KYBERNETIKA — VOLUME 52 (2016), NUMBER 1, PAGES 153–168

SOLVING MULTI-OBJECTIVE FUZZY MATRIX GAMES VIA MULTI-OBJECTIVE LINEAR PROGRAMMING APPROACH

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A class of multi-objective fuzzy matrix games is studied and it is shown that solving such a game is equivalent to solving a pair of multi-objective linear programming problems. This work generalizes an earlier study of Fernandez et al. [7] from crisp scenario to fuzzy scenario on the lines of Bector et al. [4]. Further certain difficulties with similar studies reported in the literature are also discussed.

Keywords: multi-objective game, Pareto-optimal security strategies, security level, multi-objective linear programming

Classification: 90C70, 91A40

1. INTRODUCTION

A multi-objective zero-sum matrix game is an extension of the standard two person zero-sum matrix game. In fact a competitive situation which can be modelled as a scalar zero-sum game has its counter part as a multi-objective zero-sum game when more than one scenario has to be compared simultaneously. As conflicting interests appear not only between different decision makers, but also within each individual, the study of multi-objective games becomes important.

Historically Blackwell's [2] paper was the first paper which dealt with the theory of multi-objective games as a generalization of the theory of scalar games. Shapely [14] introduced the concept of equilibrium solution in two person zero-sum multi-objective games by using the concept of Pareto-optimality. He further proved the existence of equilibrium solution by finding the correspondence between the given multi-objective game and a resulting single-objective game obtained by aggregating it with the weighting coefficients.

Zeleny [16] analyzed the maxmin and minmax values of two person zero-sum multi-objective games by aggregating multiple pay-offs in to a single payoff via weighting coefficients. Cook [5] introduced a goal vector and formulated such games as goal programming problems while Corley [6] presented the necessary and sufficient condition for optimal mixed strategies for the same. Ghose et al. [8] introduced the concept of Pareto-optimal Security Strategies for multi-objective two person zero-sum games and solved the same by the weightage average approach. Fernandez et al. [7] studied the

DOI: 10.14736/kyb-2016-1-0153

same game model as that of Ghose et al. [8] and established the equivalence between POSS and efficient solutions of a pair of multi-objective programming problems.

Though single objective two person zero-sum fuzzy matrix games have been studied extensively in the literature (see Sakawa and Nishizaki [12], Bector and Chandra [3]), the results on multi-objective scenario are rather scarce. The main contribution in this direction has been the work of Sakawa and Nishizaki [12]. Their approach has been to associate a fuzzy goal with a respective payoff matrix and define the solution in terms of maximizing the degree of minimal goal attainment for each player. Further it has also been shown in [12] that such a maxmin solution for each player can be obtained by solving a pair of primal-dual linear programming problems.

In this paper we attempt to follow a different approach. Taking motivation from Ghose et al. [8] and Fernandez et al. [7], we extend the concepts of Pareto-optimal security strategies and security levels for players to study fuzzy multi-objective matrix games and prove that these can be obtained by solving a pair of fuzzy multi-objective linear programming problems.

The paper is organized as follows. Section 2 introduces the basic definitions and reviews results with regard to (crisp) two person zero-sum multi-objective matrix games. Section 3 presents the main results of this paper where two person zero-sum fuzzy multi-objective matrix games are studied. Here the concepts of Pareto-optimal security strategies and security levels for such fuzzy games are introduced in terms of appropriate membership functions of various pay-offs. An important result of this section states that solving such a fuzzy game is equivalent to solve a pair of (crisp) multi-objective linear programming problems. Section 4 takes a relook of Nishizaki and Sakawa's model [12] and compares the same with the present model. In Section 5, the results of this paper are illustrated with a small numerical example. Some concluding remarks are furnished in Section 6.

2. REVIEW OF THE EXISTING MODEL

In this section we shall be describing a crisp multi-objective two person zero-sum game model studied by Fernandez et al. [7]. For various notations, terminology and basics related to the solution of the game we shall refer to Fernandez et al. [7].

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space and $\mathbf{R}^n_+$ be its non negative orthant. Let $A^r \in \mathbf{R}^{m\times n}, (r = 1, 2, \ldots, s,)$ be $m \times n$ real matrices, $e^T = (1, 1, \ldots, 1)$ be a vector of ones whose dimension is specified as per the specific context and further $S^m = \{x \in \mathbf{R}^m_+ \mid e^T x = 1\}$ and $S^n = \{y \in \mathbf{R}^n_+ \mid e^T y = 1\}$ are the convex polytopes.

By a two person zero-sum multi-objective matrix game G we mean

$$G = \big(S^m, S^n, A^r, (r = 1, 2, \ldots, s)\big),$$

where S^m (respectively, S^n) is the strategy space for Player I (respectively, Player II), and $A^r = [a^r_{ij}], (i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n)$, is the payoff matrix corresponding to the r^{th} criterion, (r=1,2,..., s). Also, it is a convention to assume that Player I is a maximizing player and Player II is a minimizing player. Further, for $x \in S^m,\ y \in S^n$, the expected payoff for Player I is a vector $E(x, y) = x^T Ay = [E_1(x, y), E_2(x, y), \ldots, E_s(x, y)]$

where $E_r(x,y) = x^T A^r y, (r = 1,2,\ldots,s)$. As Player I is a maximizing player and Player II is a minimizing player, the expected payoff for Player I is the expected loss for Player II. Now we have the following definitions to define the solution of the game.

