FUZZY WEIGHTED AVERAGE AS A FUZZIFIED AGGREGATION OPERATOR AND ITS PROPERTIES

Ondřej Pavlačka, Martina Pavlačková and Vladislav Hetfleiš

The weighted average is a well-known aggregation operator that is widely applied in various mathematical models. It possesses some important properties defined for aggregation operators, like monotonicity, continuity, idempotency, etc., that play an important role in practical applications. In the paper, we reveal whether and in which way such properties can be observed also for the fuzzy weighted average operator where the weights as well as the weighted values are expressed by noninteractive fuzzy numbers. The usefulness of the obtained results is discussed and illustrated by several numerical examples.

Keywords: aggregation operator, fuzzy weighted average, fuzzy numbers, fuzzy weights

Classification: 03E72, 68T37

1. INTRODUCTION

The weighted average of real numbers $x_1, \dots, x_n$ with associated weights $w_1, \dots, w_n$ is defined as

$$a_w(x_1, \dots, x_n; w_1, \dots, w_n) = \frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i}. \qquad (1)$$

Generally, the weights $w_1, w_2, \dots, w_n$ are nonnegative real numbers whose sum is different from zero. The weighted average is a well-known aggregation operator that is widely applied in various mathematical models. Particularly, as an example let us mention multi-criteria decision making (MCDM) models, where the overall evaluations of alternatives are often calculated as weighted averages of evaluations with respect to the particular criteria.

In practical applications, the input parameters in the weighted average, i. e. weights and weighted values, can be uncertain. For instance, in MCDM models the weights of criteria are often set subjectively on the basis of experts' experiences or opinions (see [22]). Information about the weighted values can be incomplete, missing, or also vague, e. g. in the case of the expert evaluation of alternatives with respect to a qualitative criterion. Such kinds of uncertainty can be sufficiently modelled by means of tools of fuzzy sets theory (see e. g. [18]).

DOI: 10.14736/kyb-2017-1-0137

The most common way for an expert how to express uncertain weights and uncertain weighted values is to describe their values separately by fuzzy numbers (more general concept where the uncertain inputs are modelled by fuzzy vectors was considered in [24]). Extension of the weighted average operation to the case where the weighted values and/or the weights are modelled by fuzzy numbers has been studied since the second half of 70's (see [1]). Since then, the fuzzy weighted average has become an important topic in both fuzzy sets theory and applications. It has been applied e. g. in risk evaluation [27, 29], in decision making [9, 13, 14, 19, 31, 32], or in information processing [33].

From the theoretical point of view, the research was focused mainly on the problems connected with computation of the fuzzy weighted average of fuzzy numbers. The case where only the weighted values $x_1, \ldots, x_n$ are expressed by fuzzy numbers while the weights remain real numbers was studied e. g. in [28]. Such fuzzy extension of a_w is easy to compute since a_w is monotone in arguments $x_1, \ldots, x_n$ and no external interactivity constraint is involved. Unlike this case, the fact that also the weights are considered to be fuzzy makes the calculation substantially more complex. The increase of complexity is caused by the fact that a_w is not monotone in arguments $w_1, \ldots, w_n$.

As was pointed out in [6, 25], in fuzzy environment it is necessary to distinguish whether the fuzzy weights model the uncertain values of nonnormalized weights, i. e. there is no interaction among the weights, or the uncertain values of normalized weights, i. e. the sum of the weights is assumed to be equal to one. For a better comprehensibility of the text, only the first situation where the fuzzy weights are noninteractive fuzzy numbers will be considered further in the paper. The calculation of the fuzzy extension of a_w in such a case was studied e. g. in [1, 5, 11, 12, 14, 15, 20, 21]. In the second situation, the assumption that the sum of $w_1, \ldots, w_n$ is equal to one implies that uncertain values of normalized weights have to be modelled only by a special structure of interactive fuzzy numbers called *a tuple of normalized fuzzy weights* (see e. g. [22, 24, 30]). The calculation of the fuzzy weighted average of fuzzy numbers with normalized fuzzy weights was studied e. g. in [4, 7, 24].

The weighted average belongs to the class of aggregation operators. The aggregation operators are mathematical objects that have the function of reducing a set of inputs into a unique representative, i. e. an output of the aggregation. An overview of definition and possible properties of aggregation operators can be found, for instance, in [2, 3, 10].

The weighted average operator possesses some important properties defined for aggregation operators, like monotonicity, continuity, idempotency, etc. These properties play an important role in practical applications. The aim of the paper is to study whether and in which way such properties can be observed also for the fuzzy weighted average operator. The obtained results can be very helpful in order to properly employ the fuzzy weighted average operator in fuzzy models. To the best of our knowledge, this is one of the first attempts to generalize the properties of aggregation operators to the case of inputs modelled by fuzzy numbers.

The paper is organized as follows. In Section 2, aggregation operators and some of their possible properties are briefly summarized. Besides that, the weighted average operator is defined and its properties are examined. In Section 3, the fuzzy weighted average operator is introduced as a sequence of fuzzy weighted averages of fuzzy numbers.

Afterwards, the properties of the fuzzy weighted average operator are studied.

2. WEIGHTED AVERAGE AS AN AGGREGATION OPERATOR

In this section, the definitions of an aggregation operator and some of its possible properties will be recalled. Afterwards, the weighted average will be introduced as an aggregation operator and its properties will be examined.

An aggregation operator is defined as a sequence of aggregation functions. In the literature, usually a collection of real numbers from the unit interval $[0,1]$ is considered as the input of the aggregation, i. e. the aggregation operators are standardly defined on $[0,1]$ (see e. g. [3]). The inputs of the weighted average operation, i. e. the weighted values $x_1,\dots,x_n$, are generally defined on $\mathbb{R}$. In particular applications, they might be restricted to a certain interval. Thus, an aggregation operator on an arbitrary nonempty interval $I \subseteq (-\infty,\infty)$ will be considered in the paper. Such a more general concept was studied e. g. in [26].

Remark 2.1. Throughout the paper, we assume that I is a nonempty real interval and we set $x^- := \inf I$ and $x^+ := \sup I$. Note that x^- and x^+ might belong to I or not, possibly with $x^- = -\infty$ or $x^+ = +\infty$.

Definition 2.2. An aggregation operator $\mathcal{A}$ on an interval I is the sequence $\{A_n\}_{n=1}^{\infty}$ of aggregation functions

$$A_n : I^n \to I$$

that satisfy the following conditions:

1) $A_1(x) = x$ for each $x \in I$.

2) If $x_i \le y_i$, for all $i = 1,\dots,n$, where $n = 2,3,\dots$, then

$$A_n(x_1,\dots,x_n) \le A_n(y_1,\dots,y_n).$$

3) For each $n \in \mathbb{N}$:

$$\lim_{(x_1,\dots,x_n)\to(x^-,\dots,x^-)} A_n(x_1,\dots,x_n) = x^-$$

$$\lim_{(x_1,\dots,x_n)\to(x^+,\dots,x^+)} A_n(x_1,\dots,x_n) = x^+.$$

Further, the following properties of aggregation operators will be considered.

Definition 2.3. We say that an aggregation operator $\mathcal{A} = \{A_n\}_{n=1}^{\infty}$ on I is

- compensative, if for each $n \in \mathbb{N}$ the following inequalities hold for any n-tuple $(x_1,\dots,x_n) \in I^n$:

$$\min\{x_1,\dots,x_n\} \le A_n(x_1,\dots,x_n) \le \max\{x_1,\dots,x_n\};$$

- idempotent, if for all $x \in I$ and all $n \in \mathbb{N}$:

$$A_n(x,\dots,x) = x;$$

- symmetric (commutative), if for all $n \in \mathbb{N}$, each vector $(x_1, \dots, x_n) \in I^n$, and any permutation σ of $\{1, \dots, n\}$:

$$A_n(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = A_n(x_1, \dots, x_n);$$

- strictly monotonic, if for all $n \in \mathbb{N}$, A_n is strictly monotonic, i.e. if for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in I^n$ such that $x_i < y_i$ for one $i \in \{1, \dots, n\}$ and $x_j = y_j$ for any $j \neq i$:

$$A_n(x_1, \dots, x_n) < A_n(y_1, \dots, y_n);$$

- stable for a linear transformation, if for each $n \in \mathbb{N}$, all $r, t \in \mathbb{R}$, and all $(x_1, \dots, x_n) \in I^n$:

$$A_n(rx_1 + t, \dots, rx_n + t) = rA_n(x_1, \dots, x_n) + t;$$

- Lipschitz with constant L (L-Lipschitz), where $L \in (0, \infty)$, if for all $n \in \mathbb{N}$ and all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in I^n$:

$$|A_n(x_1, \dots, x_n) - A_n(y_1, \dots, y_n)| \leq L \sum_{i=1}^{n} |x_i - y_i|;$$

- continuous, if for all $n \in \mathbb{N}$, the aggregation function A_n is continuous.

Remark 2.4. Continuity of an aggregation operator $\mathcal{A} = \{A_n\}_{n=1}^{\infty}$ on I means that for all $\varepsilon > 0$ and all $(x_1^0, \dots, x_n^0) \in I^n$ there exists $\delta > 0$ such that if

$$(x_1, \dots, x_n) \in I^n, \ \sum_{i=1}^{n} |x_i - x_i^0| < \delta,$$

then

$$|A_n(x_1, \dots, x_n) - A_n(x_1^0, \dots, x_n^0)| < \varepsilon.$$

Hence, it is clear that each aggregation operator Lipschitz with an arbitrary $L \in (0, \infty)$ is also continuous (but not vice-versa).

Now, let us define the weighted average as an aggregation operator and examine its properties.

Remark 2.5. Throughout the paper, let $\mathcal{W}_n$ denote the set of all n-tuples of weights, i.e.

$$\mathcal{W}_n := \left\{ (w_1, \dots, w_n) \in \mathbb{R}^n \mid w_i \geq 0, \ i = 1, \dots, n, \ \sum_{i=1}^{n} w_i \neq 0 \right\}.$$

Definition 2.6. Let $W = \{\mathbf{w}_n\}_{n=1}^{\infty}$, be a sequence of weight vectors where

$$\mathbf{w}_n := (w_{n1}, \dots, w_{nn}) \in \mathcal{W}_n, \ n = 1, 2, \dots.$$

The aggregation operator $\mathcal{A}^W = \{A_n^W\}_{n=1}^{\infty}$ defined for each $n \in \mathbb{N}$ and each $(x_1, \dots, x_n) \in \mathbb{R}^n$ by the formula

$$A_n^W(x_1, \dots, x_n) = \frac{\sum_{i=1}^{n} w_{ni} x_i}{\sum_{i=1}^{n} w_{ni}}$$

is called a weighted average operator associated with W.

