THE COVERING SEMIGROUP OF INVARIANT CONTROL SYSTEMS ON LIE GROUPS

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It is well known that the class of invariant control systems is really relevant both from theoretical and practical point of view. This work was an attempt to connect an invariant systems on a Lie group G with its covering space. Furthermore, to obtain algebraic properties of this set. Let G be a Lie group with identity e and $\Sigma \subset \mathfrak{g}$ a cone in the Lie algebra $\mathfrak{g}$ of G that satisfies the Lie algebra rank condition. We use a formalism developed by Sussmann, to obtain an algebraic structure on the covering space $\mathbf{\Gamma}(\Sigma, x), x \in G$ introduced by Colonius, Kizil and San Martin. This formalism provides a group $\widehat{G}(X)$ of exponential of Lie series and a subsemigroup $\widehat{S}(X) \subset \widehat{G}(X)$ that parametrizes the space of controls by means of a map due to Chen, which assigns to each control a noncommutative formal power series. Then we prove that $\Gamma(\Sigma, e)$ is the intersection of $\widehat{S}(X)$ with the congruence classes determined by the kernel of a homomorphism of $\widehat{S}(X)$.

Keywords: control systems, homotopy of trajectories, covering semigroup

Classification: 93C30, 14F35, 57M10

1. INTRODUCTION

An invariant control system Σ on a finite dimensional Lie group G is determined by a family $\mathcal{D}$ of differential equations given by

$$\mathcal{D} = \left\{ X_0 + \sum_{j=1}^{m} u_j X_j : u \in \mathcal{U} \right\}.$$

The drift vector field X_0 and the control vectors $X_1, \dots, X_m$ are elements of the Lie algebra $\mathfrak{g}$ of G which we think of the set of right invariant vector fields here. We consider $\mathcal{U}$ as the set of the admissible class of control that later will be formalized.

It is well known that this class of control systems is really relevant both from theoretical and practical point of view. In fact, since the beginning of the 1970s many people has been working in this kind of systems. We mention the first work in the subject by Brockett, R. [4]. Then, several mathematician started to study this system on different classes of Lie groups: Abelian, compact, nilpotent, solvable, semisimple, etc. We mention some of them [1, 3, 8, 12, 13, 14, 16, 17, 19], see also [2]. As appointed by Professor

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V. Jurdjevic, optimal control on Lie groups is a natural setting for geometry and mechanics, see [10, 11] and [18]. As a consequence, differential systems on Lie groups and their homogeneous spaces deserves to be developed. For instance, the Dubins problem [7], the brachistochrona problem [22], the control of the altitude of a satellite in orbit [9], etc., are described by invariant control systems on some particular classes of Lie groups.

Due to the importance of Σ, any information about this class of system is important. In particular, this paper deal with the connection between Σ and its covering space $\Gamma(\Sigma, x)$ for any $x \in G$. It is a preliminary theoretical work trying to establish some algebraic properties of this set in order to obtain information on Σ in return. Certainly new works in the subject will allow to show the importance of this natural connection and in particular to get relevant consequences on Σ from this construction.

Given a state $x \in G$, the covering space $\Gamma(\Sigma, x)$ for monotonic homotopy of trajectories of conic control systems has been studied in [6] and topologically determined. Actually, it has a smooth manifold structure. In this paper we show an algebraic construction on this space to explore its properties with more details. Throughout the article we consider a connected Lie group G with identity e as a state space and a cone $\Sigma \subset \mathfrak{g}$ in the Lie algebra $\mathfrak{g}$ of G. In this case, we let $E \subset \mathfrak{g}$ be the subspace spanned by Σ in the space of right invariant vector fields in G. It follows that the standard concatenation between trajectories of Σ defines a semigroup structure on the space of trajectories, which is compatible with the topology of uniform convergence on trajectories (and hence with the C^1-topology). Thus for each $x \in G$, the space of regular trajectories $\mathrm{R}(\Sigma, x)$ as well as its quotient $\Gamma(\Sigma, x)$ turns out to be a topological semigroup. However, due to the invariance of our vector fields we constrain our attention only to $\mathrm{R}(\Sigma, e)$ and $\Gamma(\Sigma, e)$, respectively.