Definition 2.1. (Security level for Player I) For a strategy $x \in S^m$, the security level of Player I corresponding to r^{th} payoff matrix is given by

$$\begin{aligned} v_r(x) &= \min_{y\in S^n} E_r(x,y) \\ &= \min_{1\le j\le n} x^T A_j^r, \end{aligned}$$

where A_j^r is the jth column of the matrix A^r. Therefore the security level for Player I is an s-tuple vector, given by

$$v(x) = [v_1(x), v_2(x), \ldots, v_s(x)].$$

Definition 2.2. (Security level for Player II) For a strategy $y \in S^n$, the security level of Player II corresponding to rth payoff matrix is given by

$$\begin{aligned} w_r(y) &= \max_{x\in S^m} E_r(x,y) \\ &= \max_{1\le i\le m} A_i^r y. \end{aligned}$$

where A_i^r is the ith row of the matrix A^r. Therefore the security level for Player II is an s-tuple vector, given by

$$w(y) = [w_1(y), w_2(y), \ldots, w_s(y)].$$

Definition 2.3. (Pareto-optimal security strategy for Player I) A strategy $x^* \in S^m$ is a Pareto-optimal security strategy (POSS) for Player I if there is no $x \in S^m$ such that

$$v(x^*) \le v(x)$$

and

$$v(x^*) \ne v(x).$$

Definition 2.4. (Pareto-optimal security strategy for Player II) A strategy $y^* \in S^n$ is a Pareto-optimal security strategy (POSS) for Player II if there is no $y \in S^n$ such that

$$w(y) \le w(y^*)$$

and

$$w(y) \ne w(y^*).$$

If x^* is a POSS for Player I, then his security level is given by $v^* = v(x^*)$. Similarly if y^* is a POSS for Player II, then his security level is given by $w^* = w(y^*)$.

Fernandez et al. [7] established the following two theorems which state that computing optimal security levels or Pareto-optimal security strategies for Player I and Player II amount to solving a pair of multi-objective programming problems.

Theorem 2.5. The strategy x^* is a POSS and v^* is the security level for Player I if and only if $(x^*,\ v^*)$ is an efficient solution to the following multi-objective programming problem:

$$(VP)_1 \qquad \max \ (v_1, v_2, \dots, v_s)$$
subject to,
$$\begin{aligned} x^T A_j^r &\geq v_r, \quad (j = 1, 2, \dots, n,\ r = 1, 2, \dots, s), \\ e^T x &= 1, \\ x &\geq 0. \end{aligned}$$

Theorem 2.6. The strategy y^* is a POSS and w^* is the security level for Player II if and only if (y^*, w^*) is an efficient solution to the following multi-objective programming problem:

$$(VP)_2 \qquad \min \ (w_1, w_2, \dots, w_s)$$
subject to,
$$\begin{aligned} (A_i^r)^T y &\leq w_r, \quad (i = 1, 2, \dots, m,\ r = 1, 2, \dots, s), \\ e^T y &= 1, \\ y &\geq 0. \end{aligned}$$

In the context of scalar two person zero-sum matrix game, it is well known that such a game can be solved by solving a pair of linear programming problems. Theorem 2.5 and Theorem 2.6 above essentially extend this classical result to multi-objective game scenario.

In the section to follow, we propose to introduce an extension of the above described crisp multi-objective two person zero sum game to the fuzzy frame work.

3. THE PROPOSED MODEL FOR A MULTI-OBJECTIVE FUZZY MATRIX GAME

Let $S^m; S^n; A^r, (r = 1, 2, \dots, s)$ be as introduced in Section 2. Let V_0^r and W_0^r be the scalars representing respectively the aspiration levels of Players I and Player II corresponding to rth payoffs respectively. The multi-objective matrix game with fuzzy goals, denoted by $MOFG$, is defined as

$$MOFG = \big(S^m, S^n, A^r, V_0^r, \gtrsim; W_0^r, \lesssim, (r = 1, 2, \dots, s)\big),$$

where '$\lesssim$' and '$\gtrsim$' are the fuzzified version of '$\leq$' and '$\geq$' respectively. Now Player I problem is to find a $x \in S^m$ such that $x^T A^r y \gtrsim V_0^r,\ \forall y \in S^n$, and Player II problem is to find a $y \in S^n$ such that $x^T A^r y \lesssim W_0^r,\ \forall x \in S^m$, $r = 1, 2, \dots, s$. In other words, the Player I problem, associated with the rth payoff matrix is

$$\text{Find } \ x \in S^m \quad \text{such that} \quad x^T A_j^r \gtrsim V_0^r, \quad (j = 1, 2, \dots, n).$$

Similarly, the Player II problem, associated with the rth payoff matrix is

$$\text{Find } \ y \in S^n \quad \text{such that} \quad A_i^r y \lesssim W_0^r, \quad (i = 1, 2, \dots, m).$$

The game $MOFG$ becomes well-defined only when a specific choice of membership functions are made to define the fuzzy inequalities '$\lesssim$' and '$\gtrsim$'. Here we shall interpret '$\lesssim$' and '$\gtrsim$' as 'essentially less than or equal to' and 'essentially more than or equal to' respectively in the sense of Zimmermann [17].