It is obvious that for any sequence of weight vectors $W = \{\mathbf{w}_n\}_{n=1}^{\infty}$, the weighted average $\mathcal{A}^W$ associated with W is compensative, idempotent, stable for a linear transformation, 1-Lipschitz, and continuous. If for any $n \in \mathbb{N}$, $w_{ni} > 0$, $i = 1, \ldots, n$, then $\mathcal{A}^W$ is strictly monotonic. $\mathcal{A}^W$ is symmetric, if and only if for any $n \in \mathbb{N}$, $w_{ni} = w_{nj}$ for all $i, j \in \{1, \ldots, n\}$. In such a case, $\mathcal{A}^W$ coincides with the arithmetic mean operator, i.e. for any $n \in \mathbb{N}$, $A_n^W(x_1, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^n x_i$.

3. FUZZY WEIGHTED AVERAGE OPERATOR AND ITS PROPERTIES

First in this section, the fuzzy weighted average operator will be defined as a sequence of fuzzy weighted averages of fuzzy numbers. Afterwards, whether and in which way the above mentioned properties of the weighted average operator are preserved in case of the fuzzy weighted average operator will be examined.

As it was mentioned in Introduction, the most common way for an expert how to express uncertain weights and uncertain weighted values is to describe their values separately by fuzzy numbers.

Definition 3.1. A fuzzy number is a fuzzy set X on $\mathbb{R}$ whose membership function $\mu_X : \mathbb{R} \to [0, 1]$ fulfils the following three conditions:

1. the set $Core\, X := \{x \in \mathbb{R} \mid \mu_X(x) = 1\}$, called the core of X, is nonempty,

2. for any $\alpha \in (0, 1]$, the set $X_\alpha := \{x \in \mathbb{R} \mid \mu_X(x) \geq \alpha\}$, called the α-cut of X, is a closed interval (the 1-cut X_1 means the core of X),

3. the set $Supp\, X := \{x \in \mathbb{R} \mid \mu_X(x) > 0\}$, called the support of X, is bounded.

Remark 3.2. The set of all fuzzy numbers will be denoted by $\mathcal{F}_N(\mathbb{R})$ throughout the paper.

Remark 3.3. Any fuzzy number X can be uniquely given by the pair of functions $\underline{x}$ and $\overline{x}$ defined on $[0, 1]$ such that $[\underline{x}(\alpha), \overline{x}(\alpha)] = X_\alpha$ for all $\alpha \in (0, 1]$ and $[\underline{x}(0), \overline{x}(0)]$ means the closure of the support of X, further denoted by X_0. For such a description of a fuzzy number, the notation $X = (\underline{x}, \overline{x})$ will be used throughout the paper. It was shown in [23] that $\underline{x}$, $\overline{x}$ are left-continuous on $(0, 1]$, right-continuous at 0, and

$$\underline{x}(\alpha) \leq \underline{x}(\beta) \leq \overline{x}(\beta) \leq \overline{x}(\alpha) \quad \text{for all } 0 \leq \alpha < \beta \leq 1. \tag{2}$$

Remark 3.4. A real number $x \in \mathbb{R}$ can be viewed as a fuzzy number $X = (\underline{x}, \overline{x})$, where $\underline{x}(\alpha) = \overline{x}(\alpha) = x$ for all $\alpha \in [0, 1]$. This enables us to understand the weighted average operator introduced in previous section as a particular case of the fuzzy weighted average operator defined below. Moreover, by this convention we can handle the cases where some input variables (weights or weighted values) are crisp and some are fuzzy.

Remark 3.5. In the crisp case, the weights $w_1, w_2, \dots, w_n$ are generally supposed to be non-negative real numbers whose sum is different from zero. It was shown in [24] that an n-tuple of fuzzy numbers $W_i = (\underline{w}_i, \overline{w}_i)$, $i = 1, 2, \dots, n$, modelling uncertain values of weights has to satisfy the following conditions:

$$\underline{w}_i(0) \geq 0 \quad \text{for all } i \in \{1, \dots, n\}, \tag{3}$$

$$\overline{w}_i(1) > 0 \quad \text{for at least one } i \in \{1, \dots, n\}. \tag{4}$$

An n-tuple of fuzzy numbers $(W_1, \dots, W_n)$ satisfying conditions (3) and (4) will be called an n-tuple of fuzzy weights hereafter. For a given $n \in \mathbb{N}$, the set of all n-tuples of fuzzy weights will be denoted by $\mathcal{W}_n^F$ throughout the paper.

At first, let us define the fuzzy weighted average of fuzzy numbers that represents the fuzzy extension of the weighted average operation a_w given by (1). It is a well-known fact that the fuzzy weighted average of fuzzy numbers cannot be computed simply by employing the standard fuzzy arithmetic operations because in formula (1), the same variables (the weights) appear both in the numerator and in the denominator. Therefore, the concept of constrained fuzzy arithmetic (see e. g. [17]) has to be applied.

Definition 3.6. Let $(W_1, \dots, W_n) \in \mathcal{W}_n^F$, and let $X_i = (\underline{x}_i, \overline{x}_i)$, $i = 1, \dots, n$, be fuzzy numbers. The fuzzy weighted average of $X_1, \dots, X_n$ with associated fuzzy weights $W_1, \dots, W_n$ is the fuzzy number

$$a_w^F(X_1, \dots, X_n; W_1, \dots, W_n) = (\underline{a}_w, \overline{a}_w)$$

such that for all $\alpha \in [0, 1]$:

$$\underline{a}_w(\alpha) = \min \left\{ \frac{\sum_{i=1}^n w_i \underline{x}_i(\alpha)}{\sum_{i=1}^n w_i} \mid w_i \in W_{i\alpha},\ i = 1, \dots, n,\ \sum_{i=1}^n w_i \neq 0 \right\}, \tag{5}$$

$$\overline{a}_w(\alpha) = \max \left\{ \frac{\sum_{i=1}^n w_i \overline{x}_i(\alpha)}{\sum_{i=1}^n w_i} \mid w_i \in W_{i\alpha},\ i = 1, \dots, n,\ \sum_{i=1}^n w_i \neq 0 \right\}. \tag{6}$$

Remark 3.7. In constrained optimization problems (5) and (6), the condition that $\sum_{i=1}^n w_i \neq 0$ is relevant only in a case where, for a given $\alpha \in [0, 1]$, $0 \in W_{i\alpha}$ for all $i \in \{1, \dots, n\}$. How can be such a case handled was analyzed in [24, Remark 19].

The usual approach for computing the fuzzy weighted average is the α-cut decomposition method proposed firstly in [5]. The input fuzzy numbers are discretized into a set of α-cuts and the boundary values of α-cuts of the fuzzy weighted average are computed applying (5) and (6). Thus, the final fuzzy weighted average is observed only approximately by connecting these α-cuts together. Various methods for solving constrained optimization problems (5) and (6) have been proposed in the literature, see e. g. [12, 14, 15, 20]; a detailed survey can be found in [21].

Now, let us define the fuzzy weighted average operator as a sequence of mappings similarly like in the case of the weighted average operator.

Definition 3.8. Let $W_F = \{\mathbf{W}_n^F\}_{n=1}^\infty$, where

$$\mathbf{W}_n^F := (W_{n1}, \dots, W_{nn}) \in \mathcal{W}_n^F, \quad n = 1, 2, \dots,$$

be a sequence of tuples of fuzzy weights. The fuzzy weighted average operator associated with W_F is the sequence $\mathcal{A}^{W_F} = \{A_n^{W_F}\}_{n=1}^\infty$, where for each $n \in \mathbb{N}$, $A_n^{W_F} : \mathcal{F}_N(\mathbb{R})^n \to \mathcal{F}_N(\mathbb{R})$ is for any n-tuple of fuzzy numbers $X_1, \dots, X_n$ given by

$$A_n^{W_F}(X_1, \dots, X_n) = a_w^F(X_1, \dots, X_n; W_{n1}, \dots, W_{nn}).$$

In the following, the way in which the three conditions from definition of an aggregation operator, namely identity, monotonicity, and the boundary condition, are reached by the fuzzy weighted average operator will be examined first.

3.1. Identity, monotonicity, and the boundary condition

Let $W = \{\mathbf{w}_n\}_{n=1}^\infty$ be an arbitrary sequence of vectors of weights and $\mathcal{A}^W = \{A_n^W\}_{n=1}^\infty$ be the weighted average operator associated with W.

According to the first condition in Definition 2.2, the first aggregation function A_1^W is an identity function, i. e.

$$A_1^W(x) = x \quad \text{for all } x \in \mathbb{R}.$$

The following theorem shows that the fuzzy weighted average operator possesses this property as well.

Theorem 3.9. Let $W_F = \{\mathbf{W}_n^F\}_{n=1}^\infty$ be an arbitrary sequence of tuples of fuzzy weights and $\mathcal{A}^{W_F} = \{A_n^{W_F}\}_{n=1}^\infty$ be the fuzzy weighted average operator associated with W_F. Then

$$A_1^{W_F}(X) = X \quad \text{for all } X \in \mathcal{F}_N(\mathbb{R}).$$

Proof. Let $X = (\underline{x}, \overline{x}) \in \mathcal{F}_N(\mathbb{R})$, and let us denote $A_1^{W_F}(X) = (\underline{a}_{w1}, \overline{a}_{w1})$. Then for all $\alpha \in [0, 1]$:

$$\underline{a}_{w1}(\alpha) = \min\left\{\frac{w_1\underline{x}(\alpha)}{w_1} \mid w_1 \in W_{11\alpha}, w_1 \neq 0\right\} = \underline{x}(\alpha),$$
$$\overline{a}_{w1}(\alpha) = \max\left\{\frac{w_1\overline{x}(\alpha)}{w_1} \mid w_1 \in W_{11\alpha}, w_1 \neq 0\right\} = \overline{x}(\alpha),$$

i. e. $A_1^{W_F}(X) = X$, which completes the proof. □

The second condition in Definition 2.2 says that the aggregation functions A_n^W, $n = 2, 3, \dots$, are non-decreasing, i. e. if $x_i \leq y_i$ for $i = 1, \dots, n$, then

$$A_n^W(x_1, \dots, x_n) \leq A_n^W(y_1, \dots, y_n).$$

For the purpose of verifying the fulfilment of this condition by the fuzzy weighted average of fuzzy numbers, it is necessary to define the ordering of fuzzy numbers.