We follow a general formalism based on exponential Lie series developed by Sussmann [20]. The idea of this formalism consists of solving the differential equation of the system formally by using indeterminate rather than the vector fields that describe Σ. To each control there corresponds a noncommutative formal power series involving iterated integrals. Actually, these series has been also considered in the literature as Chen series. This formalism gives rise a 'Lie group' of exponential Lie series and a subsemigroup that parametrizes control space. Then, the control system Σ may be regarded as an action of this group together with the specification of its subsemigroup. The main goal of the paper is to obtain $\Gamma(\Sigma, e)$ as an appropriate quotients of the semigroup of formal power series. More precisely, we prove that the covering semigroup $\Gamma(\Sigma, e)$ may be viewed as the intersection of the semigroup $\widehat{S}(X)$ of formal power series with the congruence classes determined by the kernel of the semigroup homomorphism

$$\tau : \widehat{S}(X) \to \Gamma(\Sigma, e)$$

that assigns to each control $u(\cdot)$ -for which the corresponding formal series S belongs to $\widehat{S}(X)$- the monotonic homotopy class of the induced Σ-trajectory.

2. PRELIMINARIES

This section is devoted to a general formalism of noncommuting formal power series of control functions, which will be useful for our purposes. For further details we refer the reader to the papers [20, 21] by Sussmann. The purpose of the paper is to obtain possible algebraic properties of the covering space $\Gamma(\Sigma, x)$ recently introduced in [6]. The relation is given through invariant control systems on Lie groups but we find convenient to mention first some definitions and statements from our earlier paper.

In [6] we have considered on a Riemannian manifold M the following class of differential systems

$$\frac{d\alpha}{dt} \in \Sigma(\alpha(t))$$

where Σ is a convex cone in a finite dimensional linear space E. Let us denote by $L(\Sigma)$ the smallest Lie algebra containing Σ. The main assumptions are

i) E is endowed with an inner product $\langle \cdot, \cdot \rangle$

ii) Σ is generating in the sense that it is not contained in a proper subspace of E, and

iii) Σ satisfies the Lie algebra rank condition which means that

$$L(\Sigma)(x) = T_x M, \text{ for any } x \text{ in } M.$$

We denote by $\mathcal{E}$ the Banach space of essentially bounded and measurable controls $u : [0,1] \to E$ endowed with the *ess sup-norm* $||.||_\infty$, and by $U \subset \mathcal{E}$ the convex cone formed by the controls assuming their values strictly in Σ.

Given a control $u : [0,1] \to \Sigma$ and an initial condition $x \in M$, the corresponding trajectory $\mathrm{trj}_x(u) : [0,1] \to M$ is the solution of the differential equation $\dot{x} = u(t)(x)$. That is, $\alpha = \mathrm{trj}_x(u)$ is an absolutely continuous curve in M such that $\alpha'(t) \in \Sigma(\alpha(t))$. Note that the domain of α can be taken as the unit interval $[0,1]$ due to the fact that Σ is a cone which implies that the spaces of trajectories are similar for all finite $T > 0$ and hence we reparametrize them and define in $[0,1]$ rather than $[0,T]$.

We denote the end point of a trajectory α by $e_x(u) = \mathrm{trj}_x(u)(1) = \alpha(1)$ so that we have a well defined differentiable map $e_x : \mathcal{U} \to M$. We use the notations $T(\Sigma)$ for the set of trajectories of Σ and $T(\Sigma, x, y)$ for the space of trajectories starting at x and ending at y.