In order to define the membership functions associated with the fuzzy sets defined by inequalities $\gtrsim$ and $\lesssim$, we have to associate appropriate tolerances with them. Let p_0^r and q_0^r be the positive tolerances respectively for Player I and Player II in the fuzzy inequalities, associated with the rth payoffs. Hence

$$MOFG = \big(S^m, S^n, A^r, V_0^r, p_0^r, \gtrsim; W_0^r, q_0^r, \lesssim, (r = 1, 2, \ldots, s)\big).$$

Now on following Bector et al. [3], we define the membership functions $\mu_j^r(x^T A_j^r)$, gives the degree to which $x \in S^m$ satisfies the fuzzy constraint $x^T A_j^r \gtrsim_{p_0^r} V_0^r$, $(j = 1, 2, \ldots, n)$ as follows

$$\mu_j^r(x^T A_j^r) = \begin{cases} 1, & x^T A_j^r \geq V_0^r, \\ 1 - \dfrac{V_0^r - x^T A_j^r}{p_0^r}, & V_0^r - p_0^r \leq x^T A_j^r \leq V_0^r, \\ 0, & x^T A_j^r \leq V_0^r - p_0^r. \end{cases}$$

Similarly, the membership functions $\mu_i^r(A_i^r y)$, gives the degree to which $y \in S^n$ satisfies the fuzzy constraint $A_i^r y \lesssim_{q_0^r} W_0^r, \forall (i = 1, 2, \ldots, m)$ as follows

$$\mu_i^r(A_i^r y) = \begin{cases} 1, & A_i^r y \leq W_0^r, \\ 1 + \dfrac{W_0^r - A_i^r y}{q_0^r}, & W_0^r < A_i^r y \leq W_0^r + q_0^r, \\ 0, & A_i^r > W_0^r + q_0^r. \end{cases}$$

Definition 3.1. (Security level of satisfaction for Player I) For a strategy $x \in S^m$, the security level of satisfaction for Player I corresponding to rth payoffs is

$$\alpha_r(x) = \min_{1 \leq j \leq n} \mu_j^r(x^T A_j^r).$$

Therefore the security level for Player I is an s-tuple vector, given by

$$\alpha(x) \quad = [\ \alpha_1(x), \alpha_2(x), \ldots, \alpha_s(x)\ ].$$

Definition 3.2. (Security level of satisfaction for Player II) For a strategy $x \in S^m$, the security level of satisfaction for Player II corresponding to rth payoffs is

$$\beta_r(y) = \min_{1 \leq i \leq m} \mu_i^r(A_i^r y).$$

Therefore the security level for Player II is an s-tuple vector, given by

$$\beta(y) \quad = [\beta_1(y), \beta_2(y), \ldots, \beta_s(y)].$$

Definition 3.3. (Pareto-optimal security strategy for Player I) A strategy $x^* \in S^m$ is a Pareto-optimal security strategy (POSS) for Player I if there is no $x \in S^m$ such that

$$\alpha(x^*) \quad \leq \quad \alpha(x)$$

and

$$\alpha(x^*) \quad \neq \quad \alpha(x).$$

Definition 3.4. (Pareto-optimal security strategy for Player II) A strategy $y^* \in S^n$ is a Pareto-optimal security strategy (POSS) for Player II if there is no $y \in S^n$ such that

$$\beta(y^*) \quad \leq \quad \beta(y)$$

and

$$\beta(y^*) \quad \neq \quad \beta(y).$$

If x^* is a POSS for Player I, then his security level is given by $\alpha^* = \alpha(x^*)$. Similarly if y^* is a POSS for Player II, then his security level is given by $\beta^* = \beta(y^*)$. The pair (x^*, α^*) is then understood as a solution of the given two person zero-sum fuzzy multi-objective game for Player I. The pair (y^*, β^*) is interpreted in a similar manner as a solution of the given game for Player II.

We now prove the following theorems characterizing the solutions of such fuzzy games in terms of efficient solutions of appropriate multi-objective optimization problems. For various definitions on multi-objective linear programming, including that of efficient solution, we shall refer to Steuer [13].

Theorem 3.5. The strategy x^* is a POSS and α^* is the security level for Player I if and only if $(x^*,\ \alpha^*)$ is an efficient solution to the following multi-objective programming problem:

$$(FVPI) \qquad \max \quad (\alpha_1, \alpha_2, \ldots, \alpha_s)$$
$$\text{subject to}$$
$$\begin{array}{rcll} A_j^r x + (1-\alpha_r)p_0^r & \geq & V_0^r, & (j = 1, 2, \ldots, n, r = 1, 2, \ldots, s), \\ 0 \leq \alpha_r & \leq & 1, & (r = 1, 2, \ldots, s), \\ x & \in & S^m. & \end{array}$$

P r o o f. Let x^* be a POSS for Player I. Then, there is no $x \in S^m$ such that

$$\alpha(x^*) \leq \alpha(x), \ \ \alpha(x^*) \neq \alpha(x).$$

Therefore, $\forall\ x \in S^m$, either

$$\big(\alpha_1(x^*), \alpha_2(x^*), \ldots, \alpha_s(x^*)\big) = \big(\alpha_1(x), \alpha_2(x), \ldots, \alpha_s(x)\big),$$

or there exists an index $p, 1 \leq p \leq s$, depending on x such that $\alpha_p(x) < \alpha_p(x^*)$.

