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DERIVATIVES OF HADAMARD TYPE IN SCALAR CONSTRAINED OPTIMIZATION

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Vsevolod I. Ivanov stated (Nonlinear Analysis 125 (2015), 270-289) the general second-order optimality condition for the constrained vector problem in terms of Hadamard derivatives. We will consider its special case for a scalar problem and show some corollaries for example for ℓ–stable at feasible point functions. Then we show the advantages of obtained results with respect to the previously obtained results.

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1. INTRODUCTION

Many second-order optimality conditions were stated for different optimization problems. They were very often stated in terms of generalized derivatives, see for example the monographs [23, 29, 33].

Various second-order optimality conditions have been presented for optimization problems with $C^{1,1}$ functions (see e.g. [10, 11, 17, 18, 19, 20, 26, 27, 34, 35, 36]). We recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is a $C^{1,1}$ function near $x \in \mathbb{R}^n$ if it is (Gâteaux) differentiable on some neighbourhood of x and its derivative $f'(\cdot)$ is Lipschitz there.

The authors of [2] introduced an ℓ–stable property which decreases a $C^{1,1}$–property and presented a second-order sufficient optimality condition for the unconstrained scalar problem in terms of Dini derivatives. The properties of scalar or vector functions that are ℓ–stable at some point functions and their applications in optimization were studied e.g. in [1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 15, 16, 24, 25, 28, 30, 31]. Let us remind that a second-order sufficient optimality condition for the constrained scalar problem for ℓ–stable at some point function in terms of Dini derivatives was introduced in [28].

Later, V. I. Ivanov [21] stated general necessary and sufficient conditions for the constrained vector problem in terms of Hadamard derivatives.

We will show in Sections 3 and 4 that the corollaries of the general theorem given in [21] give interesting results also for smooth classes of functions. In particular, we will devote the attention to the class of ℓ–stable at some point functions and prove that the

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corollary obtained from the general result given in [21] is tighter than the result given in [28].

2. PRELIMINARIES

Let us recall gradually the general sufficient condition of vector optimization problem obtained by Vsevolod Ivanov [21] in terms of the derivatives of Hadamard type.

Let $f : \mathbb{R}^n \to \mathbb{R}^r$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ be given, and let $C \subset \mathbb{R}^r$ and $K \subset \mathbb{R}^m$ be closed, convex and pointed cones with $\operatorname{int} C \neq \emptyset$ and $\operatorname{int} K \neq \emptyset$. For the definitions and properties of such cones, see e.g. [22, 32, 33]. We denote by $\langle a, b\rangle$ the scalar product of vectors $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$. We denote by C^* the positive polar cone of C by C^*, that is

$$C^* := \{\lambda \in \mathbb{R}^r; \langle \lambda, x\rangle \geq 0 \text{ for all } \ x \in C\}.$$

Let us consider the problem

$$\min f(x), \qquad\qquad \text{such that } g(x) \in -K. \tag{1}$$

We denote by S the feasible set, that is

$$S := \{x \in X; g(x) \in -K\}.$$

A feasible point x_0 is called an isolated local minimizer of order 2 for the problem (1) if there exist a constant A and a neighbourhood U, $x_0 \in U$, such that for all $x \in S \cap U$ there is

$$\lambda^* \in C^*, \quad \lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_r^*) \neq 0, \quad \sum_{i=1}^{r} (\lambda_i^*)^2 = 1,$$

which depends on x, with

$$\langle \lambda^*, f(x)\rangle \geq \langle \lambda^*, f(x_0)\rangle + A\|x - x_0\|^2.$$

The lower Hadamard directional derivative of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at a point $x \in \operatorname{dom} f$ in direction $u \in \mathbb{R}^n$ is defined as follows:

$$f_-^{(1)}(x; u) = \liminf_{t\downarrow 0, u'\to u} \frac{f(x + tu') - f(x)}{t}.$$

We note that if the considered function f is Lipschitz near x (i.e. there exist a neighbourhood U of x and a constant $K > 0$ such that $|f(y) - f(z)| \leq K\|y - z\|$, for every $y, z \in U$), then the lower Hadamard derivative coincides with the lower Dini derivative, i.e.

$$f_-^{(1)}(x; u) = f^\ell(x; u) := \liminf_{t\downarrow 0} \frac{f(x + tu) - f(x)}{t}. \tag{2}$$

Some other properties of the Hadamard derivative can be found in [14].