In [6] a special attention is given on a subspace of trajectories of Σ called regular. Roughly speaking, by a regular trajectory we understand a trajectory of Σ generated by a regular control u which means that $u \in \mathrm{int}(\mathcal{U})$ and the differential $d(e_x)_u$ of e_x at u is onto. We will use $R(\Sigma)$, $R(\Sigma, x)$ and $R(\Sigma, x, y)$ to stand for the sets of the respective regular trajectories.

An appropriate homotopy between regular trajectories has been defined in [6], as follows:

Definition 2.1. Let α and β be two regular trajectories in $R(\Sigma, x, y)$ for some x, y in M. We say that α and β are monotonically homotopic (and write $\alpha \simeq_m \beta$), if there exists a continuous map

$$h_t : [0,1] \to R(\Sigma, x, y), \ 0 \le t \le 1$$

such that $h_0 = \alpha$, $h_1 = \beta$.

Of course, a monotonic homotopy is a homotopy but the converse is not true in general. Since monotonic homotopic defines an equivalence relation one may define a covering of Σ as follows

Definition 2.2. Let Σ be a control system on M as above. Given an initial condition $x \in M$ we define the control covering of Σ to be the set $\Gamma(\Sigma, x)$ of equivalence classes of monotonically homotopic trajectories in $R(\Sigma, x)$, that is, $\Gamma(\Sigma, x) = R(\Sigma, x)/\simeq_m$.

It is known [6] that for a fixed $x_0 \in M$ the set $\Gamma(\Sigma, x_0)$ is a differentiable manifold of dimension $n = \dim M$ and that the projection $\varepsilon_{x_0} : \Gamma(\Sigma, x_0) \to M$ which associates to each homotopy class $[\alpha]_m$ the end-point of its representative is a local diffeomorphism. Actually the image of ε_{x_0} is the set of points in M accessible exclusively by regular controls and hence contained in the interior $\mathrm{int}\mathcal{A}(x_0)$ of the accessible set from x_0.

As is said before, we are going to relate the control covering space with the exponential Lie series formalism through invariant control systems on Lie groups.

An invariant control system Σ on a Lie group G is determined by a family $\mathcal{D}$ of differential equations given by

$$\mathcal{D} = \left\{ X_0 + \sum_{j=1}^{m} u_j X_j : u \in \mathcal{U} \right\}.$$

The drift vector field X_0 and the control vectors $X_1, \ldots, X_m$ are elements of the Lie algebra $\mathfrak{g}$ of G which we think of the set of right invariant vector fields here.

We take Σ to be the cone in the Lie algebra $\mathfrak{g}$ generated by $X_0, X_1, \ldots, X_m$. By the invariance of our vector fields it is enough to consider only Σ-trajectories starting at the identity element $e \in G$ and hence the sets $R(\Sigma, e)$, $\Gamma(\Sigma, e)$, etc. Note that the set $R(\Sigma, e)$ (and hence $\Gamma(\Sigma, e)$) becomes a topological semigroup by standard concatenation between trajectories.

We quote below a brief exposition on a general formalism of power series. Given a control function $u \in \mathcal{U}$ we use $u_1, \ldots, u_m$ to denote its components. Let us denote by $X = (X_0, X_1, \ldots, X_m)$ a finite sequence of indeterminates, by $A(X)$ the free associative algebra in $X_0, \ldots, X_m$ and by $\widehat{A}(X)$ the power series algebra. In addition to the formal power series in $\widehat{A}(X)$ one may consider truncated series. Hence, we also denote by $A^n(X)$ the free nilpotent associative algebra of step $n+1$, which is generated by monomials X_I for $|I| \leq n$ where $|I|$ means the length of I. The canonical projection $A(X) \to A^n(X)$ (resp. $\widehat{A}(X) \to A^n(X)$) is the truncation map denoted by $\widehat{t}_n$ (the same notation for both). The kernel $\ker(\widehat{t}_0) = \widehat{A}_0(X)$ of $\widehat{t}_0$ is of particular importance and the exponential map $\exp : \widehat{A}_0(X) \to 1 + \widehat{A}_0(X)$ is a well defined bijection with inverse log. For the set $A_0^n(X)$ determined by all elements of $A^n(X)$ that are linear combinations of monomials of degree greater than 0, the restricted exponential map $\exp_n : A_0^n(X) \to 1 + A_0^n(X)$ is a bijection with inverse $\log_n$.