Definition 3.10. We say that a fuzzy number $X = (\underline{x}, \overline{x})$ is less or equal than a fuzzy number $Y = (\underline{y}, \overline{y})$, denoted by $X \leq Y$, if

$$\underline{x}(\alpha) \leq \underline{y}(\alpha) \text{ and } \overline{x}(\alpha) \leq \overline{y}(\alpha) \quad \text{for all } \alpha \in [0,1].$$

If $X \leq Y$, but $Y \not\leq X$, then we say that a fuzzy number X is less than a fuzzy number Y, denoted by $X < Y$.

According to the following theorem, the fuzzy weighted average operator fulfils the generalization of the second condition to a fuzzy sets environment.

Theorem 3.11. Let $W_F = \{\mathbf{W}_n^F\}_{n=1}^{\infty}$ be an arbitrary sequence of tuples of fuzzy weights and $\mathcal{A}^{W_F} = \{A_n^{W_F}\}_{n=1}^{\infty}$ be the fuzzy weighted average operator associated with W_F. For any $n \in \{2, 3, \ldots\}$, let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be fuzzy numbers such that $X_i \leq Y_i$ for all $i \in \{1, \ldots, n\}$. Then

$$A_n^{W_F}(X_1, \ldots, X_n) \leq A_n^{W_F}(Y_1, \ldots, Y_n).$$

Proof. Let $n \in \{2, 3, \ldots\}$ be arbitrary. Let, for all $i \in \{1, \ldots, n\}$, $X_i = (\underline{x}_i, \overline{x}_i)$ and $Y_i = (\underline{y}_i, \overline{y}_i)$ be fuzzy numbers such that $\underline{x}_i(\alpha) \leq \underline{y}_i(\alpha)$ and $\overline{x}_i(\alpha) \leq \overline{y}_i(\alpha)$ for all $\alpha \in [0,1]$. Let us denote $A_n^{W_F}(X_1, \ldots, X_n) = (\underline{a}_{wn}^X, \overline{a}_{wn}^X)$ and $A_n^{W_F}(Y_1, \ldots, Y_n) = (\underline{a}_{wn}^Y, \overline{a}_{wn}^Y)$.

For any $\alpha \in [0,1]$, let $w_i^* \in W_{ni\alpha}$, $i = 1, \ldots, n$, be the weights such that

$$\begin{aligned} \underline{a}_{wn}^Y(\alpha) &= \min \left\{ \frac{\sum_{i=1}^n w_i \underline{y}_i(\alpha)}{\sum_{i=1}^n w_i} \mid w_i \in W_{ni\alpha}, i = 1, \ldots, n, \sum_{i=1}^n w_i \neq 0 \right\} \\ &= \frac{\sum_{i=1}^n w_i^* \underline{y}_i(\alpha)}{\sum_{i=1}^n w_i^*}. \end{aligned}$$

Let us denote

$$x^* = \frac{\sum_{i=1}^n w_i^* \underline{x}_i(\alpha)}{\sum_{i=1}^n w_i^*}.$$

Since $\underline{x}_i(\alpha) \leq \underline{y}_i(\alpha)$ for all $i \in \{1, \ldots, n\}$, $x^* \leq \underline{a}_{wn}^Y(\alpha)$. Further,

$$\underline{a}_{wn}^X(\alpha) = \min \left\{ \frac{\sum_{i=1}^n w_i \underline{x}_i(\alpha)}{\sum_{i=1}^n w_i} \mid w_i \in W_{ni\alpha}, i = 1, \ldots, n, \sum_{i=1}^n w_i \neq 0 \right\} \leq x^*.$$

Hence, $\underline{a}_{wn}^X(\alpha) \leq \underline{a}_{wn}^Y(\alpha)$. The proof that $\overline{a}_{wn}^X(\alpha) \leq \overline{a}_{wn}^Y(\alpha)$ would be analogous. □

Importance of this property consists, for instance, in preserving *the Pareto dominance* in fuzzy MCDM models where the fuzzy weighted average operation is applied for computation of the overall evaluations of alternatives. Let us illustrate this by the following example.

Example 3.12. Let us consider the following fuzzy MCDM model: Alternatives are evaluated with respect to n criteria; the particular evaluations are expressed by fuzzy numbers (for instance, they can represent mathematical meaning of terms from a given linguistic scale). The vaguely given information about importance of particular criteria is modelled by an n-tuple of fuzzy weights $W_1, \ldots, W_n$. The overall fuzzy evaluations of alternatives, based on which the alternatives are compared, are computed as the fuzzy weighted averages of the fuzzy evaluations with respect to the particular criteria.

Let us assume that we have two alternatives x_1 and x_2 such that their evaluations with respect to the particular criteria are given by fuzzy numbers $X_{11}, \ldots, X_{1n}$, and $X_{21}, \ldots, X_{2n}$, respectively. If x_1 is Pareto dominant with respect to x_2, i. e. if $X_{1i} \geq X_{2i}$ for all $i \in \{1, \ldots, n\}$, then, regardless of the applied n-tuple of fuzzy weights, x_1 will not be preferred by x_2 since the following relation holds:

$$a_w^F(X_{11}, \ldots, X_{1n}; W_1, \ldots, W_n) \geq a_w^F(X_{21}, \ldots, X_{2n}; W_1, \ldots, W_n).$$

Let us focus now on the third condition in Definition 2.2, i. e. on the boundary condition. In our case, $I = \mathbb{R}$, so for any $n \in \mathbb{N}$, we get

$$\lim_{(x_1,\ldots,x_n)\to(-\infty,\ldots,-\infty)} A_n^W(x_1, \ldots, x_n) = -\infty,$$
$$\lim_{(x_1,\ldots,x_n)\to(+\infty,\ldots,+\infty)} A_n^W(x_1, \ldots, x_n) = +\infty.$$

In the case of input fuzzy numbers, we have to discuss first how to express the divergence of a fuzzy number to $-\infty$ or $+\infty$. Let $X = (\underline{x}, \overline{x})$ be a fuzzy number. Then, considering the relations given in (2), we can say that the fuzzy number X diverges to $-\infty$ if $\overline{x}(0) \to -\infty$, since this implies that $\underline{x}(\alpha) \to -\infty$ and $\overline{x}(\alpha) \to -\infty$ for any $\alpha \in [0, 1]$. Analogously, we can say that X diverges to $+\infty$ if $\underline{x}(0) \to +\infty$. Next theorem shows that under such convention the fuzzy weighted average operator satisfies the generalization of the boundary condition.

Theorem 3.13. Let $W_F = \{\mathbf{W}_n^F\}_{n=1}^{\infty}$ be an arbitrary sequence of tuples of fuzzy weights and $\mathcal{A}^{W_F} = \{A_n^{W_F}\}_{n=1}^{\infty}$ be the fuzzy weighted average operator associated with W_F. Then the following holds for each $n \in \mathbb{N}$: Let $X_i = (\underline{x}_i, \overline{x}_i)$, $i = 1, \ldots, n$, be fuzzy numbers. Let us denote $A_n^{W_F}(X_1, \ldots, X_n) = (\underline{a}_{wn}, \overline{a}_{wn})$. Then

$$\lim_{(\overline{x}_1(0),\ldots,\overline{x}_n(0))\to(-\infty,\ldots,-\infty)} \overline{a}_{wn}(0) = -\infty, \tag{7}$$

and

$$\lim_{(\underline{x}_1(0),\ldots,\underline{x}_n(0))\to(+\infty,\ldots,+\infty)} \underline{a}_{wn}(0) = +\infty. \tag{8}$$

Proof. Eqs. (7) and (8) follow directly from the fact that for any n-tuple of weights $(w_1, \ldots, w_n) \in \mathcal{W}_n$ it holds that

$$\lim_{(\overline{x}_1(0),\ldots,\overline{x}_n(0))\to(-\infty,\ldots,-\infty)} \frac{\sum_{i=1}^n w_i \overline{x}_i(0)}{\sum_{i=1}^n w_i} = -\infty,$$
$$\lim_{(\underline{x}_1(0),\ldots,\underline{x}_n(0))\to(+\infty,\ldots,+\infty)} \frac{\sum_{i=1}^n w_i \underline{x}_i(0)}{\sum_{i=1}^n w_i} = +\infty.$$

□

Remark 3.14. From the practical point of view, it is worth to note that if the input fuzzy numbers $X_i = (\underline{x}_i, \overline{x}_i)$, $i = 1, \ldots, n$, are restricted to an interval I, i.e. if $[\underline{x}(0), \overline{x}(0)] \subseteq I$ for $i = 1, \ldots, n$, then we can in Theorem 3.13 replace $-\infty$ and $+\infty$ by x^- and x^+ and the theorem remains valid.

This property can be interpreted in such a way that if we aggregate only the minimal (maximal) possible inputs, then we obtain the minimal (maximal) possible output by the fuzzy weighted average operation, regardless of the applied tuple of fuzzy weights. Significance of this property is illustrated by the following example.

Example 3.15. Let us consider the fuzzy MCDM model described in Example 3.12. According to Theorem 3.13, if all the fuzzy evaluations with respect to the particular criteria tend to the worst (best) possible evaluation, then the overall fuzzy evaluation of an alternative also tend to the worst (best) one, regardless of the applied n-tuple of fuzzy weights. According to Remark 3.14, in the special case of such fuzzy MCDM model in which the fuzzy evaluations of alternatives with respect to the particular criteria are restricted to $[0, 1]$, where 0 means the worst possible and 1 the best possible evaluation, we get

$$a_w^F(0, \ldots, 0; W_1, \ldots, W_n) = 0 \text{ and } a_w^F(1, \ldots, 1; W_1, \ldots, W_n) = 1.$$

Hence, the overall fuzzy evaluation of the worst possible and of the best possible alternative are equal to 0, and to 1, respectively, regardless of the applied n-tuple of fuzzy weights.

Thus, we have shown that for any sequence of tuples of fuzzy weights the corresponding fuzzy weighted average operator satisfies generalized conditions from the definition of an aggregation operator. This fact especially means that if we aggregate in the model only fuzzy numbers restricted to some interval I (e.g. $[0, 1]$), then the resulting fuzzy weighted average is also a fuzzy number restricted to I.

In next sections, the way how can be the other properties of the weighted average operator observed for the fuzzy weighted average operator will be examined.

3.2. Compensation

The weighted average operator $\mathcal{A}^W$ associated with any sequence of weight vectors W is a compensative aggregation operator. This property means that for any $n \in \mathbb{N}$, the result of the aggregation of any n-tuple of real numbers $x_1, \ldots, x_n$ satisfies

$$\min\{x_1, \ldots, x_n\} \leq A_n^W(x_1, \ldots, x_n) \leq \max\{x_1, \ldots, x_n\}.$$

In the following theorem, it is proved that the fuzzy weighted average of fuzzy numbers can be bounded by fuzzy numbers representing the fuzzy extension of the functions minimum and maximum according to the extension principle.