The lower Hadamard subdifferential of the function $f : \mathbb{R}^n \to \mathbb{R}$ at the point $x \in \operatorname{dom} f$ is defined by the following relation:

$$\partial_-^{(1)} f(x) = \{x^* \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}); \langle x^*, u\rangle \leq f_-^{(1)}(x; u) \text{ for all directions } u \in \mathbb{R}^n\}.$$

Now, we recall the definition of the lower second-order derivative of Hadamard type, which was introduced in [21].

Definition 2.1. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an arbitrary proper extended real function. Suppose that x_1^* is a fixed element from the lower Hadamard subdifferential $\partial_-^{(1)} f(x)$ at the point $x \in \operatorname{dom} f$. Then the lower second-order derivative of Hadamard type of f at x in direction $u \in \mathbb{R}^n$ is defined as follows:

$$f_-^{(2)}(x; x_1^*; u) = \liminf_{t \downarrow 0, u' \to u} \frac{f(x + tu') - f(x) - t\langle x_1^*, u' \rangle}{t^2/2}.$$

We suppose that x_0 is a feasible point for the problem (1), i. e. x_0 is an element of feasible set S for the problem (1). Let us consider the function

$$F(x) := \max\{\langle \lambda, f(x) - f(x_0) \rangle + \langle \mu, g(x) \rangle; (\lambda, \mu) \in \Lambda\},$$

where $\Lambda := \{(\lambda, \mu); \lambda \in C^*, \mu \in K^*, \sum_{i=1}^r \lambda_i^2 + \sum_{j=1}^m \mu_j^2 = 1\}$.

Using the function F, V. I. Ivanov stated the following optimality conditions for the problem (1).

Theorem 2.2. (Ivanov [21, Theorem 5.2]) Let x_0 be a feasible point for the problem (1). Then the following claims are equivalent:

(a) x_0 is an isolated local minimizer of second-order;

(b) the following conditions hold for all $u \in \mathbb{R}^n$:

$$F_-^{(1)}(x_0; u) \geq 0 \quad \text{and} \quad F_-^{(2)}(x_0; 0; u) > 0, u \neq 0; \tag{3}$$

(c) the following conditions

$$F_-^{(1)}(x_0; u) \geq 0, \quad \forall u \in \mathbb{R}^n \tag{4}$$

and

$$u \neq 0, F_-^{(1)}(x_0; u) = 0 \Longrightarrow F_-^{(2)}(x_0; 0; u) > 0 \tag{5}$$

are satisfied.

3. SCALAR PROBLEM

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$, be given. If we put $C = \{t \in \mathbb{R};\ t \geq 0\}$, $g = (g_1, g_2, \ldots, g_m) : \mathbb{R}^n \to \mathbb{R}^m$, and $K = \{(y_1, y_2, \ldots, y_m) \in \mathbb{R}^m;\ y_1 \geq 0,\ y_2 \geq 0, \ldots, y_m \geq 0\}$ in the problem (1), we obtain the following scalar constrained problem

$$\min f(x), \qquad \text{such that } g_i(x) \leq 0, \quad i = 1, 2, \ldots, m. \tag{6}$$

Now, the feasible set can be expressed as

$$S = \{x \in \mathbb{R}^n; g_i(x) \leq 0, i = 1, 2, \ldots, m\}.$$

Because of $C^* = \{t \in \mathbb{R}; t \geq 0\}$, choosing $\lambda^* = 1$ in the definition of isolated local minimizer of order 2 for the problem (1), we can say that x_0 is a feasible point if there exist a neighbourhood U and a constant $A > 0$ such that

$$f(x) \geq f(x_0) + A\|x - x_0\|^2, \quad \forall x \in U \cap S.$$

We denote by $S_{\mathbb{R}^n}$ the unit sphere of $\mathbb{R}^n$, i. e.