It is clear that each of the algebras $A(X)$, $A^n(X)$ and $\widehat{A}(X)$ becomes a Lie algebra with the usual commutator rule $[P, Q] = PQ - QP$. In particular, we obtain the Lie subalgebras $L(X) \subset A(X)$ and $L^n(X) \subset A^n(X)$ generated by $X_0, X_1, \ldots, X_m$, and the Lie algebra $\widehat{L}(X) \subset \widehat{A}(X)$ of Lie series in $X_0, X_1, \ldots, X_m$.

Let us consider $P \in \widehat{A}(X)$ and $u \in \mathcal{U}_m$. Denote by $t(u)$ the terminal time of u. One may formally consider the following differential equation in $\widehat{A}(X)$

$$\dot{S}(t) = \left(X_0 + \sum_{i=1}^{m} u_i X_i \right) S$$

with the initial condition $S(0) = P$. A solution of the former equation is a $\widehat{A}(X)$-valued function $t \to S(t)$ such that $S(0) = P$. In particular, if $P = 1$ the solution contains iterated integrals, see Sussmann's paper, [20]. By a formal series $\mathrm{Ser}(u)$ of u we mean the solution $S(t(u))$ with initial condition $S(0) = 1$. Without loss of generality, in the sequel we consider formal series associated to regular controls rather than general control functions since our previous results in [6] were obtained in this framework.

The space of controls is regarded as a semigroup under concatenation of controls and the mapping $\mathrm{Ser} : \mathcal{U}_m \to \widehat{A}(X)$ that associates to a control the corresponding power series is a one-to-one homomorphism of semigroups (see, Lemma 3.1, [20]). We denote by $\widehat{S}(X)$ the image $\mathrm{Ser}(\mathcal{U}_m)$ which is the semigroup of noncommuting formal power series.

2.1. The group $\widehat{G}(X)$ of exponential Lie series

The elements of $\widehat{A}(X)$ that are of the form $\exp(P)$ for some $P \in \widehat{L}(X)$ are called the exponential Lie series in $X_0, X_1, \ldots, X_m$, and form the set denoted by $\widehat{G}(X)$. It follows from the Campbell–Hasdorff formula that $\widehat{G}(X)$ receives a group structure. However, the group $\widehat{G}(X)$ is an infinite dimensional Lie group while its truncated versions $G_n(X) = \widehat{T}_n(\widehat{G}(X))$ are connected simply connected and nilpotent Lie groups with Lie algebras $L^n(X) = \widehat{t}_n(L(X)) = \widehat{t}_n(\widehat{L}(X))$. Hence, for any natural number n the exponential map $\exp_n : L^n(X) \to G_n(X)$ is a global diffeomorphism. We fix, once and for all, the notations $\widehat{T}_n$ and $\widehat{t}_n$ for truncation maps on the group and algebra level, respectively. Also, we denote by $T_n : G_n(X) \to G_{n-1}(X)$ the corresponding truncation map between truncated versions of $\widehat{G}(X)$.

Since the group $\widehat{G}(X)$ is not a finite dimensional Lie group it would be interesting to focus on it at least as a topological group. Hence, we remind here the inverse limit sequences which are frequently used in topology, and define this limiting process for nilpotent approximations of $\widehat{G}(X)$, as follows.