Theorem 3.16. Let $W_F = \{\mathbf{W}_n^F\}_{n=1}^{\infty}$ be an arbitrary sequence of tuples of fuzzy weights and $\mathcal{A}^{W_F} = \{A_n^{W_F}\}_{n=1}^{\infty}$ be the fuzzy weighted average operator associated with W_F. For any $n \in \mathbb{N}$ and any n-tuple of fuzzy numbers $X_i = (\underline{x}_i, \overline{x}_i)$, $i = 1, \ldots, n$, let us consider fuzzy numbers $MIN = \left(\underline{\min}, \overline{\min}\right)$ and $MAX = (\underline{\max}, \overline{\max})$ such that for all $\alpha \in [0,1]$:

$$\underline{\min}(\alpha) = \min\{\underline{x}_1(\alpha), \ldots, \underline{x}_n(\alpha)\} \quad \text{and} \quad \overline{\min}(\alpha) = \min\{\overline{x}_1(\alpha), \ldots, \overline{x}_n(\alpha)\},$$
$$\underline{\max}(\alpha) = \max\{\underline{x}_1(\alpha), \ldots, \underline{x}_n(\alpha)\} \quad \text{and} \quad \overline{\max}(\alpha) = \max\{\overline{x}_1(\alpha), \ldots, \overline{x}_n(\alpha)\}.$$

Then
$$MIN \le A_n^{W_F}(X_1, \ldots, X_n) \le MAX.$$

P r o o f. Let $n \in \mathbb{N}$ be arbitrary. For any n-tuple of fuzzy numbers $X_1, \ldots, X_n$, let $A_n^{W_F}(X_1, \ldots, X_n) = (\underline{a}_{wn}, \overline{a}_{wn})$. Since a weighted average is a compensative operator, it holds for any n-tuple of real weights $w_1, \ldots, w_n$ that

$$\min\{\underline{x}_1(\alpha), \ldots, \underline{x}_n(\alpha)\} \le \frac{\sum_{i=1}^n w_i \underline{x}_i(\alpha)}{\sum_{i=1}^n w_i} \le \max\{\underline{x}_1(\alpha), \ldots, \underline{x}_n(\alpha)\}.$$

As
$$\underline{a}_{wn}(\alpha) = \min\left\{\frac{\sum_{i=1}^n w_i \underline{x}_i(\alpha)}{\sum_{i=1}^n w_i} \mid w_i \in W_{ni\alpha},\ i = 1, \ldots, n,\ \sum_{i=1}^n w_i \neq 0\right\},$$

it is clear that
$$\underline{\min}(\alpha) \le \underline{a}_{wn}(\alpha) \le \underline{\max}(\alpha) \quad \text{for all } \alpha \in [0,1].$$

Analogously, we would obtain that

$$\overline{\min}(\alpha) \le \overline{a}_{wn}(\alpha) \le \overline{\max}(\alpha) \quad \text{for all } \alpha \in [0,1],$$

which completes the proof. □

This property is very significant for application of the fuzzy weighted average operator in fuzzy MCDM models. It ensures that the overall fuzzy evaluation of an alternative computed as the fuzzy weighted average of the fuzzy evaluations with respect to the particular criteria do not exceeded the boundaries formed by the worst and the best particular fuzzy evaluation. Let us illustrate this by the following numerical example.

Example 3.17. An alternative x is to be evaluated with respect to four criteria. The fuzzy weights of the criteria are given by the quadruple of triangular fuzzy numbers $W_1, \ldots, W_4 \in \mathcal{F}_N([0,10])$, and the fuzzy evaluations of x with respect to the particular criteria are expressed by the triangular fuzzy numbers $X_1, \ldots, X_4 \in \mathcal{F}_N([0,1])$. The membership functions of the fuzzy weights and fuzzy weighted values are depicted in Figure 1.

The overall fuzzy evaluation of x is expressed by the fuzzy number $X \in \mathcal{F}_N([0,1])$, $X := a_w^F(X_1, \ldots, X_4; W_1, \ldots, W_4)$. In Figure 2, we can see that the fuzzy number X is between the fuzzy numbers MIN and MAX that express the fuzzy minimum and fuzzy maximum of the fuzzy evaluations $X_1, \ldots, X_4$.

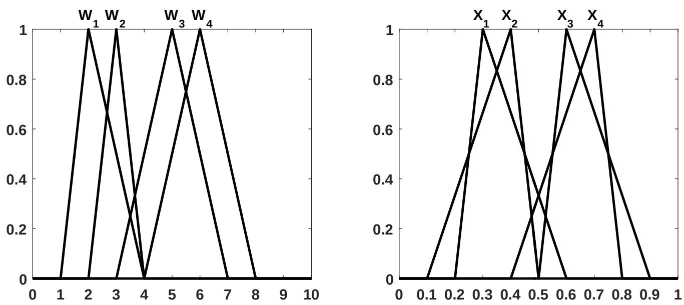

Fig. 1. Fuzzy weights $W_1, \ldots, W_4$ and fuzzy weighted values $X_1, \ldots, X_4$.

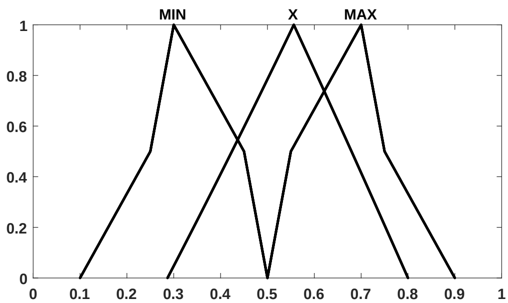

Fig. 2. Fuzzy numbers MIN, X, and MAX.

3.3. Idempotency

Another property of the weighted average operator $\mathcal{A}^W$ is idempotency. For any $n \in \mathbb{N}$ and for any n-tuple of weights, it holds that

$$A_n^W(x, \ldots, x) = x \quad \text{for all } x \in \mathbb{R}.$$

Let us show now that the fuzzy weighted average operator possesses the identical property.

Theorem 3.18. Let $W_F = \{\mathbf{W}_n^F\}_{n=1}^{\infty}$ be an arbitrary sequence of tuples of fuzzy weights and $\mathcal{A}^{W_F} = \{A_n^{W_F}\}_{n=1}^{\infty}$ be the fuzzy weighted average operator associated with W_F. Then for each $X \in \mathcal{F}_N(\mathbb{R})$ it holds that

$$A_n^{W_F}(X, \ldots, X) = X \quad \text{for all } n \in \mathbb{N}.$$

P r o o f. Let $X = (\underline{x}, \overline{x})$ be an arbitrary fuzzy number. For each $n \in \mathbb{N}$, let us denote $A_n^{W_F}(X, \dots, X) = (\underline{a}_{wn}, \overline{a}_{wn})$. Then for all $\alpha \in [0,1]$:

$$\underline{a}_{wn}(\alpha) = \min\left\{ \frac{\sum_{i=1}^n w_i \underline{x}(\alpha)}{\sum_{i=1}^n w_i} \mid w_i \in W_{ni\alpha},\ i = 1, \dots, n,\ \sum_{i=1}^n w_i \neq 0 \right\}$$
$$= \min\left\{ \underline{x}(\alpha) \frac{\sum_{i=1}^n w_i}{\sum_{i=1}^n w_i} \mid w_i \in W_{ni\alpha},\ i = 1, \dots, n,\ \sum_{i=1}^n w_i \neq 0 \right\} = \underline{x}(\alpha).$$

The equality $\overline{a}_{wn}(\alpha) = \overline{x}(\alpha)$ for all $\alpha \in [0,1]$ can be derived analogously. □

Remark 3.19. Let a real number x be viewed as a fuzzy number by convention made in Remark 3.4. Then, according to Theorem 3.18, $A_n^{W_F}(x, \dots, x) = x$ for any associated n-tuple of fuzzy weights. Thus, in such a case, the fuzziness of the weights does not affect the resulting fuzzy weighted average as it is equal to the real number x.

As it is noted in [3], idempotency is supposed to be a genuine property of aggregation operators in some areas, e. g., in MCDM (see [8]). In fuzzy MCDM models, the property observed in Theorem 3.18 can be read as follows: if all criteria are satisfied in the same fuzzy degree X, then also the overall fuzzy evaluation is X, regardless of the applied tuple of fuzzy weights. Thus, the fuzziness of the weights affects the resulting fuzzy weighted average only in the case where the fuzzy weighted values are not all the same.

3.4. Stability for a linear transformation

An important property of the weighted average operator $\mathcal{A}^W$ is stability for a linear transformation described as follows: For all $n \in \mathbb{N}$, all $r, t \in \mathbb{R}$, and all $(x_1, \dots, x_n) \in \mathbb{R}^n$:

$$A_n^W(rx_1 + t, \dots, rx_n + t) = rA_n^W(x_1, \dots, x_n) + t.$$

Let us show now that the identical property can be observed also for the fuzzy weighted average operator.

Theorem 3.20. Let $W_F = \{\mathbf{W}_n^F\}_{n=1}^\infty$ be an arbitrary sequence of tuples of fuzzy weights and $\mathcal{A}^{W_F} = \{A_n^{W_F}\}_{n=1}^\infty$ be the fuzzy weighted average operator associated with W_F. Then, for each $n \in \mathbb{N}$, all $r, t \in \mathbb{R}$, and all $(X_1, \dots, X_n) \in \mathcal{F}_N(\mathbb{R})^n$ it holds that

$$A_n^{W_F}(rX_1 + t, \dots, rX_n + t) = rA_n^{W_F}(X_1, \dots, X_n) + t. \tag{9}$$

P r o o f. Let $n \in \mathbb{N}$ and $r, t \in \mathbb{R}$ be arbitrary. Let $X_i = (\underline{x}_i, \overline{x}_i)$, $i = 1, \dots, n$, be arbitrary fuzzy numbers. Let us denote $A_n^{W_F}(X_1, \dots, X_n) = (\underline{a}_{wn}, \overline{a}_{wn})$. Then

$$rA_n^{W_F}(X_1, \dots, X_n) + t = (r\underline{a}_{wn} + t, r\overline{a}_{wn} + t).$$

Further, let us denote $A_n^{W_F}(rX_1+t,\dots,rX_n+t)=(\underline{a}_{wn}^{rt},\overline{a}_{wn}^{rt})$. For all $\alpha\in[0,1]$, we get

$$\begin{aligned}
\underline{a}_{wn}^{rt}(\alpha) &= \min\left\{\frac{\sum_{i=1}^n w_i(r\underline{x}_i(\alpha)+t)}{\sum_{i=1}^n w_i} \mid w_i\in W_{ni\alpha},\ i=1,\dots,n,\ \sum_{i=1}^n w_i\neq 0\right\}\\
&= \min\left\{r\frac{\sum_{i=1}^n w_i\underline{x}_i(\alpha)}{\sum_{i=1}^n w_i}+t \mid w_i\in W_{ni\alpha},\ i=1,\dots,n,\ \sum_{i=1}^n w_i\neq 0\right\}\\
&= r\min\left\{\frac{\sum_{i=1}^n w_i\underline{x}_i(\alpha)}{\sum_{i=1}^n w_i} \mid w_i\in W_{ni\alpha},\ i=1,\dots,n,\ \sum_{i=1}^n w_i\neq 0\right\}+t\\
&= r\underline{a}_{wn}(\alpha)+t.
\end{aligned}$$

The proof that $\overline{a}_{wn}^{rt}(\alpha)=r\overline{a}_{wn}(\alpha)+t$ for all $\alpha\in[0,1]$ would be analogous. □

Remark 3.21. Analogously, it can be shown that the equality (9) holds also if the real constants r and t are replaced by arbitrary fuzzy numbers R and T.

