KYBERNETIKA — VOLUME *51* (2015), NUMBER 2, PAGES 212–230

CHARACTERIZATIONS OF ARCHIMEDEAN n-COPULAS

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We present three characterizations of n-dimensional Archimedean copulas: algebraic, differential and diagonal. The first is due to Jouini and Clemen. We formulate it in a more general form, in terms of an n-variable operation derived from a binary operation. The second characterization is in terms of first order partial derivatives of the copula. The last characterization uses diagonal generators, which are "regular" diagonal sections of copulas, enabling one to recover the copulas by means of an asymptotic representation.

Keywords: Archimedean operation, additive generator, diagonal generator, multiplicative generator, (Archimedean) n-copula, (Archimedean) n-quasicopula

Classification: 62H20

1. INTRODUCTION

Copulas are of fundamental importance for statistics, probability theory and stochastic processes.

In Encyclopedia of Statistical Sciences [5], Fischer gave two main reasons for the interest in copulas. First, copulas may be used to present various models of stochastic dependence in distributions of random vectors under reduced influence of marginal distributions. Secondly, copulas can be a starting point for constructing families of distributions. Important sources of information about copulas are the monographs by Hutchinson and Lai [14], Joe [16] and Nelsen [22], as well as more recent publications like Durante and Sempi [4] and Rüschendorf [24].

Let $\mathbf{X} = (X_1, \dots, X_n) : (\Omega, \mathcal{F}, P) \to \mathbb{R}^n$ be an n-dimensional random vector. Let F_{X_k} be the continuous distribution function of the coordinate X_k for $k = 1, \dots, n$. We transform $\mathbf{X}$ into a random vector $\mathbf{U} = (U_1, \dots, U_n)$ by the formula $U_k = F_{X_k}(X_k)$ for $k = 1, \dots, n$.

The distribution function C which corresponds to the distribution $P^{\mathbf{U}}$ of the vector $\mathbf{U}$ (concentrated on the box $\mathbb{I}^n = [0,1]^n$) is called an *n-dimensional copula* (or *n-copula*). The vector $\mathbf{U}$ is called the *uniform representation* of $\mathbf{X}$.

DOI: 10.14736/kyb-2015-2-0212

The copula $C : \mathbb{I}^n \to \mathbb{I}$ has the following properties:

$$C(\mathbf{1}_k) = u_k \quad \text{for } k = 1, \ldots, n, \tag{1a}$$

where $\mathbf{1}_k \in \mathbb{I}^n$ has the kth coordinate u_k, and the others 1;

$$C(\mathbf{u}) = 0 \quad \text{if some } u_k \text{ is zero;} \tag{1b}$$

and

$$\sum_{\mathbf{w}} \sigma(\mathbf{w}) C(\mathbf{w}) \geq 0, \tag{1c}$$

where $\mathbf{w} = (w_1, \ldots, w_n)$ runs through the vertices of the parallelepiped $[\mathbf{u}, \mathbf{v}] = \times_{k=1}^{n} [u_k, v_k]$, $\mathbf{u} = (u_1, \ldots, u_n)$, $\mathbf{v} = (v_1, \ldots, v_n)$, $u_k \leq v_k$ for $k = 1, \ldots, n$. Here $\sigma(\mathbf{w})$ is 1 or -1 according to whether the number of equalities $u_k = v_k$ is even or odd.

Condition (1a) means that the coordinates of the random vector $\mathbf{U}$ have uniform distribution on $\mathbb{I}$. Every function $C : \mathbb{I}^n \to \mathbb{I}$ satisfying (1a)–(1c) is an n-copula. For C "sufficiently regular", condition (1c) can be easily checked by using the following lemma (see Joe [16], here reformulated):

Lemma 1. If a continuous function $C : \mathbb{I}^n \to \mathbb{I}$ satisfying (1a) and (1b) has all partial derivatives of order n on $(0, 1)^n$, then condition (1c) is equivalent to

$$\frac{\partial^n C(\mathbf{u})}{\partial u_1 \ldots \partial u_n} \geq 0 \quad \text{for all } \mathbf{u} \in (0, 1)^n.$$