$$S_{\mathbb{R}^n} = \{u \in \mathbb{R}^n; \|u\| = 1\}.$$

Theorem 3.1. Let x_0 be a feasible point for the problem (6). Suppose that for every $u \in S_{\mathbb{R}^n}$ there are $\lambda \geq 0$ and $\beta_i \geq 0$, for $i = 1, 2, \ldots, m$, such that it holds

$$\liminf_{t\downarrow 0, u'\to u} \frac{\lambda(f(x_0 + tu') - f(x_0)) + \beta_1 g_1(x_0 + tu') + \cdots + \beta_m g_m(x_0 + tu')}{t} \geq 0. \tag{7}$$

Suppose that for every $u \in S_{\mathbb{R}^n}$ with the property

$$\liminf_{t\downarrow 0, u'\to u} \frac{\lambda(f(x_0 + tu') - f(x_0)) + \beta_1 g_1(x_0 + tu') + \cdots + \beta_m g_m(x_0 + tu')}{t} = 0, \tag{8}$$

it holds

$$\liminf_{t\downarrow 0, u'\to u} \frac{\lambda(f(x_0 + tu') - f(x_0)) + \beta_1 g_1(x_0 + tu') + \cdots + \beta_m g_m(x_0 + tu')}{t^2/2} > 0. \tag{9}$$

Then x_0 is an isolated minimizer of second-order for the problem (6).

P r o o f. We can consider problem (6) as a special case of problem (1) with

$$C = \{t \in \mathbb{R}; t \geq 0\}, \quad K = \{(y_1, y_2, \ldots, y_m) \in \mathbb{R}^m; y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0\},$$

and

$$g : \mathbb{R}^n \to \mathbb{R}^m : g = (g_1, g_2, \ldots, g_m).$$

Using inequality (7), for every $u \in S_{\mathbb{R}^n}$, there exist $\lambda \geq 0$, $\beta_i \geq 0, i \in \{1, 2, \ldots, m\}$, such that $(\lambda, (\beta_1, \beta_2, \ldots, \beta_m)) \in \Lambda$ and

$$\begin{aligned} F_-^{(1)}(x_0; u) &= \liminf_{t\downarrow 0, u'\to u} \frac{F(x_0 + tu') - F(x_0)}{t} = \liminf_{t\downarrow 0, u'\to u} \frac{F(x_0 + tu')}{t} \\ &\geq \liminf_{t\downarrow 0, u'\to u} \frac{\lambda(f(x_0 + tu') - f(x_0)) + \beta_1 g_1(x_0 + tu') + \cdots + \beta_m g_m(x_0 + tu')}{t} \\ &\geq 0. \end{aligned} \tag{10}$$

Therefore, the condition (4) from Theorem 2.2 is satisfied.

Now, we suppose that for some $u \in S_{\mathbb{R}^n}$ it holds $F_-^{(1)}(x_0; u) = 0$. Then by means of formula (7) there are $\lambda \geq 0$, $\beta_i \geq 0, i \in \{1, 2, \ldots, m\}$, such that it holds

$$\begin{aligned}
0 &\leq \liminf_{t\downarrow 0, u'\to u} \frac{\lambda(f(x_0+tu')-f(x_0))+\beta_1 g_1(x_0+tu')+\cdots+\beta_m g_m(x_0+tu')}{t} \\
&\leq \liminf_{t\downarrow 0, u'\to u} \frac{F(x_0+tu')}{t} = \liminf_{t\downarrow 0, u'\to u} \frac{F(x_0+tu')-F(x_0)}{t} \\
&= F_-^{(1)}(x_0;u) = 0. \qquad (11)
\end{aligned}$$

Then, it follows from inequalities (8) and (9) that

$$\begin{aligned}
&F_-^{(2)}(x_0;0;u) = \liminf_{t\downarrow 0, u'\to u} \frac{F(x_0+tu')-F(x_0)}{t^2/2} \\
\geq\ & \liminf_{t\downarrow 0, u'\to u} \frac{\lambda(f(x_0+tu')-f(x_0))+\beta_1 g_1(x_0+tu')+\cdots+\beta_m g_m(x_0+tu')}{t^2/2} \\
>\ & 0. \qquad (12)
\end{aligned}$$

Thus also the condition (5) from Theorem 2.2 is satisfied. □

In the sequel, we will present the corollary of the previous theorem. By a critical set we will mean the set

$$D(x_0) \quad = \quad \{u \in S_{\mathbb{R}^n}; f_-^{(1)}(x_0;u) \leq 0, {g_i}_-^{(1)}(x_0;u) \leq 0 \text{ for } i \in I(x_0)\},$$

where $I(x_0) = \{i \in \{1,2,\ldots,m\}; g_i(x_0) = 0\}$.