In fuzzy models, this property ensures us that a linear transformation can be applied either to the fuzzy weighted values $X_1,\dots,X_n$, i. e. before the aggregation by the fuzzy weighted average, or to the resulting fuzzy weighted average $a_w^F(X_1,\dots,X_n;W_1,\dots,W_n)$. The result of the aggregation will be the same, regardless of the applied n-tuple of fuzzy weights. For instance, if we want to change in the fuzzy MCDM model considered in Example 3.17 the units of the evaluations of alternatives from the fuzzy degrees of satisfaction, belonging to $[0,1]$, to the fuzzy percentages of satisfaction, belonging to $[0,100]$, we can either multiply by 100 the fuzzy evaluations $X_1,\dots,X_4$ and then compute the fuzzy weighted average, or multiply by 100 the overall fuzzy evaluation X, the result will be the same.

3.5. Lipschitzianity and continuity

For any sequence of weight vectors W, the weighted average operator $\mathcal{A}^W$ is 1-Lipschitz. This means that for all $n\in\mathbb{N}$ and all $(x_1,\dots,x_n),(y_1,\dots,y_n)\in\mathbb{R}^n$:

$$|A_n^W(x_1,\dots,x_n)-A_n^W(y_1,\dots,y_n)|\leq\sum_{i=1}^n|x_i-y_i|.$$

In order to show the way in which the fuzzy weighted average operator possesses such a property, instead of the distance measure $d:\mathbb{R}^2\to\mathbb{R}_0^+$, defined for any $x,y\in\mathbb{R}$ by $d(x,y)=|x-y|$, a distance measure on the set of all fuzzy numbers has to be applied. Let us consider the popular distance measure of fuzzy numbers $d^F:\mathcal{F}_N(\mathbb{R})^2\to\mathbb{R}_0^+$, studied e. g. in [16], that is for any fuzzy numbers $X=(\underline{x},\overline{x})$ and $Y=(\underline{y},\overline{y})$ given by

$$d^F(X,Y)=\int_0^1\big(|\underline{x}(\alpha)-\underline{y}(\alpha)|+|\overline{x}(\alpha)-\overline{y}(\alpha)|\big)\,\mathrm{d}\alpha.$$

The following theorem shows that applying the metric d^F, the fuzzy weighted average operator is also 1-Lipschitz.

Theorem 3.22. Let $W_F = \{\mathbf{W}_n^F\}_{n=1}^{\infty}$ be an arbitrary sequence of tuples of fuzzy weights and $\mathcal{A}^{W_F} = \{A_n^{W_F}\}_{n=1}^{\infty}$ be the fuzzy weighted average operator associated with W_F. For all $n \in \mathbb{N}$, it holds the following: Let $X_i = (\underline{x}_i, \overline{x}_i)$ and $Y_i = (\underline{y}_i, \overline{y}_i)$, $i = 1, \ldots, n$, be fuzzy numbers. Let us denote $A_n^{W_F}(X_1, \ldots, X_n) = (\underline{a}_{wn}^X, \overline{a}_{wn}^X)$ and $A_n^{W_F}(Y_1, \ldots, Y_n) = (\underline{a}_{wn}^Y, \overline{a}_{wn}^Y)$. Then

$$\int_0^1 |\underline{a}_{wn}^X(\alpha) - \underline{a}_{wn}^Y(\alpha)| + |\overline{a}_{wn}^X(\alpha) - \overline{a}_{wn}^Y(\alpha)| \,\mathrm{d}\alpha$$
$$\leq \sum_{i=1}^n \int_0^1 |\underline{x}_i(\alpha) - \underline{y}_i(\alpha)| + |\overline{x}_i(\alpha) - \overline{y}_i(\alpha)| \,\mathrm{d}\alpha.$$

P r o o f. Let $n \in \mathbb{N}$ be arbitrary, and let $\alpha \in [0,1]$ be such that $\underline{a}_{wn}^X(\alpha) \geq \underline{a}_{wn}^Y(\alpha)$. Then

$$|\underline{a}_{wn}^X(\alpha) - \underline{a}_{wn}^Y(\alpha)| = \underline{a}_{wn}^X(\alpha) - \underline{a}_{wn}^Y(\alpha)$$
$$= \min\left\{\frac{\sum_{i=1}^n w_i \underline{x}_i(\alpha)}{\sum_{i=1}^n w_i} \mid w_i \in W_{ni\alpha},\ i = 1, \ldots, n,\ \sum_{i=1}^n w_i \neq 0\right\}$$
$$- \min\left\{\frac{\sum_{i=1}^n w_i \underline{y}_i(\alpha)}{\sum_{i=1}^n w_i} \mid w_i \in W_{ni\alpha},\ i = 1, \ldots, n,\ \sum_{i=1}^n w_i \neq 0\right\}$$
$$= \min\left\{\frac{\sum_{i=1}^n w_i \underline{x}_i(\alpha)}{\sum_{i=1}^n w_i} \mid w_i \in W_{ni\alpha}, i = 1, \ldots, n, \sum_{i=1}^n w_i \neq 0\right\}$$
$$- \frac{\sum_{i=1}^n w_i^* \underline{y}_i(\alpha)}{\sum_{i=1}^n w_i^*}$$
$$\leq \frac{\sum_{i=1}^n w_i^* \underline{x}_i(\alpha)}{\sum_{i=1}^n w_i^*} - \frac{\sum_{i=1}^n w_i^* \underline{y}_i(\alpha)}{\sum_{i=1}^n w_i^*} = \frac{\sum_{i=1}^n w_i^* (\underline{x}_i(\alpha) - \underline{y}_i(\alpha))}{\sum_{i=1}^n w_i^*}$$
$$\leq \frac{\sum_{i=1}^n w_i^* |\underline{x}_i(\alpha) - \underline{y}_i(\alpha)|}{\sum_{i=1}^n w_i^*} \leq \sum_{i=1}^n |\underline{x}_i(\alpha) - \underline{y}_i(\alpha)|.$$

If $\alpha \in [0,1]$ is such that $\underline{a}_{wn}^X(\alpha) < \underline{a}_{wn}^Y(\alpha)$, then

$$|\underline{a}_{wn}^X(\alpha) - \underline{a}_{wn}^Y(\alpha)| = -\underline{a}_{wn}^X(\alpha) + \underline{a}_{wn}^Y(\alpha)$$
$$= -\min\left\{\frac{\sum_{i=1}^n w_i \underline{x}_i(\alpha)}{\sum_{i=1}^n w_i} \mid w_i \in W_{ni\alpha},\ i = 1, \ldots, n,\ \sum_{i=1}^n w_i \neq 0\right\}$$
$$+ \min\left\{\frac{\sum_{i=1}^n w_i \underline{y}_i(\alpha)}{\sum_{i=1}^n w_i} \mid w_i \in W_{ni\alpha},\ i = 1, \ldots, n,\ \sum_{i=1}^n w_i \neq 0\right\}$$
$$= -\frac{\sum_{i=1}^n w_i^{**} \underline{x}_i(\alpha)}{\sum_{i=1}^n w_i^{**}}$$

$$+\min\left\{\frac{\sum_{i=1}^{n} w_i \underline{y}_i(\alpha)}{\sum_{i=1}^{n} w_i} \mid w_i \in W_{ni\alpha},\ i=1,\dots,n,\ \sum_{i=1}^{n} w_i \neq 0\right\}$$
$$\leq -\frac{\sum_{i=1}^{n} w_i^{**}\underline{x}_i(\alpha)}{\sum_{i=1}^{n} w_i^{**}} + \frac{\sum_{i=1}^{n} w_i^{**}\underline{y}_i(\alpha)}{\sum_{i=1}^{n} w_i^{**}} = \frac{\sum_{i=1}^{n} w_i^{**}(\underline{y}_i(\alpha)-\underline{x}_i(\alpha))}{\sum_{i=1}^{n} w_i^{**}}$$
$$\leq \sum_{i=1}^{n} |\underline{y}_i(\alpha)-\underline{x}_i(\alpha)| = \sum_{i=1}^{n} |\underline{x}_i(\alpha)-\underline{y}_i(\alpha)|.$$

Summing up, we obtain for all $n \in \mathbb{N}$ and $\alpha \in [0,1]$ that

$$|\underline{a}_{wn}^X(\alpha) - \underline{a}_{wn}^Y(\alpha)| \leq \sum_{i=1}^{n} |\underline{x}_i(\alpha)-\underline{y}_i(\alpha)|.$$

Analogously, we can obtain that

$$|\overline{a}_{wn}^X(\alpha) - \overline{a}_{wn}^Y(\alpha)| \leq \sum_{i=1}^{n} |\overline{x}_i(\alpha)-\overline{y}_i(\alpha)|.$$

Therefore,

$$\int_0^1 \big(|\underline{a}_{wn}^X(\alpha) - \underline{a}_{wn}^Y(\alpha)| + |\overline{a}_{wn}^X(\alpha) - \overline{a}_{wn}^Y(\alpha)|\big)\mathrm{d}\alpha$$
$$\leq \sum_{i=1}^{n}\int_0^1 \Big(|\underline{x}_i(\alpha)-\underline{y}_i(\alpha)| + |\overline{x}_i(\alpha)-\overline{y}_i(\alpha)|\Big)\mathrm{d}\alpha,$$

which completes the proof. □

The fact that the fuzzy weighted average operator $\mathcal{A}^{W_F}$ is 1-Lipschitz in the metric d^F implies that the following holds for all $n \in \mathbb{N}$: For all $\varepsilon > 0$ and for any fuzzy numbers $X_i^0 = (\underline{x}_i^0, \overline{x}_i^0)$, $i = 1,\dots,n$, there exists $\delta > 0$ such that if $X_i = (\underline{x}_i, \overline{x}_i)$, $i = 1,\dots,n$, are fuzzy numbers satisfying

$$\sum_{i=1}^{n}\int_0^1 \big(|\underline{x}_i(\alpha)-\underline{x}_i^0(\alpha)| + |\overline{x}_i(\alpha)-\overline{x}_i^0(\alpha)|\big)\mathrm{d}\alpha < \delta,$$

then it holds for $A_n^{W_F}(X_1^0,\dots,X_n^0) = (\underline{a}_{wn}^0, \overline{a}_{wn}^0)$ and $A_n^{W_F}(X_1,\dots,X_n) = (\underline{a}_{wn}, \overline{a}_{wn})$ that

$$\int_0^1 \big(|\underline{a}_{wn}(\alpha) - \underline{a}_{wn}^0(\alpha)| + |\overline{a}_{wn}(\alpha) - \overline{a}_{wn}^0(\alpha)|\big)\mathrm{d}\alpha < \varepsilon.$$

Hence, the fuzzy weighted average is continuous with respect to the metric d^F.