It turns out that some n-copulas can be determined by functions defined on $\mathbb{I}$. Let $\mathscr{F}$ be the family of all strictly increasing continuous bijections $f : \mathbb{I} \to \mathbb{I}$ (then $f(0) = 0$ and $f(1) = 1$). Let C be the copula corresponding to the uniform representation $\mathbf{U}$. Suppose that

$$f^{-1}[P(\{U_k \leq u_k,\, k = 1, \ldots, n\})] = \prod_{k=1}^{n} f^{-1}[P(\{U_k \leq u_k\})] \tag{2}$$

for some $f \in \mathscr{F}$, where P denotes the probability on the probability space where $\mathbf{X}$ is defined. Formula (2) can be written more simply as

$$C(\mathbf{u}) = f\Big[\prod_{k=1}^{n} f^{-1}(u_k)\Big] \quad \text{for all } \mathbf{u} \in \mathbb{I}^n. \tag{3}$$

If $\tilde{f}$ is another function in $\mathscr{F}$, it may happen that the function $\tilde{C}$ given by (3) with f replaced by $\tilde{f}$ is not an n-copula. In Theorem 1 we exhibit a class of functions f for which (3) is an n-copula. For this, we need the following notion.

Let $\mathbb{R}_+ = [0, \infty)$ and let $\mathbb{J} \subset \overline{\mathbb{R}}_+ = [0, \infty]$ be some interval. For $n \geq 1$, a continuous function $f : \mathbb{J} \to \mathbb{R}_+$ is called *n-absolutely monotone* if it is n times continuously differentiable on the interior $\mathrm{Int}(\mathbb{J})$ of $\mathbb{J}$ and satisfies the condition

$$\frac{\mathrm{d}^k f(u)}{\mathrm{d}u^k} \geq 0 \quad \text{for all } u \in \mathrm{Int}(\mathbb{J}) \text{ and } k = 1, \ldots, n.$$

Theorem 1. If $f \in \mathscr{F}$ is n-absolutely monotone, then the function (3) is an n-copula.

P r o o f. Clearly, the function (3) satisfies conditions (1a) and (1b). To check (1c), note that if C is an n-copula, then its restrictions $C^k = C|\mathbb{I}^k$ for $k = 2, \ldots, n$ are its marginal k-copulas. Hence by Lemma 1, (3) is an n-copula if and only if

$$\frac{\partial^k C^k(u_1, \ldots, u_k)}{\partial u_1 \ldots \partial u_k} = \Big\{ \prod_{j=1}^{k} \frac{\mathrm{d}f}{\mathrm{d}u}[f^{-1}(u_j)] \Big\}^{-1} \Big\{ \sum_{j=1}^{k} a_j \frac{\mathrm{d}^j f}{\mathrm{d}u^j} \Big[\prod_{j=1}^{k} f^{-1}(u_j) \Big] \Big[\prod_{j=1}^{k} f^{-1}(u_j) \Big]^{j-1} \Big\} \geq 0 \quad (4)$$

for all $(u_1, \ldots, u_k) \in (0,1)^k$ and all $k = 2, \ldots, n$. Here the coefficients a_j for $j = 1, \ldots, n$ are suitable positive integers. Since the function df/du is positive, the n-absolute continuity of f entails (4). □

From (2) one can also obtain another form of an n-copula, which may be called "additive", as opposed to the "multiplicative" form (3). Taking the logarithm of both sides of (2), one can express C in terms of the function $g = -\ln f^{-1}$. The value of g at $u = 0$ is understood to be $\lim_{u \to 0_+} g(u)$. We obtain the representation

$$C(\mathbf{u}) = g^{-1}\Big[\sum_{k=1}^{n} g(u_k) \Big] \quad \text{for all } \mathbf{u} \in \mathbb{I}^n. \quad (5)$$

Clearly, the function (5) corresponding to $g = \ln \tilde{f}^{-1}$ for $\tilde{f} \in \mathscr{F}$ need not be an n-copula. If $g(0) < \infty$, the function g in (5) should be replaced by its pseudoinverse.