In other words, for any $x \in S^m$, either

$$\min_{1\le j\le n} \mu_j^r(x^T A_j^r) = \min_{1\le j\le n} \mu_j^r(x^{*T} A_j^r), \quad (r = 1, 2, \dots, s),$$

or there exists an index $p, 1 \le p \le s$, such that

$$\min_{1\le j\le n} \mu_j^p(x^T A_j^r) < \min_{1\le j\le n} \mu_j^p(x^{*T} A_j^r)$$

Hence from the definition of efficient solution (Steuer [13]), x^* is an efficient solution of the multi-objective programming problem

$$\max_{x\in S^m} (\min_{1\le j\le n} \mu_j(x^T A_j^1), \min_{1\le j\le n} \mu_j(x^T A_j^2) \dots \min_{1\le j\le n} \mu_j(x^T A_j^s)).$$

Now using the representation of various membership functions $\mu_j(x^T A_j^r), (r = 1, 2, \dots, s)$, we get i. e.,

$$\begin{array}{lrcll} (FVPI) & \multicolumn{4}{l}{\max \quad (\alpha_1, \alpha_2, \dots, \alpha_s)} \\ & \multicolumn{4}{l}{\text{subject to}} \\ & A_j^r x + (1-\alpha_r) p_0^r & \ge & V_0^r, & (j = 1, 2, \dots, n, r = 1, 2, \dots, s), \\ & 0 \le \alpha_r & \le & 1, & (r = 1, 2, \dots, s), \\ & x & \in & S^m, & \end{array}$$

where we have used the notation $\alpha_r = \alpha_r(x),\ (r = 1, 2, \dots, s)$.

Conversely, suppose that an efficient solution $(x^*, \alpha^* = \alpha(x^*))$ of $(FVPI)$ is not a POSS for Player I. Then, there exists $x \in S^m$, such that

$$\alpha(x^*) \le \alpha(x), \ \ \alpha(x^*) \ne \alpha(x). \tag{1}$$

Notice that by definitions of $\alpha_r(x)$ and $\mu_j^r(x^T A_j^r),\ (r = 1, 2, \dots, s; j = 1, 2, \dots, n),\ (x, \alpha(x))$ is feasible to $(FVPI)$. Thus (1) contradicts the assumption that (x^*, α^*) is an efficient solution of $(FVPI)$. □

Theorem 3.6. The strategy y^* is a POSS and β^* is the security level for Player II if and only if (y^*, β^*) is an efficient solution to the following multi-objective programming problem

$$\begin{array}{lrcll} (FVPII) & \multicolumn{4}{l}{\max \quad (\beta_1, \beta_2, \dots, \beta_s)} \\ & \multicolumn{4}{l}{\text{subject to}} \\ & A_i^r y - (1-\beta_r) q_0^r & \le & W_0^r, & (i = 1, 2, \dots, m,\ r = 1, 2, \dots, s), \\ & 0 \le \beta_r & \le & 1, & (r = 1, 2, \dots, s), \\ & y & \in & S^n. & \end{array}$$

Proof. The proof follows on the lines of the proof of Theorem 3.6. □

From the above discussion we observe that solving $MOFG$ is equivalent to solving the crisp pair of multi-objective linear programming problems $(FVPI)$ and $(FVPII)$

for Player I and Player II, respectively. Also if $(x^*, \alpha^* = (\alpha_1^*, \alpha_2^*, \ldots, \alpha_s^*))$ is an efficient solution of $(FVPI)$ then x^* is POSS for Player I and $\alpha_r^*(r = 1, 2, \ldots, s)$ is the degree to which the aspiration level V_0^r of Player I is met by choosing to play the strategy x^*. Similar interpretation can also be given to an optimal solution $(y^*, \beta^* = (\beta_1^*, \beta_2^*, \ldots, \beta_s^*))$ of the problem $(FVPII)$.

Remark 3.7. Here in this game POSS is defined for each player as a best strategy, within the strategy profile of that player without considering the strategy profile of other player. Whereas in Pareto Nash Equilibrium (PNE) [9] a strategy $s^* = (x^*, y^*) \in S^m \times S^n$ is called PNE if $\nexists s = (x, y) \in S^m \times S^n$ such that

$$u_i(s) \geq u_i(s^*), \forall i = 1, 2$$

i.e.,

$$u_i(x, y) \geq u_i(x^*, y^*), \forall i = 1, 2,$$

where u_i is the utility function of the i^{th} player. Therefore

$$u_1(x, y) = u_2(x, y) = (E_1(x, y), E_2(x, y), \ldots, E_s(x, y)),$$

whereas in our case

$$v(x) = (\underset{y \in S^n}{min} E_1(x, y), \underset{y \in S^n}{min} E_2(x, y), \ldots, \underset{y \in S^n}{min} E_s(x, y)),$$

and

$$w(y) = (\underset{x \in S^m}{max} E_1(x, y), \underset{x \in S^m}{max} E_2(x, y), \ldots, \underset{x \in S^m}{max} E_s(x, y)).$$

Hence $\forall x \in S^m,\ y \in S^n$

$$v(x) \leq u_1(x, y) \leq w(y)$$

and

$$v(x) \leq u_2(x, y) \leq w(y)$$

where $\leq$ is taken in vectors as component wise. Now $\forall x \in S^m,\ y \in S^n$

$$v(x^*) \leq u_1(x^*, y), \quad u_1(x, y^*) \leq w(y^*),$$

and

$$v(x^*) \leq u_2(x^*, y), \quad u_2(x, y^*) \leq w(y^*).$$

Thus the two notations POSS and PNE are different.