As it is written in [3], continuous aggregation operators are usually applied in engineering applications, reflecting the property that a "small" error in inputs cannot cause a "big" error in the output. That is, this property is a guaranty for a certain consistency and for a non chaotic behaviour. In case of the fuzzy weighted average operator, it means that regardless of the applied tuple of fuzzy weights if the fuzzy weighted values are changed only slightly, the resulting fuzzy weighted average will be close to the original one.

3.6. Strict monotonicity

In case of positive weights, the weighted average operator $\mathcal{A}^W$ is strictly monotone, i. e. for each $n \in \mathbb{N}$, if we have $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and $(y_1, \ldots, y_n) \in \mathbb{R}^n$ such that $x_i < y_i$ for one $i \in \{1, \ldots, n\}$ and $x_j = y_j$ for all $j \in \{1, \ldots, n\}$, $j \neq i$, then

$$A_n^W(x_1, \ldots, x_n) < A_n^W(y_1, \ldots, y_n).$$

Let us show now that this property can be observed for the fuzzy weighted average operator as well.

Remark 3.23. We say that fuzzy weights $W_i = (\underline{w}_i, \overline{w}_i)$, $i = 1, \ldots, n$, are positive if $\underline{w}_i(0) > 0$ for all $i \in \{1, \ldots, n\}$.

Theorem 3.24. Let $W_F^+ = \{\mathbf{W}_n^{+F}\}_{n=1}^{\infty}$ be an arbitrary sequence of tuples of positive fuzzy weights and $\mathcal{A}^{W_F^+} = \{A_n^{W_F^+}\}_{n=1}^{\infty}$ be the fuzzy weighted average operator associated with W_F^+. Let, for all $n \in \mathbb{N}$, $X_i = (\underline{x}_i, \overline{x}_i)$ and $Y_i = (\underline{y}_i, \overline{y}_i)$, $i = 1, \ldots, n$, be fuzzy numbers such that $X_i < Y_i$ for one $i \in \{1, \ldots, n\}$ and $X_j = Y_j$ for all $j \in \{1, \ldots, n\}$, $j \neq i$. Then

$$A_n^{W_F^+}(X_1, \ldots, X_n) < A_n^{W_F^+}(Y_1, \ldots, Y_n).$$

Proof. Let $n \in \mathbb{N}$ be arbitrary. Let us denote $A_n^{W_F^+}(X_1, \ldots, X_n) = (\underline{a}_{wn}^X, \overline{a}_{wn}^X)$ and $A_n^{W_F^+}(Y_1, \ldots, Y_n) = (\underline{a}_{wn}^Y, \overline{a}_{wn}^Y)$. Let us assume that there exists $\alpha \in [0, 1]$ such that $\underline{x}_i(\alpha) < \underline{y}_i(\alpha)$. Then

$$\begin{aligned}
\underline{a}_{wn}^X(\alpha) &= \min\left\{ \frac{\sum_{i=1}^n w_i \underline{x}_i(\alpha)}{\sum_{i=1}^n w_i} \mid w_i \in W_{ni\alpha},\ i = 1, \ldots, n \right\} \\
&= \min\left\{ \frac{\sum_{\substack{j=1 \\ j\neq i}}^n w_j \underline{x}_j(\alpha) + w_i \underline{x}_i(\alpha)}{\sum_{i=1}^n w_i} \mid w_i \in W_{ni\alpha},\ i = 1, \ldots, n \right\} \\
&< \min\left\{ \frac{\sum_{\substack{j=1 \\ j\neq i}}^n w_j \underline{y}_j(\alpha) + w_i \underline{y}_i(\alpha)}{\sum_{i=1}^n w_i} \mid w_i \in W_{ni\alpha},\ i = 1, \ldots, n \right\} = \underline{a}_{wn}^Y(\alpha).
\end{aligned}$$

The proof that $\overline{a}_{wn}^X(\alpha) < \overline{a}_{wn}^Y(\alpha)$ if $\overline{x}_i(\alpha) < \overline{y}_i(\alpha)$ would be analogous. $\square$

The importance of this property can be illustrated by the following example that is based on the fuzzy MCDM model considered in Example 3.12.

Example 3.25. Let the fuzzy weights $W_1, \ldots, W_n$ expressing the importance of the particular criteria be positive. Let the fuzzy evaluations of the two alternatives x_1 and x_2 with respect to the particular criteria be expressed by the fuzzy numbers $X_{11}, \ldots, X_{1n}$, and $X_{21}, \ldots, X_{2n}$, respectively. Let, for some $i \in \{1, \ldots, n\}$, $X_{1i} > X_{2i}$, and $X_{1j} = X_{2j}$ for all $j \in \{1, \ldots, n\} \setminus \{i\}$, i. e. the alternative x_1 is better than x_2 with respect to the ith criterion and is equal with x_2 with respect to the rest of the criteria. Then the overall fuzzy evaluation of x_1 will be better than the overall fuzzy evaluation of x_2 because the following relation holds:

$$a_w^F(X_{11},\dots,X_{1n};W_1,\dots,W_n) > a_w^F(X_{21},\dots,X_{2n};W_1,\dots,W_n).$$

3.7. Symmetry

As it was mentioned before, the weighted average operator $\mathcal{A}^W$ associated with a sequence of weight vectors W is symmetric, if and only if for any $n \in \mathbb{N}$, $w_{ni} = w_{nj}$ for all $i, j \in \{1,\dots,n\}$. In such a case, $\mathcal{A}^W$ coincides with the arithmetic mean operator, i. e. for any $n \in \mathbb{N}$, $A_n^W(x_1,\dots,x_n) = \frac{1}{n}\sum_{i=1}^n x_i$ for all $x_1,\dots,x_n \in \mathbb{R}$.

The following theorem shows that if the particular fuzzy weights are the same in each tuple of fuzzy weights, then the fuzzy weighted average operator is also symmetric.

Theorem 3.26. Let $E_F = \{\mathbf{E}_n^F\}_{n=1}^\infty$ be a sequence of tuples of fuzzy weights $\mathbf{E}_n^F = (W_{1n},\dots,W_{nn})$ such that $W_{ni} = W_{nj}$ for all $i, j \in \{1,\dots,n\}$. Let $\mathcal{A}^{E_F} = \{A_n^{E_F}\}_{n=1}^\infty$ be the fuzzy weighted average operator associated with E_F. Then for all $n \in \mathbb{N}$, each n-tuple of fuzzy numbers $(X_1,\dots,X_n)$, and any permutation σ of $\{1,\dots,n\}$ it holds that

$$A_n^{E_F}(X_{\sigma(1)},\dots,X_{\sigma(n)}) = A_n^{E_F}(X_1,\dots,X_n).$$

Proof. Let $n \in \mathbb{N}$ be arbitrary. Let $W \in \mathcal{F}_N(\mathbb{R})$ be such that $W_{ni} = W$ for $i = 1,\dots,n$, i.e. $\mathbf{E}_n^F = (W,\dots,W)$. Let $X_i = (\underline{x}_i, \overline{x}_i)$, $i = 1,\dots,n$, be arbitrary fuzzy numbers, and let σ be arbitrary permutation of $\{1,\dots,n\}$. Let us denote $A_n^{E_F}(X_1,\dots,X_n) = (\underline{a}_{wn}, \overline{a}_{wn})$ and $A_n^{E_F}(X_{\sigma(1)},\dots,X_{\sigma(n)}) = (\underline{a}_{wn}^\sigma, \overline{a}_{wn}^\sigma)$. Then for all $\alpha \in [0,1]$ we get

$$\underline{a}_{wn}(\alpha) = \min\left\{ \frac{\sum_{i=1}^n w_i \underline{x}_i(\alpha)}{\sum_{i=1}^n w_i} \mid w_i \in W_\alpha,\ i = 1,\dots,n,\ \sum_{i=1}^n w_i \neq 0 \right\}$$

$$= \min\left\{ \frac{\sum_{i=1}^n w_i \underline{x}_{\sigma(i)}(\alpha)}{\sum_{i=1}^n w_i} \mid w_i \in W_\alpha,\ i = 1,\dots,n,\ \sum_{i=1}^n w_i \neq 0 \right\} = \underline{a}_{wn}^\sigma(\alpha),$$

since the range W_α for the admissible values of the weights does not depend on i. The proof that $\overline{a}_{wn}(\alpha) = \overline{a}_{wn}^\sigma(\alpha)$ for all $\alpha \in [0,1]$ would be analogous. □

In practical applications, this property means that if the uncertain values of the weights are expressed by the equal fuzzy numbers, then, analogously as in the crisp case, the resulting fuzzy weighted average does not depend on the sequence of the fuzzy weighted values. However, next theorem shows that in contrast to the crisp case, the fuzzy weighted average operator associated with a sequence of tuples of equal fuzzy weights does not coincide with the fuzzy arithmetic mean operator $\mathcal{M}$. The fuzzy arithmetic mean operator is defined in the following way: $\mathcal{M} = \{M_n\}_{n=1}^\infty$ where for any $n \in \mathbb{N}$, M_n represents the fuzzy arithmetic mean of fuzzy numbers; it is given for any fuzzy numbers $X_i = (\underline{x}_i, \overline{x}_i)$, $i = 1,\dots,n$, by

$$M_n(X_1,\dots,X_n) = \left(\frac{1}{n}\sum_{i=1}^n \underline{x}_i, \frac{1}{n}\sum_{i=1}^n \overline{x}_i \right).$$

Theorem 3.27. Let $E_F = \{\mathbf{E}_n^F\}_{n=1}^{\infty}$ be a sequence of tuples of fuzzy weights $\mathbf{E}_n^F = (W_{1n}, \dots, W_{nn})$ such that $W_{ni} = W_{nj}$ for all $i, j \in \{1, \dots, n\}$. Let $\mathcal{A}^{E_F} = \{A_n^{E_F}\}_{n=1}^{\infty}$ be the fuzzy weighted average operator associated with E_F. Let $\mathcal{M} = \{M_n\}_{n=1}^{\infty}$ be the fuzzy arithmetic mean operator. Then for all $n \in \mathbb{N}$ and for each n-tuple of fuzzy numbers $(X_1, \dots, X_n)$:

$$M_n(X_1, \dots, X_n) \subseteq A_n^{E_F}(X_1, \dots, X_n), \tag{10}$$

where the equality $M_n(X_1, \dots, X_n) = A_n^{E_F}(X_1, \dots, X_n)$ holds, if and only if at least one of the following three conditions is satisfied:

1. $n = 1$,
2. $\mathbf{E}_n^F = (w, \dots, w)$, where $w \in \mathbb{R}^+$,
3. $X_i = X_j$ for all $i, j \in \{1, \dots, n\}$.