Let $\mathscr{G}$ be the family of all strictly decreasing convex functions $g : \mathbb{I} \to [0, \infty]$ with $g(1) = 0$. Clearly, $g(0) = \lim_{u \to 0_+} g(u)$. For $g \in \mathscr{G}$ we define its *pseudoinverse* by

$$g^{\leftarrow}(x) = \inf\{t \in \mathbb{I} : g(t) \geq x,\ x \in [0, \infty]\}.$$

The pseudoinverse $g^{\leftarrow}$ coincides with the usual inverse g^{-1} if and only if $g(0) = \infty$.

We now introduce the notion of an n-monotone function. For $n \geq 1$, a continuous function $\psi : \mathbb{J} \to \mathbb{R}_+$ is called *n-monotone* if ψ is n times differentiable on $\mathrm{Int}(\mathbb{J})$ and satisfies the condition

$$(-1)^k \frac{\mathrm{d}^k \psi(x)}{\mathrm{d}x^k} \geq 0 \quad \text{for all } x \in \mathrm{Int}(\mathbb{J}) \text{ and } k = 1, \ldots, n.$$

If ψ is C^∞ and satisfies the last condition for all $k = 1, \ldots, n$, then it is called *completely monotone.*

Kimberling [18] showed that for each $n \geq 2$, (5) is an n-copula if and only if $g(0) = \infty$ and $g^{\leftarrow}$ is completely monotone. Nelsen [22] observed that for (5) to be an n-copula, it suffices that $g^{\leftarrow}$ be n-monotone. Recently, McNeil and Nešlehová [21] gave a general solution of the problem of when (5) is an n-copula. We recall it below in a modified form.

Theorem 2. The function (5) is an n-copula if and only if:

(a) $g \in \mathscr{G}$,

(b) $g^{\leftarrow}$ is $(n-2)$-monotone,

(c) $(-1)^{n-2}\frac{\mathrm{d}^{n-2}g^{\leftarrow}}{\mathrm{d}x^{n-2}}$ is decreasing and convex on $(0, g(0))$.

A copula of the form (5) is called an *n-dimensional Archimedean copula* (or an *Archimedean n-copula*) with *additive generator g*. The generator g is *strict* (resp. *nonstrict*) if $g(0) = \infty$ (resp. $g(0) < \infty$).

If (3) is an n-copula, then f is called its *multiplicative generator.* The additive and multiplicative generators of an Archimedean n-copula are related by the formula

$$g = -\ln f^{-1}. \tag{6}$$

Remark 1. The additive generator is uniquely determined up to a positive multiplicative constant.

The family of all functions g with properties (a) – (c) of Theorem 2 will be denoted by $\mathscr{G}_{n-2}$.

Let $\overset{*}{\mathscr{G}}_n$ consist of all additive generators g whose pseudoinverse $g^{\leftarrow}$ is n-monotone and $\frac{\mathrm{d}^n g^{\leftarrow}}{\mathrm{d}x^n}$ is a continuous function. Denote by $\overset{*}{\mathscr{G}}{}_n^{-}$ (resp. $\overset{*}{\mathscr{G}}{}_n^{0}$) the family of all $g \in \overset{*}{\mathscr{G}}_n$ such that $\lim_{u\to 1_-}\frac{\mathrm{d}g(u)}{\mathrm{d}u}$ equals -1 (resp. 0).

In the last few years the role of Archimedean copulas is systematically increasing, especially as regards their applications in practical areas, including econometrics, insurance, finance, risk management and survival analysis. Nelsen [22] devoted to them Chapter 4 of his monograph. In his opinion, an increasing role of Archimedean copulas is mainly due to their simplicity, ease of constructing and several nice properties. A systematic investigation of Archimedean copulas was initiated by Genest and MacKay [8, 9]. Recently, substantial progress was obtained by McNeil and Nešlehová[21]. Among other things, they pointeọut that the pseudoinverses of elements of $\mathscr{G}_{n-2}$ are Williamson n-functions (Williamson [27]), related to the notion of l_1-norm symmetric distribution introduced by Fang and Fang [6]. McNeil and Nešlehová [21] explained the unique role of such distributions in the study of Archimedean n-copulas.

