359e+13)$
ω_v 1(0.014246) ***
α_1 1474.7(11.841) ***
α_2 $-8.2627e+05(12192)$ ***

$\tilde{s}_0$ $-$ 5.9748(0.020342) ***
s_N $-0.001624(1.1325e-05)$ ***
s_S 0.02855(0.0011552) ***

ϕ_0 0(0) **
$\phi_{\Delta N}$ $-0.0050195(3.6664e-05)$ ***
ϕ_N $4.9935e-08(3.9065e-08)$
$\phi_{S,1}$ $-2.5328e+05(1.1494e+20)$
$\phi_{S,2}$ $-2.4135e+05(1.0479e+20)$

n_0 396.08(9.7161) ***
n_N $-$ 1.2386(0.1736) ***
n_S $-25569(1.1499e+19)$

XOM / ARCA

Volume/s: 86.567
Trades/s: 0.81679
Avg. spread: 0.014917

ω_σ $0.95(8.1009e+12)$
ω_v 1(0.0056599) ***
α_1 934.99(3.1889) ***
α_2 7983.5(3911.8) *

$\tilde{s}_0$ $-$ 5.7256(0.0060374) ***
s_N $-0.024096(4.61e-05)$ ***
s_S 0.021896(0.00034323) ***

ϕ_0 $-$ 0(0) ***
$\phi_{\Delta N}$ $-0.02135(6.1542e-05)$ ***
ϕ_N $5.5859e-06(8.0615e-07)$ ***
$\phi_{S,1}$ $-1.5334e+06(5.0362e+20)$
$\phi_{S,2}$ $1.5346e+06(5.0555e+20)$

n_0 41.901(3.5983) ***
n_N $-$ 4.8412(1.4614) ***
n_S $-81937(2.6208e+19)$

XOM / NASDAQ T

Volume/s: 114.54
Trades/s: 0.98704
Avg. spread: 0.015841

ω_σ $0.95(3.7084e+13)$
ω_v 1(0.0034162) ***
α_1 911.21(6.0542) ***
α_2 8284.3(10181)

$\tilde{s}_0$ $-$ 4.8612(0.0067711) ***
s_N $-0.0011634(2.3809e-05)$ ***
s_S 0.046558(0.00042438) ***

ϕ_0 $-$ 0(0) ***
$\phi_{\Delta N}$ $-0.019435(6.4418e-05)$ ***
ϕ_N $-1.563e-06(3.5772e-07)$ ***
$\phi_{S,1}$ $-21938(6.3972e+19)$
$\phi_{S,2}$ $5.9243e+05(8.642e+20)$

n_0 $-$ 42.272(4.5421) ***
n_N $-$ 3.4918(0.79933) ***
n_S $-973.61(1.5786e+18)$

XOM / CBOE

Volume/s: 0.23657
Trades/s: 0.0017438
Avg. spread: 1.1452

ω_σ 0.085351(0.0013758) ***
ω_v 0.86287(0.0013625) ***
α_1 $-$ 16.428(0.4417) ***
α_2 525.92(7.6943) ***

$\tilde{s}_0$ $-$ 0.36012(0.0094829) ***
s_N $-$ 0.6677(0.028159) ***
s_S 0.23972(0.00083618) ***

ϕ_0 $-$ 0(0) ***
$\phi_{\Delta N}$ 0.043113(0.021867) *
ϕ_N 0.0011916(0.00015559) ***
$\phi_{S,1}$ $1.0205e+07(88056)$ ***
$\phi_{S,2}$ $-9.9257e+06(82612)$ ***

n_0 5.0641(0.29297) ***
n_N $-$ 2.3629(0.86311) **
n_S $-$ 15.015(1.9858) ***

XOM / BATS

Volume/s: 73.509
Trades/s: 0.73198
Avg. spread: 0.016484

ω_σ $0.95(8.7076e+12)$
ω_v 1(0.0039107) ***
α_1 678.01(5.6187) ***
α_2 $-$ 66251(2977.7) ***

$\tilde{s}_0$ $-$ 4.3373(0.011628) ***
s_N 0.0035177(0.00012081) ***
s_S 0.082578(0.00080128) ***