The duality in classical multi-objective problems is not straightforward and can be studied by converting both problems into their scalar counter parts using weighted sum approach with same weights. In the same spirit, consider the weights $\lambda_r \geq 0, \sum_{r=1}^{s} \lambda_r = 1, (r = 1, 2, \ldots, s)$, associated with the objective functions of $(FVPI)$ and $(FVPII)$ and obtain their scalar counter parts as $(FVPI)_1$ and $(FVPII)_2$ respectively as follows

$$
\begin{array}{llrcll}
(FVPI)_1 & \max & \multicolumn{4}{l}{\displaystyle\sum_{r=1}^{s} \lambda_r \alpha_r} \\
& \text{subject to} & & & & \\
& & A_j^r x + (1-\alpha_r) p_0^r & \geq & V_0^r, & (j = 1,2,\ldots,n, r = 1,2,\ldots,s), \\
& & 0 \leq \alpha_r, \lambda_r & \leq & 1, & (r = 1,2,\ldots,s), \\
& & \sum_{r=1}^{s} \lambda_r = 1,\ x & \in & S^m, &
\end{array}
$$

and

$$
\begin{array}{llrcll}
(FVPII)_1 & \max & \multicolumn{4}{l}{\displaystyle\sum_{r=1}^{s} \lambda_r \beta_r} \\
& \text{subject to} & & & & \\
& & A_i^r y - (1-\beta_r) q_0^r & \leq & W_0^r, & (i = 1,2,\ldots,m,\ r = 1,2,\ldots,s), \\
& & 0 \leq \beta_r, \lambda_r & \leq & 1, & (r = 1,2,\ldots,s), \\
& & \sum_{r=1}^{s} \lambda_r = 1,\ y & \in & S^n. &
\end{array}
$$

The following modified weak duality theorem between $(FVPI)_1$ and $(FVPII)_1$ establish the duality relation between $(FVPI)$ and $(FVPII)$, in fuzzy sense.

Theorem 3.8. (Modified weak duality theorem) Let $(x, \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s), \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s))$ and $(y, \beta = (\beta_1, \beta_2, \ldots, \beta_s), \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s))$ be feasible for $(FVPI)_1$ and $(FVPII)_1$ respectively. Then,

$$\sum_{r=1}^{s} \lambda_r (\alpha_r - 1) p_0^r + \sum_{r=1}^{s} \lambda_r (\beta_r - 1) q_0^r \leq \sum_{r=1}^{s} \lambda_r (W_0^r - V_0^r).$$

Proof. The proof follows on the lines of Theorem 5.3.1 of Bector et al. [4]. □

The $MOFG$ has also been investigated by Nishazaki and Sakawa [11] with a different approach than the one we have proposed above. While we have introduced the solution for two players in the sense of POSS, Nishazaki and Sakawa [11] defined them in conventional optimality. However, we notice an error in [11] which we propose to rectify in the next section.

4. NISHIZAKI AND SAKAWA'S MODEL FOR MOFG

We now briefly describe Nishizaki and Sakawa's [11] model for multi-objective two person zero-sum fuzzy matrix game with fuzzy goals. We assume that each player has set a goal for each of the expected payoff and the attainment of these goals is described by fuzzy sets whose membership functions can be interpreted as the degree of attainment of those goals. Let for each $(r = 1, 2, \ldots, s)$, $\underline{a}^r$ and $\overline{a}^r$ be the pay-offs giving the best degree of satisfaction and the worst degree of satisfaction with respect to rth expected payoff. We next explain the model of [11] for $MOFG$.

Let $D^r = \{x^T A^r y : (x, y) \in S^m \times S^n\}$ be the set of expected values corresponding to the rth payoff matrix A^r. Then the fuzzy sets on D^r for Player I and Player II are characterized by the membership functions $\mu_1^r : D^r \to [0,1]$ and $\mu_2^r : D^r \to [0,1]$, respectively.