Alsina et al. [1] introduced the notion of quasicopula (in two dimensions) in an algebraic way. This notion was later characterized by Genest et al. [10]. The definition and the characterization were generalized to the multivariate case by Cuculescu and Theodorescu [2]; we will use this characterization as an alternative definition.

Lemma 2. A function $Q : \mathbb{I}^n \to \mathbb{I}$ is an n-quasicopula if and only if it satisfies, apart from (1a) and (1b), the following conditions:

- Q is nondecreasing in each variable separately,

- Q satisfies the Lipschitz condition

$$|Q(\mathbf{u}) - Q(\mathbf{v})| \leq \sum_{k=1}^{n} |u_k - v_k| \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbb{I}^n.$$

Clearly, every n-copula is an n-quasicopula. A quasicopula that is not a copula is called *proper.* We distinguish a certain subclass of n-quasicopulas, easy to characterize (see Nelsen et al. [23]).

Theorem 3. A function $Q : \mathbb{I}^n \to \mathbb{I}$ of the form

$$Q(\mathbf{u}) = g^{\leftarrow}\Big[\sum_{k=1}^{n} g(u_k)\Big] \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbb{I}^n \tag{7}$$

is a quasicopula if and only if $g \in \mathscr{G}$. The quasicopula (7) is called the *Archimedean n-quasicopula with additive generator g.*

Example 1. The function $g(u) = 1 - u$ belongs to $\mathscr{G}$. For $n > 2$, to g corresponds the proper Archimedean n-quasicopula

$$Q(\mathbf{u}) = \max\Big[\sum_{k=1}^{n} u_k - (n-1), 0\Big] \quad \text{for all } \mathbf{u} \in \mathbb{I}^n.$$

In the next three sections we give, among other things, characterizations of Archimedean n-copulas. Two of them are fully functional, in the sense that from the analytic form of the n-copula (an n-variate function) we can obtain its additive generator. Both those characterizations have analogues for Archimedean n-quasicopulas.

2. ALGEBRAIC STRUCTURE

In this section we give the first of the announced characterizations of Archimedean n-copulas. It is mainly theoretical, although it does shed some light on their structure. We will precede it by a discussion of the "algebraic nature" of Archimedean copulas. We will also consider one of their properties, justifying the terminology used.

Ling [20] gave an elegant characterization of two-dimensional Archimedean copulas. We recall it below in a slightly modified form.

Lemma 3. Let C be a two-dimensional copula. Then the following conditions are equivalent:

(a_1) C is Archimedean.

(a_2) C satisfies the conditions

$$C(C(u_1, u_2), u_3) = C(u_1, C(u_2, u_3)) \quad \text{for all } (u_1, u_2, u_3) \in \mathbb{I}^3, \tag{8a}$$
$$C(u, u) < u \quad \text{for all } u \in (0, 1). \tag{8b}$$

The formula (8a) has a simple interpretation. The copula C can be thought of as a binary operation in $\mathbb{I}$. Then (8a) states that the operation is associative. Together with the symmetry of C, this means that $(\mathbb{I}, C)$ is a commutative semigroup, with neutral element 1. A hypothetical generalization of Lemma 3 to n dimensions requires introducing a notion of associativity for n-ary operations; any such notion will be called *n-associativity.* We recall the popular definition of n-associativity from Dudek and Trokhimenko [3]:

$$\begin{aligned} &C(u_1, \dots, u_{k-1}, C(u_k, \dots, u_{k+n-1}), u_{k+n}, \dots, u_{2n-1}) \\ &\qquad = C(u_1, \dots, u_k, C(u_{k+1}, \dots, u_{k+n}), u_{k+n+1}, \dots, u_{2n-1}) \end{aligned} \tag{9a}$$

for all $u_1, \dots, u_{2n-1} \in \mathbb{I}$ and $k \in \{2, \dots, n-1\}$. For $n = 2$, formula (9a) reduces to (8a). More complicated notions of n-associativity can be found in Gluskin [11, 12, 13]. Note that an n-ary operation which is an Archimedean copula satisfies the n-associativity condition (9a) and the following condition:

$$C(u_1, \dots, u_{k-1}, \dot{u}, u_{k+1}, \dots, u_n) = C^{n-1}(u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_n) \tag{9b}$$

for all $(u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_n) \in \mathbb{I}^{n-1}$, where $\dot{u} = 1$ and $k \in \{2, \dots, n-1\}$. Here C^{n-1} is the marginal $(n-1)$-copula of C. Formula (9b) suggests that $\dot{u}$ can be regarded as the neutral element of the operation C. Hence $(\mathbb{I}, C)$ is a commutative semigroup. A generalization of Lemma 3 to n dimensions for associativity (9a) was given by Stupňanová and Kolesárová [25, Theorem 4.2].

Nelsen [22] showed that a two-dimensional Archimedean copula, restricted to $(0,1)^2$ and regarded as a binary operation, has the Archimedean property. We will extend the latter notion to n-ary operations, and then give an analogue of the above fact for Archimedean n-copulas. Let T be a binary operation on $\mathbb{I}$. Then we define the powers u_T^n of $u \in \mathbb{I}$ according to T by induction:

$$u_T^1 = u, \quad u_T^{k+1} = T(u, u_T^k) \quad \text{for } k \geq 1.$$

We say that T has the *Archimedean property* if for every pair $(u_1, u_2) \in \mathbb{I}^2$ there exists a positive integer k such that $(u_1)_T^k < u_2$. This notion is extended to n-ary operations T on $\mathbb{I}$ as follows. We define u_T^k by

$$u_T^1 = u, \quad u_T^{k+1} = T(u, u, \dots, u, u_T^k) \quad \text{for } k \geq 1.$$

If for every $\mathbf{u} \in \mathbb{I}^n$ there exists a positive integer k such that

$$(u_j)_T^k < u_n \quad \text{for } j = 1, \dots, n-1, \tag{9c}$$

then we say that T has the *Archimedean property.*

Lemma 4. Let C be an Archimedean n-copula with additive generator g. Then for every $\mathbf{u} \in (0,1)^n$ there exists a positive integer k such that (9c) holds.

Proof. Let $u \in (0,1)$. Then $u_C^k = g^{\leftarrow}[m_k g(u)]$ where $m_k = (n-1)k - n + 2$. We have $u_C^k \to 0$ (pointwise convergence on $(0,1)$) as $k \to \infty$. Hence for every $\varepsilon > 0$ there exists a positive integer $K(\varepsilon, u)$ such that $u_C^k < \varepsilon$ for all $k > K(\varepsilon, u)$.

Let now $\mathbf{u} \in (0,1)^n$. Setting $\varepsilon = u_n$, we get $(u_j)_C^{1+K(u_n,u_j)} < u_n$ for $j = 1, \ldots, n-1$. The integer $k = 1 + \max[K(u_n, u_1), K(u_n, u_2), \ldots, K(u_n, u_{n-1})]$ satisfies (9c). □

Lemma 4 justifies calling copulas C of the form (5) Archimedean. The term "two-dimensional Archimedean copula" is attributed to Ling [20].

We now give a characterization of Archimedean n-copulas which is due to Jouini and Clemen [17]. We will formulate it in terms of a certain algebraic property of n-ary operations. To this end, for a binary operation $T : \mathbb{I}^2 \to \mathbb{I}$ we define, for any fixed integer $n \geq 3$, a finite sequence of iterations $T_k : \mathbb{I}^k \to \mathbb{I}$ of T as follows:

$$\begin{aligned} &T_2 = T, \\ &T_k(u_1, \ldots, u_k) = T(T_{k-1}(u_1, \ldots, u_{k-1}), u_k) \quad \text{for } k = 3, \ldots, n-1. \end{aligned}$$

We say that an n-ary operation $C : \mathbb{I}^n \to \mathbb{I}$ *derives from the binary operation* T if $T_n = C$.

Theorem 4. Let C be an n-copula. Then the following conditions are equivalent:

(A_1) C is Archimedean.