Definition 2.3. An inverse sequence of the groups $G_n(X)$ and the mappings T_n is a pair $(G_n(X), T_n)$, which can be represented by means of the diagram

$$\cdots \xleftarrow{T_{n-1}} G_{n-1}(X) \xleftarrow{T_n} G_n(X) \xleftarrow{T_{n+1}} \cdots \quad .$$

The projective limit

$$G_\infty(X) = \{(g_0, g_1, \ldots) : T_n(g_n) = g_{n-1}, \text{ for each } n \in \mathbb{N}\}$$

of the inverse sequence $(G_n(X), T_n)$ is a topological subgroup of the product group $\Pi_n G_n(X)$. Furthermore, let (H_n, f_n) be another inverse sequence of topological groups and continuous mappings. By a mapping Φ of (H_n, f_n) to $(G_n(X), T_n)$ we understand a collection $\{\varphi_n\}$ of continuous mappings $\varphi_n : H_n \to G_n(X)$ such that

$$\varphi_{n-1} \circ f_n = T_n \circ \varphi_n, \text{ for each } n \in \mathbb{N}.$$

We denote by $\mathcal{P}_\infty$ the product topology on $G_\infty(X)$ called the projective topology. As we said before, the semigroup $\widehat{S}(X)$ may be viewed as the control semigroup $\mathcal{U}_m$ embedded in $\widehat{G}(X)$ since $\mathrm{Ser}(u)$ always belongs to $\widehat{G}(X)$. It should be noted that we do not assume here that the component u_0 of $u(\cdot)$ to be identically 1 as it was in [20]. This means that we take into account systems of the form

$$\dot{x} \ = \pm X_0(x) + \sum_{j=1}^{m} u_j X_j(x), \quad x \in G$$

for which the mapping Ser is actually an injection of $\widehat{S}(X)$ onto $\widehat{G}(X)$.

2.2. Projective topology on $\widehat{G}(X)$

Let us first prove the following result

Proposition 2.4. The group $\widehat{G}(X)$ of exponential of Lie series is topologically isomorphic to the projective limit of its nilpotent approximations. Hence, is a connected topological group.

P r o o f. One may think of the group $\widehat{G}(X)$ itself together with its identity map, say $\widehat{1}$, as an inverse system $(\widehat{G}(X), \widehat{1})$ over a one element index set, and consider the mapping

$$\Phi = \{\widehat{T}_n\}_{n\in\mathbb{N}} : (\widehat{G}(X), \widehat{1}) \to (G_n(X), T_n).$$

It follows that such a mapping induces an unique continuous mapping $\varphi_\infty : \widehat{G}(X) \to G_\infty(X)$ of the limits as follows. For each $\exp(P)$ in $\widehat{G}(X)$ the map φ_∞ is defined by

$$\varphi_\infty(\exp(P)) = (\widehat{T}_1(\exp(P))), \widehat{T}_2(\exp(P)), \ldots).$$

Clearly, φ_∞ is indeed a topological isomorphism. Connectedness assertion should follow from the fact that any connected topological group is generated by a neighborhood of its identity element. $\square$

Inverse sequences can be defined in general for sets with binary operations. Let Ser_n be the finite version of the mapping Ser. Denote by $S_n(X) = \mathrm{Ser}_n(\mathcal{U}_m)$, the nilpotent versions of the semigroup $\widehat{S}(X)$. Hence, one can define in a similar way a projective topology on the semigroup $\widehat{S}(X)$ that actually coincides with the subspace topology of $\widehat{G}(X)$.

3. QUOTIENT SEMIGROUPS OF $\widehat{S}(X)$

In this section we present the main result of the paper, namely, the covering semigroup $\Gamma(\Sigma, e)$ can be expressed as appropriate quotients of the semigroup $\widehat{S}(X)$. We refer the reader to the Ljapin´s book (see, Chap. VII, [15]) for the basic definitions and the proposition listed below.

Definition 3.1. A relation $\sim$ in a semigroup S is said to be a congruence on S if it is both right and left stable. That is, for any $x, y \in S$, $x \sim y$ implies $xz \sim yz$ and $zx \sim zy$, respectively, for every z in S.