Corollary 3.2. Let x_0 be a feasible point for the problem (6). If for every $u \in D(x_0)$ there exist $\lambda \geq 0$ and $\beta_i \geq 0$, $i \in I(x_0) = \{i_1, i_2, \ldots, i_s\} \subset \{1,2,\ldots,m\}$, such that

$$\liminf_{t\downarrow 0, u'\to u} \frac{\lambda(f(x_0+tu')-f(x_0))+\beta_{i_1} g_{i_1}(x_0+tu')+\cdots+\beta_{i_s} g_{i_s}(x_0+tu')}{t} = 0, \quad (13)$$

and

$$\liminf_{t\downarrow 0, u'\to u} \frac{\lambda(f(x_0+tu')-f(x_0))+\beta_{i_1} g_{i_1}(x_0+tu')+\cdots+\beta_{i_s} g_{i_s}(x_0+tu')}{t^2/2} > 0, \quad (14)$$

then x_0 is an isolated minimizer of second-order for problem (6).

Proof. If $u \in S_{\mathbb{R}^n}$ is not a critical direction, i.e. $u \notin D(x_0)$, then there are two possibilities:

Case 1. If

$$\liminf_{t\downarrow 0, u'\to u} \frac{f(x_0+tu')-f(x_0)}{t} > 0,$$

then we put $\lambda = 1$ and $\beta_i = 0$ for every $i \in \{1,2,\ldots,m\}$. Hence,

$$\begin{aligned}
&\liminf_{t\downarrow 0, u'\to u} \frac{\lambda(f(x_0+tu')-f(x_0))+\beta_1 g_1(x_0+tu')+\cdots+\beta_m g_m(x_0+tu')}{t} \\
=\ & \liminf_{t\downarrow 0, u'\to u} \frac{f(x_0+tu')-f(x_0)}{t} > 0,
\end{aligned}$$

and the condition (7) from Theorem 3.1 is satisfied.

Case 2. If

$$\liminf_{t\downarrow 0, u'\to u} \frac{g_{i_0}(x_0+tu') - g_{i_0}(x_0)}{t} > 0,$$

for some $i_0 \in I(x_0)$, then we put $\lambda = 0$, $\beta_i = 0$ for $i \in \{1, 2, \ldots, m\} \setminus \{i_0\}$, and $\beta_{i_0} = 1$. Hence,

$$\begin{aligned} &\liminf_{t\downarrow 0, u'\to u} \frac{\lambda(f(x_0+tu') - f(x_0)) + \beta_1 g_1(x_0+tu') + \cdots + \beta_m g_m(x_0+tu')}{t} \\ = &\liminf_{t\downarrow 0, u'\to u} \frac{g_{i_0}(x_0+tu') - g_{i_0}(x_0)}{t} > 0, \end{aligned}$$

and the condition (7) from Theorem 3.1 is also satisfied.

If $u \in S_{\mathbb{R}^n}$ is a critical direction, i. e. $u \in D(x_0)$, then we put $\beta_i = 0$ for $i \in \{1, 2, \ldots, m\} \setminus I(x_0)$ and the conditions (13) and (14) mean that the conditions (8) and (9) from Theorem 3.1 are satisfied. Therefore, x_0 is an isolated minimizer of second-order for problem (6). □

If the considered functions are Gâteaux differentiable at the considered feasible point x_0 and Lipschitz near x_0, then we can state the following corollary. We recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is Gâteaux differentiable at x_0, if there exists a linear continuous functional $f'(x_0)$ such that

$$f'(x_0)h = \lim_{t\to 0} \frac{f(x_0+th) - f(x_0)}{t}$$

for every $h \in S_{\mathbb{R}^n}$.