Nishizaki and Sakawa [11] took the membership functions $\mu_1^r(x^T A^r y)$ and $\mu_2^r(x^T A^r y)$ for all $(r = 1, 2, \ldots, s)$ as linear and aggregated the same by the 'min' operator. They defined them as follows

$$\mu_1^r(x^T A^r y) = \begin{cases} 0, & x^T A^r y \le \underline{a}^r, \\ 1 - \dfrac{\overline{a}^r - x^T A^r y}{\overline{a}^r - \underline{a}^r}, & \underline{a}^r < x^T A^r y \le \overline{a}^r, \\ 1, & x^T A^r y \ge \overline{a}^r, \end{cases}$$

and

$$\mu_2^r(x^T A^r y) = \begin{cases} 1, & x^T A^r y \le \underline{a}^r, \\ 1 - \dfrac{x^T A^r y - \underline{a}^r}{\overline{a}^r - \underline{a}^r}, & \underline{a}^r < x^T A^r y \le \overline{a}^r, \\ 0, & x^T A^r y \ge \overline{a}^r, \end{cases}$$

where

$$\underline{a}^r = \min_{x \in S^m} \min_{y \in S^n} x^T A^r y = \min_{1 \le i \le m} \min_{1 \le j \le n} a_{ij}^r,$$

$$\overline{a}^r = \max_{x \in S^m} \max_{y \in S^n} x^T A^r y = \max_{1 \le i \le m} \max_{1 \le j \le n} a_{ij}^r.$$

Therefore the membership functions for the aggregated fuzzy goals for Player I and Player II are respectively

$$\mu_1(x^T A y) = \min_{1 \le r \le s} \mu_1^r(x^T A^r y)$$

and

$$\mu_2(x^T A y) = \min_{1 \le r \le s} \mu_2^r(x^T A^r y).$$

Now employing Bellman and Zadeh [1] decision making criterion, the maximin value with respect to the degree of attainment of the aggregated fuzzy goals for Player I is

$$\max_{x \in S^m} \min_{y \in S^n} \mu_1(x^T A y).$$

Similarly the maximin value with respect to the degree of attainment of the aggregated fuzzy goals for Player II is

$$\max_{x \in S^m} \min_{y \in S^n} \mu_2(x^T A^r y).$$

The following theorems from [11] provides the optimization model for the solution of the game.

Theorem 4.1. The maximin solution of Player I with respect to a degree of attainment of the aggregated fuzzy goal is equal to an optimal solution to the following linear programming problem.

$$
\begin{array}{llrcll}
(CPI) & \max & \lambda & & & \\
& \text{subject to} & & & & \\
& & \dfrac{x^T A_j^r}{\overline{a}^r - \underline{a}^r} - \dfrac{\underline{a}^r}{\overline{a}^r - \underline{a}^r} & \geq & \lambda, & (j = 1, 2. \dots n,\ r = 1, 2, \dots, s), \\
& & 0 \leq \lambda & \leq & 1, & \\
& & x & \in & S^m. &
\end{array}
$$

Theorem 4.2. The maximin solution of Player II with respect to the degree of attainment of the aggregated fuzzy goal is equal to an optimal solution to the following linear programming problem.

$$
\begin{array}{llrcll}
(CPII) & \min & \sigma & & & \\
& \text{subject to} & & & & \\
& & \dfrac{A_i^r y}{\overline{a}^r - \underline{a}^r} - \dfrac{\underline{a}^r}{\overline{a}^r - \underline{a}^r} & \leq & \sigma, & (i = 1, 2, \dots, m,\ r = 1, 2, \dots, s), \\
& & 0 \leq \sigma & \leq & 1, & \\
& & y & \in & S^n & .
\end{array}
$$

Remark 4.3. The optimization problem $(CPII)$ for Player II as obtained in Nishizaki and Sakawa [11] is not fully correct. We can verify that the relevant objective function should be 'to maximize $(1 - \sigma)$', i. e. '$1 - minimize(\sigma)$'. For the single payoff case (i. e., when r=1), this point has been well explained in Bector et al. [3] and Bector and Chandra [4]. Therefore problem (CPI) and $((CPII)$ with the corrected objective function $\max(1 - \sigma)$ can not be dual to each other in the crisp sense. In fact it can be shown that these are dual to each other in the fuzzy sense (Bector and Chandra [3, 4]) only.

Remark 4.4. In this approach all pay-offs are aggregated together via the 'min' operator. Therefore the optimal solution (x^*, λ^*) of (CPI) gives only the maximum degree of the aggregated fuzzy goal. But it does not provide any information about the performance of individual goals. In our approach presented in Section 3, we obtain the best possible compromised degree for each of the individual goals (in the sense of efficiency) and hence have some freedom to redefine our aspiration levels so as to obtain the most satisfying solution. Similar arguments hold for Player II as well.

Remark 4.5. Let $V_0^r = \overline{a}^r$, $W_0^r = \underline{a}^r$, $p_r = q_r = \overline{a}^r - \underline{a}^r$, $\alpha_1 = \alpha_2 = \ldots = \alpha_s = \lambda(\text{say})$ and $\beta_1 = \beta_2 = \ldots = \beta_s = 1 - \sigma(\text{say})$, then $(FVPI)$ reduces to (CPI) and $(FVPII)$ reduces to $(CPII)$. The assumption that both players have same tolerance level $\overline{a}^r - \underline{a}^r$, $(r = 1, 2, \dots, s)$, appears to be very restrictive.

Remark 4.6. In [11], for $\overline{a}^r = \max_{1 \leq i \leq m} \max_{1 \leq j \leq n} a_{ij}^r$ and $\underline{a}^r = \min_{1 \leq i \leq m} \min_{1 \leq j \leq n} a_{ij}^r$ $(r = 1, 2, \dots, s)$. But this does not seem to be correct. For $(r = 1, 2, \dots, s)$, it has been shown in [4], that it amounts to saying that A^r is a constant matrix.