Definition 3.2. Suppose that f is a homomorphism of the semigroup S onto the semigroup H and that h is a homomorphism of H onto the semigroup T. We say that the homomorphism f is a right divisor of the homomorphism $g = hf$ defined by $g(x) = h(f(x))$, x in S. Concerning g one says that it is divided on the right by f. We shall in this case write $f \ \mathfrak{r} \ g$.

We note that only homomorphisms of one and the same semigroup can lie in the relation of right divisibility $\mathfrak{r}$. If for the homomorphisms h_1, h_2 and h_3 one has $h_1 \ \mathfrak{r} \ h_2$ and $h_2 \ \mathfrak{r} \ h_3$, then one has also $h_1 \ \mathfrak{r} \ h_3$ (transitivity of $\mathfrak{r}$).

Proposition 3.3. Let h_1 and h_2 be two homomorphism of a semigroup H. For $h_1 \ \mathfrak{r} \ h_2$ it is necessary and sufficient that if $h_1(x) = h_1(y)$ for any $x, y \in H$, then $h_2(x) = h_2(y)$. On the other hand, if h_1 and h_2 are such that from $h_1(x) = h_1(y)$ it always follows that $h_2(x) = h_2(y)$, then for the semigroup $h_1(H)$ one may define, uniquely, a homomorphism f of it onto the semigroup $h_2(H)$ such that $h_2 = fh_1$. Thus, $h_1 \ \mathfrak{r} \ h_2$.

Proof. See [15]. □

We need for later references an appropriate version of the evaluation map e_{x_0} in [6] adopted to formal power series.

Definition 3.4. We define the evaluation map $\widehat{e}$ on $\widehat{G}(X)$ to be the map $\widehat{e} : \widehat{G}(X) \to G$ that sends a power series $S \in \widehat{G}(X)$ to the end point of the trajectory induced by $u \in \mathcal{U}_m$ for which $\mathrm{Ser}(u) = S$.

For our purposes it would be interesting if $\widehat{e}$ is an onto continuous homomorphism of topological (semi)groups. Note that the image of $\widehat{e}$ is nothing else than the semigroup $S_\Sigma(e)$ from the identity. Hence we state the following

Lemma 3.5. The evaluation map $\widehat{e} : \widehat{G}(X) \to G$ is an onto homomorphism of topological groups which is continuous with respects to the projective topology on $\widehat{G}(X)$.

Proof. Since Σ is a cone satisfying the Lie algebra rank condition then the map $\widehat{e}$ is surjective. It follows that $\widehat{e}$ is a homomorphism since the concatenation of controls leads concatenation of their respective trajectories. On the other hand, it is well known that a homomorphism $h : T_1 \to T_2$ of topological groups is continuous if and only if it is continuous at the identity e_{T_1} of T_1. Consequently, the map $\widehat{e}$ is continuous with respect to the projective topology if and only if for any $n \in \mathbb{N}$ the restriction $\widehat{e}_n : G_n(X) \to G$ is continuous at the identity.

Now, given $P \in \widehat{L}(X)$ we know that $\widehat{e}(\exp(P)) = \mathrm{trj}(u)(1)$, where $\mathrm{trj}(u)$ is the solution of the differential equation $\dot{S} = P \cdot S$ in $\widehat{G}(X)$ with the initial condition $S(0) = 1$. The truncation map sends a solution of the former differential equation to a solution of the same differential equation, regarded now as evolving in $G_n(X)$. Being the solution of a differential equation, $\widehat{e}_n$ is continuous at the identity for each n. This finishes the proof. $\square$