Corollary 3.3. Let x_0 be a feasible point for the problem (6) and we suppose that the functions f and g_i, $i \in I(x_0)$, are Gâteaux differentiable at x_0 and Lipschitz near x_0. If for every $u \in D(x_0)$ there exist $\lambda \geq 0$ and $\beta_i \geq 0$, $i \in I(x_0) = \{i_1, i_2, \ldots, i_s\} \subset \{1, 2, \ldots, m\}$, such that

$$\lambda f'(x_0)u + \beta_{i_1} g'_{i_1}(x_0)u + \beta_{i_s} g'_{i_s}(x_0)u = 0, \tag{15}$$

and

$$\liminf_{t\downarrow 0, u'\to u} \frac{\lambda(f(x_0+tu') - f(x_0)) + \beta_{i_1} g_{i_1}(x_0+tu') + \cdots + \beta_{i_s} g_{i_s}(x_0+tu')}{t^2/2} > 0, \tag{16}$$

then x_0 is an isolated minimizer of second-order for problem (6).

Proof. We will show that the condition (13) from Corollary 3.2 is satisfied. If $i \in I(x_0)$, then $g_i(x_0) = 0$. Thus for every $u \in D(x_0)$ we have

$$\begin{aligned} &\liminf_{t\downarrow 0, u'\to u} \frac{\lambda(f(x_0+tu') - f(x_0)) + \beta_{i_1} g_{i_1}(x_0+tu') + \cdots + \beta_{i_s} g_{i_s}(x_0+tu')}{t} \\ = &\liminf_{t\downarrow 0, u'\to u} \frac{\lambda(f(x_0+tu') - f(x_0)) + \beta_{i_1}(g_{i_1}(x_0+tu') - g_{i_1}(x_0)) + \cdots + \beta_{i_s}(g_{i_s}(x_0+tu') - g_{i_s}(x_0))}{t}, \end{aligned} \tag{17}$$

where λ and β_{i_j}, $j \in \{1,\dots,s\}$, are those for which in the assumptions of Corollary 3.3 the validity of formulas (15) and (16) is supposed for the considered $u \in D(x_0)$. Since the functions f and g_i, $i \in I(x_0)$, are Lipschitz near x_0, it holds

$$\begin{aligned} &\liminf_{t\downarrow 0, u'\to u} \frac{\lambda(f(x_0+tu')-f(x_0)) + \beta_{i_1}(g_{i_1}(x_0+tu')-g_{i_1}(x_0)) + \cdots + \beta_{i_s}(g_{i_s}(x_0+tu')-g_{i_s}(x_0))}{t} \\ =\ &\liminf_{t\downarrow 0} \frac{\lambda(f(x_0+tu')-f(x_0)) + \beta_{i_1}(g_{i_1}(x_0+tu')-g_{i_1}(x_0)) + \cdots + \beta_{i_s}(g_{i_s}(x_0+tu')-g_{i_s}(x_0))}{t}. \end{aligned} \tag{18}$$

Finally, using the Gâteaux differentiability of f and g_i, $i \in I(x_0)$, we obtain

$$\begin{aligned} &\liminf_{t\downarrow 0} \frac{\lambda(f(x_0+tu')-f(x_0)) + \beta_{i_1}(g_{i_1}(x_0+tu')-g_{i_1}(x_0)) + \cdots + \beta_{i_s}(g_{i_s}(x_0+tu')-g_{i_s}(x_0))}{t} \\ =\ &\lambda f'(x_0)u + \beta_{i_1} g'_{i_1}(x_0)u + \cdots + \beta_{i_s} g'_{i_s}(x_0)u. \end{aligned} \tag{19}$$

It folows from the formulas (17), (18), (19), and (15) that the condition (13) is satisfied. Because of the conditions (14) and (16) are the same, by Corollary 3.2 x_0 is an isolated minimizer of second-order for problem (6). □

4. ℓ–STABLE FUNCTIONS

In this section we recall some notions concerning ℓ–stability and state for this class of functions the optimality conditions for problem (6). We also compare our result with the previous result obtained for ℓ–stable at some point functions by S. J. Li and S. Xu [28].

We have already introduced the Dini lower derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point $x \in \mathbb{R}^n$ in the direction $u \in \mathbb{R}^n$ in formula (2), and mentioned that it equals to $f_-^{(1)}(x;u)$ if f is Lipschitz near x.

We recall the definition of ℓ–stable at some point function which was introduced in [2].