ϕ_0 $-$ 0(0) ***
$\phi_{\Delta N}$ $-$ 0.030682(0.00014181) ***
ϕ_N $-6.7286e-06(8.0554e-07)$ ***
$\phi_{S,1}$ $-5.7162e+05(1.7723e+20)$
$\phi_{S,2}$ $4.3218e+05(1.332e+20)$

n_0 $-$ 171.81(2.3484) ***
n_N $-$ 10.087(0.98432) ***
n_S $2415(8.7173e+17)$

D.2. MSFT

MSFT / ISE

Volume/s: 15.51570
Trades/s: 0.05852
Avg. spread: 0.01086

ω_σ 0.05183(0.066963)
ω_v 0.91097(0.014496) ***
α_1 222.78(16.434) ***
α_2 56274(10488) ***

$\tilde{s}_0$ $-$ 5.9451(0.040337) ***
s_N 0.040779(0.00058426) ***
s_S 0.049948(0.0020674) ***

ϕ_0 $-$ 0(0)
$\phi_{\Delta N}$ $-$ 0.012552(6.2532$e-$05) ***
ϕ_N $-$ 2.0493$e-$06(1.0396$e-$06) *
$\phi_{S,1}$ 7.6087$e+$05(6.436$e+$05)
$\phi_{S,2}$ $-$ 6.8694$e+$05(5.8646$e+$05)

n_0 $-$ 35.949(80.939)
n_N $-$ 2.2009(1.5806)
n_S 124.21(386.03)

MSFT / NASD OB

Volume/s: 15.555
Trades/s: 0.062039
Avg. spread: 0.015007

ω_σ 0.94998(1.0911$e+$11)
ω_v 1(0.022246) ***
α_1 $-$ 400.56(9.6241) ***
α_2 3.9123$e+$05(2714.3) ***

$\tilde{s}_0$ $-$ 5.9264(0.042675) ***
s_N 0.045934(0.0058773) ***
s_S 0.037898(0.0023373) ***

ϕ_0 $-$ 0(0) ***
$\phi_{\Delta N}$ $-$ 0.020949(0.00013103) ***
ϕ_N $-$ 1.6842$e-$05(3.1455$e-$06) ***
$\phi_{S,1}$ 1.0413$e+$06(3.8469$e+$18)
$\phi_{S,2}$ $-$ 9.219$e+$05(3.4058$e+$18)

n_0 $-$ 4.509(1.9475) *
n_N $-$ 11.147(3.1107) ***
n_S 10.646(9.2742$e+$14)

MSFT / NSE

Volume/s: 4.3454
Trades/s: 0.02742
Avg. spread: 0.017626

ω_σ 0.043958(0.0060394) ***
ω_v 0(0.018443)
α_1 374.75(1.474) ***
α_2 $-$ 8212.5(54cm21) ***

$\tilde{s}_0$ $-$ 8.4278(0.025502) ***
s_N 0.036478(0.0042794) ***
s_S $-$ 0.098783(0.0023787) ***

ϕ_0 $-$ 0(0) ***
$\phi_{\Delta N}$ $-$ 0.033465(0.00039222) ***
ϕ_N $-$ 5.6632$e-$06(3.9276$e-$06)
$\phi_{S,1}$ 4.7512$e+$05(41206) ***
$\phi_{S,2}$ 1.4588$e+$05(16672) ***

n_0 2.4794(25.867)
n_N $-$ 2.2667(1.25) *
n_S $-$ 0.95859(36.216)

MSFT / NASD ADF

Volume/s: 4.4333
Trades/s: 0.021722
Avg. spread: 0.077888

ω_σ 0.065717(0.0033765) ***
ω_v 0.84524(0.0034994) ***
α_1 155.64(2.1107) ***
α_2 $-$ 3768.9(140.85) ***

$\tilde{s}_0$ $-$ 2.7232(0.026542) ***
s_N 0.015255(0.0021335) ***
s_S 0.16088(0.0015159) ***

ϕ_0 $-$ 0(0) ***
$\phi_{\Delta N}$ $-$ 0.016447(0.00048424) ***
ϕ_N $-$ 1.3724$e-$06(3.5257$e-$06)
$\phi_{S,1}$ 2.7998$e+$06(90649) ***
$\phi_{S,2}$ $-$ 2.5635$e+$06(81901) ***

n_0 34.393(5.8984) ***
n_N $-$ 4.4514(0.91891) ***
n_S $-$ 66.016(28.5) *

MSFT / Chicago

Volume/s: 1.0754
Trades/s: 0.0021385
Avg. spread: 0.21988

ω_σ 0.064128(0.0018865) ***
ω_v 0.91767(0.0016575) ***
α_1 50.013(0.46573) ***
α_2 $-$ 239.82(10.865) ***

$\tilde{s}_0$ $-$ 1.0415(0.011989) ***
s_N $-$ 0.039426(0.0042225) ***
s_S 0.18947(0.000755) ***