P r o o f. Let $n \in \mathbb{N}$ be arbitrary. Let $W \in \mathcal{F}_N(\mathbb{R})$ be such that $W_{ni} = W$ for all $i \in \{1, \dots, n\}$, i.e. $\mathbf{E}_n^F = (W, \dots, W)$. Let $X_i = (\underline{x}_i, \overline{x}_i)$, $i = 1, \dots, n$, be arbitrary fuzzy numbers. Let us denote $A_n^{E_F}(X_1, \dots, X_n) = (\underline{a}_{wn}, \overline{a}_{wn})$ and $M_n(X_1, \dots, X_n) = (\underline{m}_n, \overline{m}_n)$. For any $\alpha \in [0, 1]$:

$$\frac{1}{n}\sum_{i=1}^{n} \underline{x}_i(\alpha) \in \left\{ \frac{\sum_{i=1}^{n} w_i \underline{x}_i(\alpha)}{\sum_{i=1}^{n} w_i} \mid w_i \in W_\alpha,\ i = 1, \dots, n,\ \sum_{i=1}^{n} w_i \neq 0 \right\},$$

because $\frac{1}{n}\sum_{i=1}^{n} \underline{x}_i(\alpha) = \frac{\sum_{i=1}^{n} w \underline{x}_i(\alpha)}{\sum_{i=1}^{n} w}$, where $w \in W_\alpha$, $w \neq 0$. Hence

$$\min \left\{ \frac{\sum_{i=1}^{n} w_i \underline{x}_i(\alpha)}{\sum_{i=1}^{n} w_i} \mid w_i \in W_\alpha,\ i = 1, \dots, n,\ \sum_{i=1}^{n} w_i \neq 0 \right\} \leq \frac{1}{n}\sum_{i=1}^{n} \underline{x}_i(\alpha),$$

i.e. $\underline{a}_{wn}(\alpha) \leq \underline{m}_n(\alpha)$. Analogously, it can be shown that

$$\max \left\{ \frac{\sum_{i=1}^{n} w_i \overline{x}_i(\alpha)}{\sum_{i=1}^{n} w_i} \mid w_i \in W_\alpha,\ i = 1, \dots, n,\ \sum_{i=1}^{n} w_i \neq 0 \right\} \geq \frac{1}{n}\sum_{i=1}^{n} \overline{x}_i(\alpha),$$

i.e. $\overline{a}_{wn}(\alpha) \geq \overline{m}_n(\alpha)$. Therefore, $M_n(X_1, \dots, X_n) \subseteq A_n^{E_F}(X_1, \dots, X_n)$. Further, let us examine the cases when $M_n(X_1, \dots, X_n) = A_n^{E_F}(X_1, \dots, X_n)$.

For $n = 1$, the equality is obvious since $M_1(X_1) = X_1 = A_1^{E_F}(X_1)$ for any $X_1 \in \mathcal{F}_N(\mathbb{R})$. Let $n \geq 2$ and $\alpha \in [0, 1]$ be arbitrary. If $W_\alpha = \{w\}$, where $w \in \mathbb{R}^+$, then

$$\underline{a}_{wn}(\alpha) = \min \left\{ \frac{\sum_{i=1}^{n} w \underline{x}_i(\alpha)}{\sum_{i=1}^{n} w} \right\} = \frac{1}{n}\sum_{i=1}^{n} \underline{x}_i(\alpha) = \underline{m}_n(\alpha)$$

and

$$\overline{a}_{wn}(\alpha) = \max \left\{ \frac{\sum_{i=1}^{n} w \overline{x}_i(\alpha)}{\sum_{i=1}^{n} w} \right\} = \frac{1}{n}\sum_{i=1}^{n} \overline{x}_i(\alpha) = \overline{m}_n(\alpha).$$

If $\underline{x}_i(\alpha) = \underline{x}(\alpha)$ and $\overline{x}_i(\alpha) = \overline{x}(\alpha)$ for all $i \in \{1, \dots, n\}$, then, applying Theorem 3.18, $\underline{a}_{wn}(\alpha) = \underline{x}(\alpha) = \frac{1}{n}\sum_{i=1}^{n} \underline{x}(\alpha) = \underline{m}_n(\alpha)$ and $\overline{a}_{wn}(\alpha) = \overline{x}(\alpha) = \frac{1}{n}\sum_{i=1}^{n} \overline{x}(\alpha) = \overline{m}_n(\alpha)$. Thus, if $\mathbf{E}_n^F = (w, \dots, w)$, where $w \in \mathbb{R}^+$, or if $X_i = X_j$ for all $i, j \in \{1, \dots, n\}$, then $A_n^{E_F}(X_1, \dots, X_n) = M_n(X_1, \dots, X_n)$.

Finally, for $n \geq 2$ and for some $\alpha \in [0, 1]$, let us assume that $W_\alpha = [\underline{w}(\alpha), \overline{w}(\alpha)]$, where $\underline{w}(\alpha) < \overline{w}(\alpha)$, and that $X_{i\alpha} \neq X_{j\alpha}$ for some $i, j \in \{1, \dots, n\}$, $i \neq j$. If $\underline{x}_i(\alpha) < \underline{x}_j(\alpha)$, then obviously $\underline{a}_{wn}(\alpha) < \frac{1}{n}\sum_{k=1}^{n} \underline{x}_k(\alpha) = \underline{m}_n(\alpha)$, as for instance

$$\frac{\overline{w}(\alpha)\underline{x}_i(\alpha) + \sum_{k=1, k\neq i}^{n} \underline{w}(\alpha)\underline{x}_k(\alpha)}{\overline{w}(\alpha) + (n-1)\underline{w}(\alpha)} < \frac{1}{n}\sum_{k=1}^{n} \underline{x}_k(\alpha).$$

Analogously, it can be shown that if $\overline{x}_i(\alpha) < \overline{x}_j(\alpha)$, then $\overline{a}_{wn}(\alpha) > \frac{1}{n}\sum_{k=1}^{n} \overline{x}_k(\alpha) = \overline{m}_n(\alpha)$. Hence, $M_n(X_1, \dots, X_n)_\alpha$ is a strict subset of $A_n^{E_F}(X_1, \dots, X_n)_\alpha$. □

The fuzziness of the equal fuzzy weights affects the fuzziness of the resulting fuzzy weighted average the more, the more the fuzzy weighted values differ from each other. In the extreme case where the fuzzy weighted values are identical, the fuzziness of the equal fuzzy weights remains hidden due to the idempotency of the fuzzy weighted average operator. The problem is illustrated by the following numerical example.

Example 3.28. For $n = 4$, let all the fuzzy weights $W_1, \dots, W_4$ be equal to the triangular fuzzy number $W = (\underline{w}, \overline{w})$, where $\underline{w}(\alpha) = 3 + 2\alpha$ and $\overline{w}(\alpha) = 7 - 2\alpha$ for any $\alpha \in [0, 1]$. Let us consider two quadruples of fuzzy weighted values, $X_1, \dots, X_4$ and $Y_1, \dots, Y_4$, whose membership functions are depicted in Figure 3.

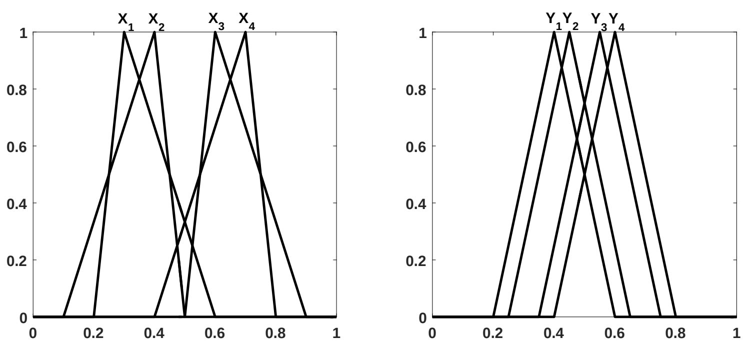

Fig. 3. Fuzzy weighted values $X_1, \dots, X_4$ and $Y_1, \dots, Y_4$.

The fuzzy arithmetic means $M_4(X_1, \dots, X_4)$ and $M_4(Y_1, \dots, Y_4)$ coincide; they are equal to the fuzzy number $M = (\underline{m}, \overline{m})$, where $\underline{m}(\alpha) = 0.3 + 0.2\alpha$ and $\overline{m}(\alpha) = 0.7 - 0.2\alpha$ for any $\alpha \in [0, 1]$. However, $X_1, \dots, X_4$ differ more from each other than $Y_1, \dots, Y_4$. Therefore, the fuzzy weighted average $A_X^W = a_w^F(X_1, \dots, X_4; W, \dots, W)$ is more uncertain than $A_Y^W = a_w^F(Y_1, \dots, Y_4; W, \dots, W)$; this can be easily seen from Figure 4, where the membership functions of A_X^W, A_Y^W and M are depicted.

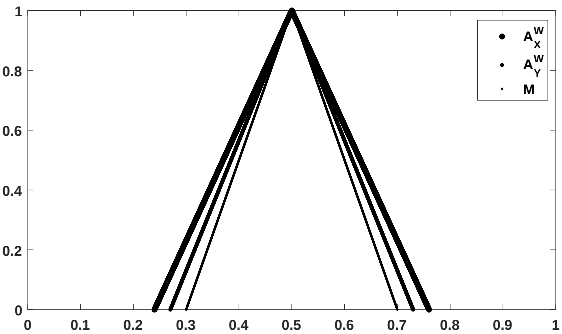

Fig. 4. Fuzzy numbers A_X^W, A_Y^W and M.