Definition 4.1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called ℓ–stable at $x \in \mathbb{R}^n$ if there exist a neighbourhood U of x and $L > 0$ such that

$$|f^\ell(y;u) - f^\ell(x;u)| \le L\|y-x\|, \quad \forall y \in U, \forall u \in S_{\mathbb{R}^n}.$$

We note that the class of ℓ–stable at some point functions was introduced to weaken $C^{1,1}$–property in some optimization problems. It was shown in [2] that the class of functions that are ℓ–stable at some point properly contains the class of functions that are $C^{1,1}$ near this point.

We recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is strictly differentiable at $x \in \mathbb{R}^n$ if there exists a linear continuous functional $f'_s(x)$ such that

$$f'_s(x)u = \lim_{y\to x, t\downarrow 0} \frac{f(y+th)-f(y)}{t}, \quad \forall u \in S_{\mathbb{R}^n},$$

and the limit is uniform with respect to $u \in S_{\mathbb{R}^n}$.

It is easy to show that if a function $f : \mathbb{R}^n \to \mathbb{R}$ is strictly differentiable at $x \in \mathbb{R}^n$, then it is also Gâteaux differentiable at x and $f'_s(x) = f'(x)$.

Proposition 4.2. (Bednařík and Pastor [2]) If a function $f : \mathbb{R}^n \to \mathbb{R}$ is ℓ–stable at $x \in \mathbb{R}^n$, then it is strictly differentiable at x and Lipschitz near x.

Definition 4.3. The second-order lower Dini directional derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ at $x \in \mathbb{R}^n$ in direction $u \in \mathbb{R}^n$ is defined as

$$f'^{\ell}(x;u) = \liminf_{t \downarrow 0} \frac{f(x+tu) - f(x) - tf^{\ell}(x;h)}{t^2/2}.$$

The following proposition follows from the proof of Proposition 6.3 given in [21].

Proposition 4.4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be ℓ–stable at $x \in \mathbb{R}^n$. If for every $u \in S_{\mathbb{R}^n}$ we have $f'(x)u = 0$, then

$$f'^{\ell}(x;u) = f_-^{(2)}(x;0;u), \quad \forall u \in S_{\mathbb{R}^n}.$$

We define the Lagrange function for the problem (6):

$$L(x) = f(x) + \sum_{i \in I(x_0)} \beta_i g_i(x), \quad \forall x \in \mathbb{R}^n.$$

If the functions f and g_i, $i \in I(x_0)$, are ℓ–stable at x_0, then also the function L is ℓ–stable at x_0 by Lemma 3 from [12].

Now, we can formulate the following sufficient optimality condition for the problem (6) when the considered functions are ℓ–stable at the feasible point.

Corollary 4.5. Let x_0 be a feasible point for the problem (6) and suppose that the functions f and g_i, $i \in I(x_0)$ are ℓ–stable at x_0. Suppose that there are $\beta_i \geq 0$, $i \in \{1, 2, \ldots, m\}$, such that for each $u \in S_{\mathbb{R}^n}$ it holds

$$L'(x_0)u = 0 \tag{20}$$

and moreover,

$$L'^{\ell}(x_0;u) > 0, \quad \forall u \in D(x_0). \tag{21}$$

Then x_0 is an isolated minimizer of second-order for problem (6).

Proof. We will show that the assumptions of Corollary 3.3 are satisfied. By Proposition 4.2 the functions f and g_i, $i \in I(x_0)$, are Gâteaux differentiable at x_0 and Lipschitz near x_0. The condition (20) implies the condition (15) immediately (with $\lambda = 1$).

We notice that $g_i(x_0) = 0$ for every $i \in I(x_0)$. Then, since the Lagrange function L is ℓ–stable at $x_0 \in \mathbb{R}^n$, by Proposition 4.4 and formula (20) we have $L'^{\ell}(x_0;u) = L_-^{(2)}(x_0;0,u)$ and the condition (21) implies the condition (16).

Summarizing the previous considerations, all assumptions of Corollary 3.3 are satisfied and it means that x_0 is an isolated minimizer of second-order for problem (6).