ϕ_0 $-$ 0(0) ***
$\phi_{\Delta N}$ $-$ 0.0067275(0.0011691) ***
ϕ_N 3.5934$e-$05(1.2277$e-$05) **
$\phi_{S,1}$ 6.1937$e+$06(95814) ***
$\phi_{S,2}$ $-$ 5.9308$e+$06(89319) ***

n_0 $-$ 22.61(2.8023) ***
n_N 1.7243(1.6636)
n_S 88.14(14.309) ***

MSFT / ARCA

Volume/s: 58.437
Trades/s: 0.30532
Avg. spread: 0.010469

ω_σ 0.049997(0.14235)
ω_v 0.9437(0.0073105) ***
α_1 $-$ 46.117(15.878) **
α_2 2.4978$e+$05(9822.2) ***

$\tilde{s}_0$ $-$ 4.0192(0.028369) ***
s_N 0.0011309(0.00034024) ***
s_S 0.10424(0.0019419) ***

ϕ_0 $-$ 0(0)
$\phi_{\Delta N}$ $-$ 0.015401(1.8155$e-$05) ***
ϕ_N 2.2894$e-$06(7.8214$e-$07) **
$\phi_{S,1}$ 5.7362$e+$05(1.1006$e+$06)
$\phi_{S,2}$ $-$ 5.5472$e+$05(1.0682$e+$06)

n_0 $-$ 297.25(879.73)
n_N 0.45912(2.0278)
n_S 1643.8(3243.9)

MSFT / CBOE

Volume/s: 0.081456
Trades/s: 0.0005438
Avg. spread: 0.17358

ω_σ 0.07184(0.0050463) ***
ω_v 1(0.0045003) ***
α_1 32.128(0.55648) ***
α_2 $-$ 75.827(4cm993) ***

$\tilde{s}_0$ $-$ 4.2534(0.031282) ***
s_N $-$ 0.8546(0.073113) ***
s_S 0.02747(0.0011386) ***

ϕ_0 $-$ 0(0) ***
$\phi_{\Delta N}$ $-$ 0.28355(0.020379) ***
ϕ_N 0.00010124(0.00023608)
$\phi_{S,1}$ 4.5651$e+$06(4cm9$e+$05) ***
$\phi_{S,2}$ $-$3.6996$e+$06(2.4034$e+$05) ***

n_0 0.86831(0.35448) **
n_N $-$ 1.7913(1.5784)
n_S $-$ 4.0915(2.4088) *

MSFT / BATS

Volume/s: 87.766
Trades/s: 0.5192
Avg. spread: 0.010857

ω_σ 0.047686(0.10571)
ω_v 0.93997(0.017539) ***
α_1 289.15(17.314) ***
α_2 90485(10185) ***

$\tilde{s}_0$ $-$ 5.7613(0.044663) ***
s_N 0.015231(0.00023101) ***
s_S 0.053509(0.0026472) ***

ϕ_0 $-$ 0(0)
$\phi_{\Delta N}$ $-$ 0.017061(4.7478$e-$05) ***
ϕ_N $-$ 6.7627$e-$07(4.3914$e-$07)
$\phi_{S,1}$ 5.8381$e+$05(7.8721$e+$05)
$\phi_{S,2}$ $-$ 5.851$e+$05(7.9192$e+$05)

n_0 $-$ 59.903(167.9)
n_N $-$ 5.2091(1.3578) ***
n_S 153.83(807.19)

D.3. GE

GE on ISE

Volume/s: 58.47722
Trades/s: 0.13993
Avg. spread: 0.01042

ω_σ 0.05000(0.10717)
ω_v 0.98542(0.0051133) ***
α_1 115.01(6.0202) ***
α_2 $-$ 4511.7(1798.9) **

$\tilde{s}_0$ $-$ 1.6382(0.017086) ***
s_N $-$ 0.0089245(0.00011842) ***
s_S 0.15541(0.0017559) ***

ϕ_0 $-$ 0(0)
$\phi_{\Delta N}$ $-$ 0.0085401(3.74$e-05$) ***
ϕ_N 6.4443$e-06$(9.7995$e-07$) ***
$\phi_{S,1}$ 1.4619$e+06$(2.0482$e+06$)
$\phi_{S,2}$ $-$ 1.1426$e+06$(1.6029$e+06$)

n_0 $-$ 435.06(1027)
n_N $-$ 1.5806(2.1401)
n_S 2531.6(3805.3)