5. EXAMPLE

Let us consider the numerical example as taken by Cook [5] and latter on cited by Nishizaki and Sakawa [11]. Here we solve all numerical problems using GAMS [10].

Example 5.1. Consider the multi-objective matrix game having payoff matrices

$$A^1 = \begin{pmatrix} 2 & 5 & 1 \\ -1 & -2 & 6 \\ 0 & 3 & -1 \end{pmatrix}, A^2 = \begin{pmatrix} -3 & 7 & 2 \\ 0 & -2 & 0 \\ 3 & -1 & 6 \end{pmatrix}, A^3 = \begin{pmatrix} 8 & 2 & 3 \\ -5 & 6 & 0 \\ -3 & 1 & 6 \end{pmatrix},$$

as the cost matrix, the time matrix and the productivity matrix respectively.

Solution by the proposed model. We now solve this problem with the same data as taken by Nishizaki and Sakawa [11]. Thus $V_0^1 = \overline{a}^1 = 6, W_0^1 = \underline{a}^1 = -2, p_0^1 = q_0^1 = \overline{a}^1 - \underline{a}^1 = 8$; $V_0^2 = \overline{a}^2 = 7, W_0^2 = \underline{a}^2 = -3, p_0^2 = q_0^2 = \overline{a}^2 - \underline{a}^2 = 10$; $V_0^3 = \overline{a}^3 = 8, W_0^3 = \underline{a}^3 = -5, p_0^3 = q_0^3 = \overline{a}^3 - \underline{a}^3 = 13$. Hence the problem $(FVPI)$ for Player I is

$$\begin{array}{ll} \text{Max} & (\alpha_1, \alpha_2, \alpha_3) \\ \text{subject to} & \end{array}$$

$$\begin{aligned} 2x_1 - x_2 - 8\alpha_1 &\geq -2, \\ 5x_1 - 2x_2 + 3x_3 - 8\alpha_1 &\geq -2, \\ x_1 + 6x_2 - x_3 - 8\alpha_1 &\geq -2, \\ -3x_1 + 3x_3 - 10\alpha_2 &\geq -3, \\ 7x_1 - 2x_2 - x_3 - 10\alpha_2 &\geq -3, \\ 2x_1 - 6x_3 - 10\alpha_2 &\geq -3, \\ 8x_1 - 5x_2 - 3x_3 - 13\alpha_3 &\geq -5, \\ 2x_1 + 6x_2 + x_3 - 13\alpha_3 &\geq -5, \\ 3x_1 + 6x_3 - 13\alpha_3 &\geq -5, \\ 0 \leq \alpha_1, \alpha_2, \alpha_3 &\leq 1 \\ x &\in S^3. \end{aligned}$$

The Pareto-optimal security strategies with corresponding security levels for Player I are depicted in Table 1.

#	x_1^*	x_2^*	x_3^*	α_1^*	α_2^*	α_3^*
1	0.875	0.125	0.0	0.4531	0.0375	0.5769
2	0.8098	0.125	0.0651	0.4368	0.0766	0.5719
3	0.7446	0.125	0.1303	0.4205	0.1157	0.5668
4	0.6794	0.125	0.1955	0.4042	0.1548	0.5618
5	0.6142	0.125	0.2607	0.3879	0.1939	0.5568
6	0.5491	0.125	0.3258	0.3716	0.2330	0.5518

Tab. 1.

Similarly for the same choice of tolerance and aspiration levels, the problem $(FVPII)$

for Player II is

$$
\begin{array}{ll}
\text{Max} & (\beta_1, \beta_2, \beta_3) \\
\text{subject to} & \\
\end{array}
$$

$$
\begin{array}{rcl}
2y_1 + 5y_2 + y_3 + 8\beta_1 & \leq & 6, \\
-y_1 - 2y_2 + 6y_3 + 8\beta_1 & \leq & 6, \\
3y_2 - y_3 + 8\beta_1 & \leq & 6, \\
-3y_1 + 7y_2 + 2y_3 + 10\beta_2 & \leq & 7, \\
-2y_2 + 10\beta_2 & \leq & 7, \\
3y_1 - y_2 + 6y_3 + 10\beta_2 & \leq & 7, \\
8y_1 + 2y_2 + 3y_3 + 13\beta_3 & \leq & 8, \\
-5y_1 + 6y_2 + 13\beta_3 & \leq & 8, \\
-3y_1 + y_2 + 6y_3 + 13\beta_3 & \leq & 8, \\
0 \leq \beta_1, \beta_2, \beta_3 & \leq & 1, \\
y & \in & S^3.
\end{array}
$$

The Pareto-optimal security strategies with corresponding security levels for Player II are depicted in Table 2.

#	y_1^*	y_2^*	y_3^*	β_1^*	β_2^*	β_3^*
1	0.625	0.0	0.375	0.5468	0.2875	0.1442
2	0.6299	0.0249	0.3451	0.5337	0.3064	0.1442
3	0.6349	0.0498	0.3152	0.5207	0.3253	0.1442
4	0.6399	0.0747	0.2853	0.5076	0.3442	0.1442
5	0.6449	0.0996	0.2554	0.4945	0.3632	0.1442
6	0.6499	0.1245	0.2255	0.4814	0.3821	0.1442

Tab. 2.