□

We will compare the previous result with the result given in [28] where the authors also stated the second-order sufficient optimality condition for the problem (6) with ℓ–stable functions. S.J. Li and S. Xu considered the Lagrange function

$$\hat{L}(x) = f(x) + \sum_{i=1}^{m} \beta_i g_i(x), \quad \forall x \in \mathbb{R}^n,$$

where $\beta_i \geq 0$, $i = 1, \ldots, m$. Then they separated the set $I(x) = \{i; g_i(x) = 0\}$ into the sets

$$M(x) = \{i \in I(x); \beta_i = 0\}$$

and

$$J(x) = \{i \in I(x); \beta_i > 0\}.$$

Finally, they defined the set

$$E(x) = \{u \in S_{\mathbb{R}^n}; g_i'(x_0)u \leq 0, \forall i \in M(x), g_i'(x_0)u = 0, \forall i \in J(x)\},$$

and stated the following theorem.

Theorem 4.6. (Li and Xu [28, Theorem 3.2]) Let x_0 be a feasible point for the problem (6) and we suppose that the functions f and g_i, $i \in \{1, 2, \ldots, m\}$, are ℓ–stable at x_0. We suppose that there are $\beta_i \geq 0$, $i \in \{1, 2, \ldots, m\}$, such that for each $u \in S_{\mathbb{R}^n}$ it holds

$$\sum_{i=1}^{m} \beta_i g_i(x_0) = 0, \tag{22}$$

and

$$\hat{L}'(x_0)u = 0. \tag{23}$$

Moreover, we suppose that

$$\hat{L}'^{\ell}(x_0; u) > 0, \quad \forall u \in E(x_0). \tag{24}$$

Then x_0 is an isolated minimizer of second-order for problem (6).

Proof. We will prove that Theorem 4.6 follows from Corollary 4.5. From formula (22) it follows that $\beta_i = 0$ for every $i \in \{1, 2, \ldots, m\} \setminus I(x_0)$ and thus $\hat{L}(x) = L(x)$. Now, it suffices to show that

$$D(x_0) \subset E(x_0). \tag{25}$$

We notice that for ℓ–stable functions

$$\begin{aligned} D(x_0) &= \{u \in S_{\mathbb{R}^n}; f_-^{(1)}(x_0; u) \leq 0, {g_i}_-^{(1)}(x_0; u) \leq 0 \text{ for } i \in I(x_0)\}, \\ &= \{u \in S_{\mathbb{R}^n}; f'(x_0)u \leq 0, g_i'(x_0)u \leq 0 \text{ for } i \in I(x_0)\}, \end{aligned}$$

and

$$E(x_0) = \{u \in S_{\mathbb{R}^n}; g_i'(x_0)u \leq 0, \forall i \in M(x_0), g_i'(x_0)u = 0, \forall i \in J(x_0)\}.$$

So, let us consider that $d \in S_{\mathbb{R}^n}$ such that $d \notin E(x_0)$. Then there are two possibilities.

- Case 1. There exists $i_0 \in M(x_0)$ such that $g'_{i_0}(x_0)d = g_{i_0}{}_{-}^{(1)}(x_0; d) > 0$. Then $d \notin D(x_0)$.

- Case 2. There exists $i_0 \in J(x_0)$ such that $g'_{i_0}(x_0)d \neq 0$. If we suppose that $d \in D(x_0)$, then $g'_{i_0}(x_0)d < 0$, $f'(x_0)d \leq 0$ and $g'_i(x_0)d \leq 0$ for every $i \in I(x_0) \setminus \{i_0\}$.

  Then, $L'(x_0)d = \hat{L}'(x_0)d < 0$, but it is a contradiction with the formula (23). Therefore, $d \notin D(x_0)$.

Summarizing the previous considerations, we have that $d \notin E(x_0)$ implies $d \notin D(x_0)$. Thus we proved the formula (25). □

Remark 4.7. It seems that the only advantage of Corollary 4.5 with respect to Theorem 4.6 is the fact that the ℓ–stability of the functions g_i is required only for $i \in I(x_0)$.

On the other hand, supposing moreover in Corollary 4.5 that all functions g_i, for $i \in \{1, 2, \ldots, m\}$ are ℓ–stable at x_0, Theorem 4.6 is equivalent to Corollary 4.5. Indeed, having in mind the previous proof, it suffices to show that $E(x_0) \subset D(x_0)$. So, let us consider $d \in S_{\mathbb{R}^n}$ such that $d \notin D(x_0)$. Then there are two possibilities.