Suppose that h is any homomorphism of a semigroup H. It is well known that one can define in H a relation $\mathfrak{c}$ by putting

$$x \ \mathfrak{c} \ y \quad \text{if} \quad h(x) = h(y). \tag{1}$$

Reflexivity, symmetry and transitivity of this relation are evident. The equivalence $\mathfrak{c}$ is called the equivalence corresponding to the given homomorphism h. It is also clear that the equivalence $\mathfrak{c}$ is two-side stable, and hence a congruence. This way, to each homomorphism of the semigroup H there corresponds some two-sided stable equivalence, i. e., a congruence. It follows that the set $H/\mathfrak{c}$ of all $\mathfrak{c}$-classes is a semigroup, called the quotient semigroup of the semigroup H modulo $\mathfrak{c}$. Moreover, there exists a homomorphism π (in fact, the natural homomorphism of H onto $H/\mathfrak{c}$) to which the equivalence (resp. congruence) $\mathfrak{c}$ corresponds.

Suppose we are given two homomorphisms h_1 and h_2 of a semigroup H, and that $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are the corresponding equivalences (resp. congruences). It follows from the Proposition 3.3 that if $h_2 \ \mathfrak{r} \ h_1$ holds for the homomorphisms, i. e., if h_2 is a right divisor of h_1, then the equation $h_2(x) = h_2(y)$ $(x, y \in H)$ always implies $h_1(x) = h_1(y)$, and for the equivalence (resp. congruence) relations we have $\mathfrak{c}_2 \subseteq \mathfrak{c}_1$. Conversely, suppose that $\mathfrak{c}_2 \subseteq \mathfrak{c}_1$. Therefore, for the homomorphisms h_1 and h_2 themselves, we find that for some homomorphism ψ we have $h_1 = \psi h_2$, i. e., $h_2 \ \mathfrak{r} \ h_1$. This means that the relation $\mathfrak{r}$ of right divisibility between homomorphisms is induced by the partial ordering of the corresponding equivalences (resp. congruences). See [15] for further details.

It follows by means of the correspondence $\mathcal{U}_m \simeq \widehat{S}(X)$ that the map $\mathrm{trj} : \mathcal{U}_m \to \mathrm{R}(\Sigma, e)$ which associates to a control u its corresponding trajectory is a homomorphism of the semigroup $\widehat{S}(X)$. Similarly, if we compose trj with the canonical projection

$$\pi : \mathrm{R}(\Sigma, e) \to \Gamma(\Sigma, e) = \mathrm{R}(\Sigma, e)/ \simeq_m$$

we obtain the mapping $\tau : \mathcal{U}_m \to \Gamma(\Sigma, e)$ as a homomorphism of the same semigroup $\widehat{S}(X)$. Following the arguments mentioned above we define in $\widehat{S}(X)$ a relation of congruence $\mathfrak{h}_m$ by putting

$$S \ \mathfrak{h}_m \ P \quad \text{if} \quad S \in P \ker(\tau), \tag{2}$$

whenever $S, P \in \widehat{S}(X)$. Analogously, in $\widehat{S}(X)$ we also define a congruence $\mathfrak{h}$ corresponding to the homomorphism $\widehat{e} : \widehat{S}(X) \to S_\Sigma(e)$ as follows:

$$S \ \mathfrak{h} \ P \quad \text{if} \quad S \in P \ker(\widehat{e}). \tag{3}$$

It turns out that from the standard theory of semigroups we obtain the following results whose proof will be omitted.

Proposition 3.6. Let $\widehat{e}$ and τ denote the homomorphisms of the semigroup $\widehat{S}(X)$ with the corresponding congruences $\mathfrak{h}$ and $\mathfrak{h}_m$ as above. Denote by π_1 and π_2 the natural homomorphisms of $\widehat{S}(X)$ onto $\widehat{S}(X)/\mathfrak{h}$ and $\widehat{S}(X)/\mathfrak{h}_m$, respectively. There exist the isomorphisms $\varepsilon_1 : \widehat{S}(X)/\mathfrak{h} \to S_\Sigma(e)$ and $\varepsilon_2 : \widehat{S}(X)/\mathfrak{h}_m \to \Gamma(\Sigma, e)$ such that $\widehat{e} = \varepsilon_1\pi_1$ and $\tau = \varepsilon_2\pi_2$.