(A_2) C restricted to $\mathbb{I}^2$ is a two-dimensional Archimedean copula T with additive generator in $\mathscr{G}_{n-2}$. Moreover, the n-ary operation C derives from the (binary) operation T.

Proof. This follows from Theorem 2 and the definitions. □

3. DIFFERENTIAL CHARACTERIZATION

Here we will give a useful characterization of Archimedean n-copulas in terms of their first order partial derivatives.

Theorem 5. Suppose that $n \geq 2$ and C is an n-copula all of whose first order partial derivatives are continuous on $\mathrm{Int}(\operatorname{supp} C)$. Then the following conditions are equivalent:

(B_1) C is Archimedean.

(B_2) There exists a continuous function $\psi : (0,1) \to (0, \infty)$ with a finite left limit at 1 such that

$$\psi(u_{k_1}) \frac{\partial C(\mathbf{u})}{\partial u_{k_2}} = \psi(u_{k_2}) \frac{\partial C(\mathbf{u})}{\partial u_{k_1}} \tag{10}$$

for all $\mathbf{u} \in \mathrm{Int}(\operatorname{supp} C)$ and $k_1, k_2 \in \{1, \ldots, n\}$.

Moreover, if both (B_1) and (B_2) are valid, then the function ψ determines the additive generator g of C by

$$g(u) = \int_u^1 \psi(s)\, \mathrm{d}s \quad \text{for all } u \in \mathbb{I}. \tag{11}$$

P r o o f. $(\mathrm{B}_1) \Rightarrow (\mathrm{B}_2)$. Let g be the additive generator of C. By Theorem 2, for $n \geq 3$ the functions $\mathrm{d}g/\mathrm{d}u$ and $\mathrm{d}g^{\leftarrow}/\mathrm{d}x$ are continuous on $(0,1)$ and $(0,\infty)$ respectively. The function $\psi = -\mathrm{d}g/\mathrm{d}u$ is positive, has a finite left limit at $u=1$ and satisfies (10).

In the case of $n=2$, note that since g is convex, it is differentiable on $(0,1)$ except possibly at countably many points. For every $u_1 \in (0,1)$ there exists $u_2 \in (0,1)$ such that g is differentiable at $C(u_1,u_2)$. The derivative of the right-hand side of the equality $g(u_1)+g(u_2) = g(C(u_1,u_2))$ equals $\frac{\mathrm{d}g}{\mathrm{d}u}(C(u_1,u_2))\frac{\partial C(u_1,u_2)}{\partial u_1}$. Hence $\frac{\mathrm{d}g}{\mathrm{d}u}(u_1)$ exists. The continuity of $\frac{\mathrm{d}g}{\mathrm{d}u}$ follows from convexity and differentiability of g (we have already used that for $n \geq 3$).

$(\mathrm{B}_2) \Rightarrow (\mathrm{B}_1)$. We extend ψ by setting $\psi(0) = \lim_{s\to 0_+}\psi(s)$ and $\psi(1) = \lim_{s\to 1_-}\psi(s)$. Here $\psi(0)$ may be infinite, but $\psi(1)$ is finite. For every $u \in (0,1]$ the integral $\int_u^1 \psi(s)\,\mathrm{d}s$ exists. Hence the function g defined by (11) is positive on $(0,1)$, strictly decreasing, C^1 and such that $g(1)=0$. Additionally we define $g(0) = \lim_{u\to 0_+} g(u)$. Clearly, g has a pseudoinverse $g^{\leftarrow}$ which is C^1 on $(0, g(0))$.