- Case 1. There exists $i_0 \in I(x_0)$ such that $g'_{i_0}(x_0)d > 0$. Then $d \notin E(x_0)$.

- Case 2. It holds $f'(x_0)d > 0$. If we suppose that $d \in E(x_0)$, then $g'_i(x_0)d = 0$ for every $i \in J(x_0)$ and because of $\beta_i = 0$ for every $i \in M(x_0)$, we have $\beta_i g'_i(x_0)d = 0$ for every $i \in I(x_0)$. Summarizing the previous facts, we obtain $L'(x_0)d > 0$, but it is a contradiction with the formula (20).

Finishing our paper, we present an example which illustrates the advantage of Corollary 3.3 with respect to Theorem 4.6 and Corollary 4.5.

Example 4.8. We define an objective function f as follows

$$f(x) = \begin{cases} \int_0^{|x|} t(1 + \sin(\ln t))dt & , \quad \text{if } x \neq 0, \\ 0 & , \quad \text{if } x = 0. \end{cases}$$

Let us consider the constrained programming problem (6)

$$\begin{aligned} &\min f(x), \\ &\text{such that} \quad g_1(x) = x^{\frac{4}{3}} \leq 0, \quad g_2(x) = x^3 \leq 0, \quad g_3(x) = 2x - 5 \leq 0. \end{aligned}$$

Since $f'(x)h = x(\frac{19}{20} + \sin(\ln|x|)h$ for $x \neq 0$, $h \in \mathbb{R}$, and $f'(0) = 0$, f is $C^{1,1}$ function. The functions g_2 and g_3 are C^2 functions. Therefore, f, g_2 and g_3 are also ℓ–stable functions at 0. On the other hand, the function g_1 is C^1 function, but it is not ℓ–stable at 0.

Thus, to verify that 0 is an isolated local minimizer of order 2 we cannot use neither Corollary 4.5 nor Theorem 4.6.

But we can use Corollary 3.3. We notice that $I(0) = \{1, 2\}$, $S = \{0\}$ and $D(0) = \{-1, 1\}$ because $f'(0) = g_1'(0) = g_2'(0) = 0$. To satisfy the conditions of Corollary 3.3 we need to find $\lambda > 0$, $\beta_1 \geq 0$, and $\beta_2 \geq 0$ such that

$$\lambda f'(0)u + \beta_1 g_1'(0)u + \beta_2 g_2'(0) = 0,$$

for $u = \pm 1$. Since $f'(0) = g_1'(0) = g_2'(0) = 0$, we can consider $\lambda = 1$, $\beta_1 = 0$ and $\beta_2 = 1$. Now, we check the condition (16) from Corollary 3.3. At first, we note that it is easy to calculate

$$f(tu') = \frac{t^2u'^2}{2} + \frac{1}{5}t^2u'^2(2\sin(\ln|tu'|) - \cos(\ln|tu'|)), \quad t \in \mathbb{R}.$$

Then

$$\begin{aligned}
& \liminf_{t\downarrow 0, u'\to 1} \frac{f(tu') + g_2(tu')}{t^2/2} \\
= \quad & \liminf_{t\downarrow 0, u'\to 1} \frac{\frac{t^2u'^2}{2} + \frac{1}{5}t^2u'^2(2\sin(\ln|tu'|) - \cos(\ln|tu'|)) + t^3u'^3}{t^2/2} \\
= \quad & \liminf_{t\downarrow 0, u'\to 1} \left(u'^2 + \frac{2}{5}u'^2(2\sin(\ln|tu'|) - \cos(\ln|tu'|)) + \frac{tu'^3}{2}\right) \\
= \quad & 1 - \frac{2\sqrt{5}}{5} > 0.
\end{aligned}$$

Analogously, also

$$\liminf_{t\downarrow 0, u'\to -1} \frac{f(tu') + g_2(tu')}{t^2/2} > 0.$$

Thus, the assumptions of Corollary 3.3 are satisfied and it means that 0 is an isolated minimizer of second-order.

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