We have a simple corollary of the Proposition 3.3 as follows:

Lemma 3.7. Let $\widehat{e}$ and τ denote the two homomorphisms of the semigroup $\widehat{S}(X)$ as defined above. Then, the end-point mapping

$$\varepsilon : \Gamma(\Sigma, e) \to S_\Sigma(e) \subset G \text{ defined by } [\gamma]_m \to \gamma(1)$$

is the unique semigroup homomorphism of $\Gamma(\Sigma, e)$ onto $S_\Sigma(e)$ such that $\widehat{e} = \varepsilon\tau$.

Proof. Let S and P belong to $\widehat{S}(X)$ and let $\alpha = \mathrm{trj}(u)$ and $\beta = \mathrm{trj}(v)$ such that $S = \mathrm{Ser}(u)$ and $P = \mathrm{Ser}(v)$. It follows that $\tau, \widehat{e} \in \mathrm{Hom}(\widehat{S}(X), \cdot)$ are such that

$$\tau(S) = \tau(P) \text{ implies } \widehat{e}(S) = \widehat{e}(P).$$

Indeed, $\tau(S) = \tau(P)$ (or, equivalently $S\ \mathfrak{h}_m\ P$) means that α is monotonically homotopic to β while $\widehat{e}(S) = \widehat{e}(P)$ (or, equivalent $S\ \mathfrak{h}\ P$) says that they are homotopic as paths. Since monotonic homotopy is a homotopy it is clear that $\tau(S) = \tau(P)$ always implies $\widehat{e}(S) = \widehat{e}(P)$. By Proposition 3.3 the homomorphism f of $\tau(\widehat{S}(X))$ onto $\widehat{e}(\widehat{S}(X))$ such that $\widehat{e} = f\tau$ is uniquely defined. It follows that $f = \varepsilon$ since ε already satisfies $\widehat{e} = \varepsilon\tau$. □

However, for the end-point mapping $\varepsilon : \Gamma(\Sigma, e) \to S_\Sigma(e)$ being an isomorphism it is necessary and sufficient that if $\widehat{e}(S) = \widehat{e}(P)$ for any $S, P \in \widehat{S}(X)$, then $\tau(S) = \tau(P)$. In other word, $\widehat{e}\ \mathfrak{r}\ \tau$ if homotopy of paths implies monotonic homotopy, which is in general not true.

We have obtained up to now the semigroup $\Gamma(\Sigma, e)$ of monotonic homotopy as the factor semigroup $\widehat{S}(X)/\ker(\tau)$ and also the system semigroup $S_\Sigma(e)$ as the factor semigroup $\widehat{S}(X)/\ker(\widehat{e})$.

Now, we are willing to establish the main results of the paper.

Theorem 3.8. Keep the notations and assumptions as before. The covering semigroup $\Gamma(\Sigma, e)$ may be viewed as the intersection of the semigroup $\widehat{S}(X)$ of formal power series with the congruence classes determined by the kernel of the homomorphism τ of the semigroup $\widehat{S}(X)$ such that under the homomorphism ε one has $\mathfrak{h}_m \subseteq \mathfrak{h}$. That is,

$$\Gamma(\Sigma, e) = \widehat{S}(X) \cap [\ker(\tau)].$$

We also have $\Gamma(\Sigma, e) = \widehat{S}(X) \cap [\ker(\widehat{e})]$ such that $\mathfrak{h} \subseteq \mathfrak{h}_m$ whenever ε is an isomorphism.

4. CONCLUSION

The main results of the paper in Theorem 3.8 shows a way to compute the covering semigroup of an invariant control system through the formal power series associated to the semigroup of the system. It is a preliminary theoretical work trying to establish some algebraic properties of this set in order to obtain information of the system in return. At this point, we are not able to exhibit examples. However, we hope that new works in the subject will allow to show the importance of this natural connection and in particular to get relevant information on the system from this construction.

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