GE on NASD OB

Volume/s: 14.445
Trades/s: 0.051471
Avg. spread: 0.013025

ω_σ 0.049965(0.028969) *
ω_v 0.9157(0.010515) ***
α_1 17.966(7.0558) **
α_2 39606(1857.5) ***

$\tilde{s}_0$ $-$ 4.4864(0.03378) ***
s_N $-$ 0.10196(0.0018156) ***
s_S 0.07572(0.0021302) ***

ϕ_0 $-$ 0(0) *
$\phi_{\Delta N}$ $-$ 0.048436(0.00034748) ***
ϕ_N 1.8813$e-05$(4.3037$e-06$) ***
$\phi_{S,1}$ 1.2406$e+06$(3.1599$e+05$) ***
$\phi_{S,2}$ $-$1.0206e+06(2.6586e+05) ***

n_0 99.5(49.375) *
n_N $-$ 13.359(1.6544) ***
n_S $-$ 314.73(119.95) **

GE on NSE

Volume/s: 53.203
Trades/s: 0.080859
Avg. spread: 0.015904

ω_σ 0.95(5.1851$e+09$)
ω_v 1(0.020567) ***
α_1 131.05(5.525) ***
α_2 1905.8(805.75) **

$\tilde{s}_0$ $-$ 5.1914(0.041067) ***
s_N 0.02047(0.00039725) ***
s_S 0.046802(0.0025796) ***

ϕ_0 $-$ 0(0) ***
$\phi_{\Delta N}$ $-$ 0.0052975(1.5484$e-05$) ***
ϕ_N $-$ 5.4966$e-07$(6.2755$e-07$)
$\phi_{S,1}$ 1.4369$e+06$(2.1952$e+17$)
$\phi_{S,2}$ $-$ 1.3306$e+06$(2.0319$e+17$)

n_0 $-$ 6.9068(9.8679)
n_N 0.51362(2.1661)
n_S $-$ 611.64(3.8078$e+14$)

GE on NASD ADF

Volume/s: 30.757
Trades/s: 0.095042
Avg. spread: 0.040949

ω_σ 0.94974(3.0689$e+08$)
ω_v 1(0.01761) ***
α_1 130.29(2.8319) ***
α_2 $-$ 2086.1(172.41) ***

$\tilde{s}_0$ $-$ 4.599(0.041907) ***
s_N $-$ 0.0028928(0.0010147) **
s_S 0.059258(0.0027775) ***

ϕ_0 $-$ 0(0) ***
$\phi_{\Delta N}$ $-$ 0.020868(0.00018253) ***
ϕ_N 3.778$e-06$(1.5339$e-06$) **
$\phi_{S,1}$ 2.3346$e+06$(1.9568$e+16$)
$\phi_{S,2}$ $-$ 2.2646$e+06$(1.8962$e+16$)

n_0 $-$ 33.712(6.5207) ***
n_N $-$ 6.6238(0.9183) ***
n_S 732.59(8.5141$e+12$)

GE on Chicago

Volume/s: 3.7557
Trades/s: 0.010167
Avg. spread: 0.035606

ω_σ 0.94961(2.0005$e+07$)
ω_v 1(0.0062228) ***
α_1 70.561(0.75797) ***
α_2 $-$ 406.97(9.2038) ***

$\tilde{s}_0$ $-$ 2.3243(0.022114) ***
s_N $-$ 0.29927(0.0045229) ***
s_S 0.10119(0.0015389) ***

ϕ_0 $-$ 0(0) ***
$\phi_{\Delta N}$ $-$ 0.045754(0.0012499) ***
ϕ_N 0.00012536(2.2774$e-05$) ***
$\phi_{S,1}$ 3.3047$e+06$(1.6352$e+15$)
$\phi_{S,2}$ $-$ 3.434$e+06$(1.6926$e+15$)

n_0 9.142(1.2257) ***
n_N $-$ 12.571(2.4598) ***
n_S $-$ 31.283(2.5398$e+11$)

GE on NYSE

Volume/s: 327.09
Trades/s: 0.53195
Avg. spread: 0.011136

ω_σ 0.95(1.8282$e+10$)
ω_v 1(0.0023119) ***
α_1 401.48(0.65944) ***
α_2 $-$ 69615(190.01) ***

$\tilde{s}_0$ $-$ 2.5455(0.0031375) ***
s_N 0.0017114(4.0711$e-06$) ***
s_S 0.036345(0.00023) ***