Here for $\lambda_1 = 1/2,\ \lambda_2 = 1/3,\ \lambda_3 = 1/6$ the modified weak duality theorem is satisfied by first Pareto-optimal security strategies with corresponding security levels of Player I and Player II. Now let us solve the above problem by Nishizaki and Sakawa's [11] model.

Solution by Nishizaki and Sakawa's model. On following their notations, as given in Section 4, we have $\overline{a}^1 = 6, \underline{a}^1 = -2, \overline{a}^1 - \underline{a}^1 = 8;\ \ \overline{a}^2 = 7, \underline{a}^2 = -3, \overline{a}^2 - \underline{a}^2 = 10;\ \ \overline{a}^3 = 8, \underline{a}^3 = -5, \overline{a}^3 - \underline{a}^3 = 13$. Therefore the Player I problem is

$$
\begin{array}{ll}
\text{Max} & \lambda \\
\text{subject to} & \\
\end{array}
$$

$$
\begin{array}{rcl}
2x_1 - x_2 - 8\lambda & \geq & -2, \\
5x_1 - 2x_2 + 3x_3 - 8\lambda & \geq & -2, \\
x_1 + 6x_2 - x_3 - 8\lambda & \geq & -2,
\end{array}
$$

$$\begin{aligned}
-3x_1 + 3x_3 - 10\lambda &\geq -3,\\
7x_1 - 2x_2 - x_3 - 10\lambda &\geq -3,\\
2x_1 + 6x_3 - 10\lambda &\geq -3,\\
8x_1 - 5x_2 - 3x_3 - 13\lambda &\geq -5,\\
2x_1 + 6x_2 + x_3 - 13\lambda &\geq -5,\\
3x_1 + 6x_3 - 13\lambda &\geq -5,\\
0 \leq \lambda &\leq 1,\\
x &\in S^3.
\end{aligned}$$

The optimal solution is obtained as ($x_1^* = 0.3860,\ x_2^* = 0.1250,\ x_3^* = 0.48897,\ \lambda^* = 0.33088$). Similarly Player II problem is

$$\begin{aligned}
&\text{Min} \quad \sigma\\
&\text{subject to}
\end{aligned}$$

$$\begin{aligned}
2y_1 + 5y_2 + y_3 - 8\sigma &\leq -2,\\
-y_1 - 2y_2 + 6y_3 - 8\sigma &\leq -2,\\
3y_2 - y_3 - 8\sigma &\leq -2,\\
-3y_1 + 7y_2 + 2y_3 - 10\sigma &\leq -3,\\
-2y_2 - 10\sigma &\leq -3,\\
3y_1 - y_2 + 6y_3 - 10\sigma &\leq -3,\\
8y_1 - 2y_2 + 3y_3 - 13\sigma &\leq -5,\\
-5y_1 + 6y_2 + -13\sigma &\leq -5,\\
-3y_1 + y_2 + 6y_3 - 13\sigma &\leq -5,\\
0 \leq \sigma &\leq 1,\\
y &\in S^3.
\end{aligned}$$

The optimal solution is ($y_1^* = 0.25595,\ y_2^* = 0.34685,\ y_3^* = 0.39720,\ \sigma^* = 0.5804$). But as mentioned in Remark 1, the objective function value should be $(1-\sigma^*) = 1-0.5804 = 0.4196$.

Remark 5.2. Here it is to be noted that for strategy each POSS, $\max(\alpha_1^*, \alpha_2^*, \alpha_3^*) \geq \lambda^*$ and $\max(\beta_1^*, \beta_2^*, \beta_3^*) \geq 1-\sigma^*$. Thus for the first POSS in Table 1, has $\max(\alpha_1^*, \alpha_2^*, \alpha_3^*) = 0.5769 > \lambda^* = 0.3308$. Also for the first POSS in Table 2, $\max(\beta_1^*, \beta_2^*, \beta_3^*) = 0.5468 > (1-\sigma^*) = 0.4196$. Further under the same circumstances, the players have better confidence corresponding to at least one payoff criterion in the proposed model as compared to Nishizaki and Sakawa's [11] model, and this payoff criterion can be identified by our proposed model.

6. CONCLUSIONS

A new model is presented for studying multi-objective zero-sum fuzzy matrix games. This model is in the sprit of Fernandez et al. [7] and it solves the given fuzzy game by solving a pair of (crisp) multi-objective linear programming problems. Since the model gives the best possible (i. e., most satisfying) degree for the individual players payoff meeting the specified aspiration levels, it provides more flexibility to the decision maker. Certain difficulties with the existing model of Nishizaki and Sakawa [11] are also pointed out. It should be of interest to extend these results to situations where elements of matrices $A^r, (r = 1, 2, \ldots, s)$ are given in terms of fuzzy numbers. The scalar fuzzy matrix games have been studied by [15] using fuzzy relational approach. It would be interesting and challenging to explore this approach for study multi-objective matrix games.

ACKNOWLEDGEMENT

The authors are thankful to Professor Suresh Chandra and Professor Aparna Mehra Department of Mathematics, IIT Delhi for their valuable suggestions and comments on this work. The authors are grateful to the referees for their valuable comments.

(Received December 5, 2014)

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