For instance, if the fuzzy weighted values $X_1, \ldots, X_4$ and $Y_1, \ldots, Y_4$ represent the fuzzy evaluations of the two alternatives x and y with respect to the given four criteria in a fuzzy MCDM model, i. e. if the fuzzy weighted averages A_X^W and A_Y^W express the overall fuzzy evaluations of x and y, then the greater uncertainty of A_X^W corresponds to the fact that despite the same mean, the fuzzy evaluations of x with respect to the particular criteria are more inconsistent and therefore, the overall fuzzy evaluation of x is more dependant to the values of the weights of criteria. Thus, this information will affect the final ranking of the alternatives.

4. CONCLUSION

In the paper, the properties of the fuzzy weighted average operator where the weights as well as the weighted values are expressed by noninteractive fuzzy numbers were examined. First, it was shown that the fuzzy weighted average operator fulfils the fuzzy extension of the three conditions that characterize an aggregation operator. Afterwards, it was revealed the way in which the fuzzy weighted average operator preserve the properties of the crisp weighted average operator, namely compensation, idempotency, stability for linear transformation, lipschitzianity, continuity, strict monotonicity in case of positive fuzzy weights, and symmetry in case of equal fuzzy weights. The usefulness of the obtained results was discussed and illustrated by several examples.

Further research in this area could be focused on the properties of other kinds of fuzzy weighted average operator, e. g. when the special structure of interactive fuzzy numbers called normalized fuzzy weights is applied, or when the weights and/or the weighted values are described by fuzzy vectors (see [24] for more details). It is also worth to study the properties of the fuzzy extension of other aggregation operators, like fuzzy OWA operator, etc.

ACKNOWLEDGEMENT

Research has been supported by the project No. GA 14-02424S *Methods of operations research for decision support under uncertainty* of the Grant Agency of the Czech Republic. The support is greatly acknowledged.

(Received February 8, 2016)

REFERENCES

[1] S. Baas and H. Kwakernaak: Rating and ranking of multiple–aspect alternatives using fuzzy sets. Automatica *13* (1977), 47–58. DOI:10.1016/0005-1098(77)90008-5

[2] G. Beliakov, A. Pradera, and T. Calvo: Aggregation Functions: A Guide for Practitioners. Springer–Verlag, Berlin 2007.

[3] T. Calvo, G. Mayor, and R. Mesiar (eds.): Aggregation Operators: New Trends and Applications. Physica–Verlag, Heidelberg 2002. DOI:10.1007/978-3-7908-1787-4

[4] L. M. De Campos, J. F. Heute, and S. Moral: Probability intervals: a tool for uncertain reasoning. Int. J. Uncertainty, Fuzziness and Knowledge-Based Systems *2* (1994), 167–196. DOI:10.1142/s0218488594000146

[5] W. M. Dong and F. S. Wong: Fuzzy weighted averages and implementation of the extension principle. Fuzzy Sets and Systems *21* (1987), 183–199. DOI:10.1016/0165-0114(87)90163-1

[6] D. Dubois: Fuzzy weighted averages and fuzzy convex sums: Authors response. Fuzzy Sets and Systems *213* (2013), 106–108. DOI:10.1016/j.fss.2012.09.003

[7] D. Dubois and H. Prade: Additions of interactive fuzzy numbers. IEEE Trans. Automat. Control *26*(1981), 4, 926–936. DOI:10.1109/tac.1981.1102744

[8] J. C. Fodor and M. Roubens: Fuzzy Preference Modelling and Multicriteria Decision Support. Kluwer Academic Publishers, Dordrecht 1994. DOI:10.1007/978-94-017-1648-2_3

[9] S. Ghazinoory, A. E. Zadeh, and A. S. Kheirkhah: Application of fuzzy calculations for improving portfolio matrices. Inform. Sci. *180* (2010), 9, 1582–1590. DOI:10.1016/j.ins.2010.01.012

[10] M. Grabisch, J. L. Marichal, R. Mesiar, and E. Pap: Aggregation Functions. Cambridge University Press, Cambridge 2009. DOI:10.1017/cbo9781139644150

[11] Y.-Y. Guh, C. C. Hong, K. M. Wang, and E. S. Lee: Fuzzy weighted average: A max-min paired elimination method. Comp. Math. Appl. *32* (1996), 115–123. DOI:10.1016/0898-1221(96)00171-x

[12] Y. Y. Guh, C. C. Hon, and E. S. Lee: Fuzzy weighted average: The linear programming approach via Charness and Coopers rule. Fuzzy Sets and Systems *117* (2001), 1, 157–160. DOI:10.1016/s0165-0114(98)00333-9

[13] K. C. Hung, P. Julian, T. Chien, and W. T. H. Jin: A decision support system for engineering design based on an enhanced fuzzy MCDM approach. Expert Systems Appl. *37* (1) (2010), 202–213. DOI:10.1016/j.eswa.2009.04.069

[14] P. T. Chang and K. C. Hung: Applying thefFuzzy–weighted–average approach to evaluate network security systems. Comp. Math. Appl. *49* (2005), 1797–1814. DOI:10.1016/j.camwa.2004.10.042

[15] C. Kao and S. T. Liu: Fractional programming approach to fuzzy weighted average. Fuzzy Sets and Systems *120* (2001), 3, 435–444. DOI:10.1016/s0165-0114(99)00137-2

[16] A. Kaufmann and M. M. Gupta: Introduction to Fuzzy Arithmetic: Theory and Applications. (Second edition) Van Nostrand Reinhold, New York 1991.

[17] G. J. Klir and Y. Pan: Constrained fuzzy arithmetic: Basic questions and some answers. Soft Computing *2* (1998), 100–108. DOI:10.1007/s005000050038

[18] G. J. Klir and B. Yuan: Fuzzy Sets and Fuzzy Logic: Theory and Applications. Prentice Hall, New Jersey, 1996.

[19] C. H. Lin, B. Tan, and P. J. Hsieh: Application of the fuzzy weighted average in strategic portfolio management. Decision Sciences *36* (2005), 3, 489–511. DOI:10.1111/j.1540-5414.2005.00081.x

[20] F. Liu and J. M. Mendel: Aggregation using the fuzzy weighted average as computed by the Karnik–Mendel algorithms. IEEE Trans. Fuzzy Systems *16* (2008), 1, 1–12. DOI:10.1109/tfuzz.2007.896229

[21] X. Liu, J. M. Mendel, and D. Wu: Analytical solution methods for the fuzzy weighted average. Inform. Sci. *187* (2012), 151–170. DOI:10.1016/j.ins.2011.10.006

[22] O. Pavlačka and J. Talašová: Application of the Fuzzy Weighted Average of Fuzzy Numbers in Decision Making Models. In: New Dimensions in Fuzzy Logic and Related Technologies. Vol II. (M. Štěpnička,, V. Novák, and U. Bodenhofer, eds.) Proc. 5th EUSFLAT Conference, Ostravská univerzita, Ostrava 2007, pp. 455–462.

[23] O. Pavlačka and J. Talašová: Fuzzy vectors as a tool for modeling uncertain multidimensional quantities. Fuzzy Sets and Systems *161* (11) (2010), 1585–1603. DOI:10.1016/j.fss.2009.12.008

[24] O. Pavlačka: Modeling uncertain variables of the weighted average operation by fuzzy vectors. Inform. Sci. *181* (22) (2011), 4969–4992. DOI:10.1016/j.ins.2011.06.022

[25] O. Pavlačka: Note on the lack of equality between fuzzy weighted average and fuzzy convex sum. Fuzzy Sets and Systems *213* (2013), 102–105. DOI:10.1016/j.fss.2012.08.003

[26] R. G. Ricci and R. Mesiar: Multi-attribute aggregation operators. Fuzzy Sets and Systems *181* (2011), 1–13. DOI:10.1016/j.fss.2011.06.008

[27] T. Sachs and R. L. K. Tiong: Quantifying qualitative information on risks: development of the QQIR method. J. Construction Engrg. Management-Asce *135* (2009), 1, 56–71. DOI:10.1061/(asce)0733-9364(2009)135:1(56)

[28] J. Talašová: NEFRIT – Multicriteria Decision Making Based on Fuzzy Approach. Central Europ. J. Oper. Res. *8* (2000), 4, 297–319.

[29] Y. M. Wang and T. M. S. Elhag: Fuzzy TOPSIS method based on alpha level sets with an application to bridge risk assessment. Expert Systems Appl. *31* (2006), 2, 309–319. DOI:10.1016/j.eswa.2005.09.040

[30] Y. M. Wang and T. M. S. Elhag: On the normalization of interval and fuzzy weights. Fuzzy Sets and Systems *157* (2006), 2456–2471. DOI:10.1016/j.fss.2006.06.008

[31] C. C. Wei and H. W. Chang: A new approach for selecting portfolio of new product development projects. Expert Systems Appl. *38* (2011), 1, 429–434. DOI:10.1016/j.eswa.2010.06.081

[32] W. Y. Wu, C. H. Lin, J. Y. Kung, and C. T. Lin: A new fuzzy TOPSIS for fuzzy MADM problems under group decisions. J. Intelligent and Fuzzy Systems *18* (2007), 2, 109–115.

[33] Q. Xu, L. Ma, W. F. Nie, P. Li, J. W. Zhang, and J. Z. Sun: Adaptive fuzzy weighted average filter for synthesized image. In: Computational Science and Its Applications – ICCSA 2005, Part 3, Lecture Notes in Computer Science, vol. 3482, Springer–Verlag, Berlin 2005, pp. 292–298. DOI:10.1007/11424857_32

Ondřej Pavlačka, Department of Mathematical Analysis and Applications of Mathematics, Faculty of Science, Palacký University Olomouc, 17. listopadu 12, 771 46 Olomouc. Czech Republic.
e-mail: ondrej.pavlacka@upol.cz

Martina Pavlačková, Department of Mathematical Analysis and Applications of Mathematics, Faculty of Science, Palacký University Olomouc, 17. listopadu 12, 771 46 Olomouc. Czech Republic.
e-mail: martina.pavlackova@upol.cz

Vladislav Hetfleiš, Department of Mathematical Analysis and Applications of Mathematics, Faculty of Science, Palacký University Olomouc, 17. listopadu 12, 771 46 Olomouc. Czech Republic.
e-mail: vladislav.hetfleis@upol.cz