ϕ_0 $-$ 0(0) **
$\phi_{\Delta N}$ $-$ 0.0040295(1.2378$e-05$) ***
ϕ_N 1.4787$e-06$(3.3802$e-07$) ***
$\phi_{S,1}$ $-$ 6.0035$e+05$(4.2306$e+17$)
$\phi_{S,2}$ $-$ 5.1887$e+06$(3.1578$e+18$)

n_0 $-$ 117.67(34.702) ***
n_N $-$ 3.2554(1.9402) *
n_S 2.0426$e+05$(1.2446$e+17$)

GE on ARCA

Volume/s: 181.35
Trades/s: 0.62776
Avg. spread: 0.010746

ω_σ 0.94999(4.0059$e+09$)
ω_v 1(0.0047224) ***
α_1 596.6(0.6428) ***
α_2 $-$ 1.1469$e+05$(168.82) ***

$\tilde{s}_0$ $-$ 3.6532(0.0043017) ***
s_N 0.005801(4.2141$e-06$) ***
s_S 0.019062(0.00027262) ***

ϕ_0 $-$ 0(0)
$\phi_{\Delta N}$ $-$ 0.0066309(1.402$e-05$) ***
ϕ_N $-$ 4.4906$e-07$(4.0948$e-07$)
$\phi_{S,1}$ 8.6993$e+07$(1.2133$e+19$)
$\phi_{S,2}$ 8.233$e+07$(1.1391$e+19$)

n_0 $-$ 106.56(32.759) ***
n_N $-$ 5.6455(1.5924) ***
n_S $-$ 1.0665$e+06$(1.4819$e+17$)

GE on NASDAQ T

Volume/s: 293.88
Trades/s: 0.84622
Avg. spread: 0.01035

ω_σ 0.95(5.7593$e+10$)
ω_v 1(0.0012615) ***
α_1 433.97(0.38833) ***
α_2 $-$ 63431(107.18) ***

$\tilde{s}_0$ $-$ 2.0757(0.0020717) ***
s_N 0.0033185(2.3404$e-06$) ***
s_S 0.043354(0.00015339) ***

ϕ_0 $-$ 0(0)
$\phi_{\Delta N}$ $-$ 0.0075907(1.4062$e-05$) ***
ϕ_N 5.0327$e-08$(5.6622$e-07$)
$\phi_{S,1}$ 1.8891$e+06$(3.6913$e+18$)
$\phi_{S,2}$ $-$ 8.8483$e+06$(1.7359$e+19$)

n_0 $-$ 63.698(31.068) *
n_N $-$ 6.4363(2.5091) **
n_S $-$ 67683(1.3368$e+17$)

GE on CBOE

Volume/s: 1.781
Trades/s: 0.0022853
Avg. spread: 0.056185

ω_σ 0.94187(2.7481$e+06$)
ω_v 1(0.0092024) ***
α_1 38.611(1.1982) ***
α_2 $-$ 109.06(28.624) ***

$\tilde{s}_0$ $-$ 2.6066(0.027653) ***
s_N $-$ 0.43132(0.012505) ***
s_S 0.1035(0.0023159) ***

ϕ_0 $-$ 0(0) ***
$\phi_{\Delta N}$ $-$ 0.036403(0.0016178) ***
ϕ_N 0.00031822($4cm327e-05$) ***
$\phi_{S,1}$ $4cm78e+06$(2.5388$e+14$)
$\phi_{S,2}$ $-$ 3.9319$e+06$(2.3965$e+14$)

n_0 8.1851(1.9947) ***
n_N $-$ 15.071(3.474) ***
n_S 20.363(3.1185$e+10$)

GE on BATS

Volume/s: 115.91
Trades/s: 0.54623
Avg. spread: 0.010922

ω_σ	$0.95(1.2363e+11)$
ω_v	$1(0.0032372)$ ***
α_1	$416.27(1.0944)$ ***
α_2	$-63840(287.03)$ ***
$\tilde{s}_0$	$-3.7054(0.0059007)$ ***
s_N	$0.015501(1.8261e-05)$ ***
s_S	$0.049293(0.00043825)$ ***
ϕ_0	$-0(0)$
$\phi_{\Delta N}$	$-0.021838(2.8481e-05)$ ***
ϕ_N	$-5.2713e-07(7.9203e-07)$
$\phi_{S,1}$	$7.1366e+05(2.8686e+18)$
$\phi_{S,2}$	$-1.5651e+06(6.3608e+18)$
n_0	$-40.825(10.561)$ ***
n_N	$-3.8637(1.2664)$ **
n_S	$707.46(3.2538e+15)$