We introduce new variables $\mathbf{v} = (v_1,\ldots,v_n)$, where $v_k = g(u_k)$ for $k=1,\ldots,n$. Denote by $\tilde{C}(\mathbf{v})$ the function obtained from the n-copula C after this change of variables. Note that the system (10) is equivalent to

$$\frac{\partial \tilde{C}(\mathbf{v})}{\partial v_{k_1}} = \frac{\partial \tilde{C}(\mathbf{v})}{\partial v_{k_2}} \quad \text{for } k_1,k_2 \in \{1,\ldots,n\}. \tag{12}$$

The variable $\mathbf{v}$ in (12) runs through $\mathrm{Int}(\operatorname{supp}\tilde{C})$. Our proof amounts to showing that

$$\tilde{C}(\mathbf{v}) = g^{\leftarrow}\Big(\sum_{k=1}^{n} v_k\Big) \quad \text{for all } \mathbf{v} \in \operatorname{supp}\tilde{C}. \tag{13}$$

We have

$$\begin{aligned} \tilde{C}(v_1,0,\ldots,0) &= g^{\leftarrow}(v_1),\\ \tilde{C}(0,v_2,0,\ldots,0) &= g^{\leftarrow}(v_2),\ \ldots,\\ \tilde{C}(0,0,\ldots,0,v_n) &= g^{\leftarrow}(v_n). \end{aligned}$$

Moreover, $\tilde{C}(\mathbf{v}) = 0$ if at least one coordinate of $\mathbf{v}$ is equal to $g(0)$.

We decompose $\mathbb{R}^n$ as $X \oplus Y$ where $X = \{(x,\ldots,x) : x \in \mathbb{R}\}$ and $Y = \{(y_1,\ldots,y_n) : \sum_{i=1}^n y_i = 0\}$. Let $(\mathbf{e}_1,\ldots,\mathbf{e}_n)$ be the canonical basis of $\mathbb{R}^n$. Then the vector $\mathbf{a} = \sum_{k=1}^n \mathbf{e}_k$ spans X, and $(\mathbf{b}_1,\ldots,\mathbf{b}_{n-1})$, where $\mathbf{b}_k = \mathbf{e}_k - \mathbf{e}_{k+1}$ for $k=1,\ldots,n-1$, is a basis of Y. The vectors $\mathbf{y} = (y_1,\ldots,y_n) \in Y$ have a representation $y_1 = t_1$, $y_k = t_k - t_{k-1}$ for $k=2,\ldots,n-1$, and $y_n = -t_{n-1}$, where the t_k run through $\mathbb{R}$. Every $\mathbf{v} \in \operatorname{supp}\tilde{C}$ decomposes as $\mathbf{v} = \mathbf{x}_{\mathbf{v}} + \mathbf{y}_{\mathbf{v}}$, where $\mathbf{x}_{\mathbf{v}} \in X$ and $\mathbf{y}_{\mathbf{v}} \in Y$. Then the right-hand side of (13) takes the form

$$g^{\leftarrow}\Big(\sum_{k=1}^{n} v_k\Big) = g^{\leftarrow}\Big[\sum_{k=1}^{n}(x+y_k)\Big] = g^{\leftarrow}(nx).$$

Hence $g^{\leftarrow}(\sum_{k=1}^{n} v_k)$ depends only on $\mathbf{x_v}$ (hence on x). It turns out that so does $\tilde{C}$. To see this, we have to check that the strong derivative $D\tilde{C}$ restricted to Y is the zero form. Indeed,

$$\begin{aligned}[D\tilde{C}(\mathbf{v})]\Big(\sum_{k=1}^{n-1} t_k \mathbf{b}_k\Big) &= \sum_{k=1}^{n-1} t_k [D\tilde{C}(\mathbf{v})](\mathbf{b}_k) \\ &= \sum_{k=1}^{n-1} t_k \left[\frac{\partial \tilde{C}(\mathbf{v})}{\partial v_k} - \frac{\partial \tilde{C}(\mathbf{v})}{\partial v_{k+1}}\right] = 0.\end{aligned}$$

We have used (12) and the continuity of the partial derivatives $\partial\tilde{C}(\mathbf{v})/\partial v_k$. The equality $[D\tilde{C}(\mathbf{v})]|Y = 0$ means that $\tilde{C}$ only depends on $\mathbf{x_v}$ (hence on x). Therefore $\tilde{C}(\mathbf{v}) = \tilde{C}(\mathbf{x_v} + \mathbf{y_v}) = \tilde{C}(\mathbf{x_v})$. In particular this holds for

$$\mathbf{x_v} = (x, \dots, x) \quad \text{and} \quad \mathbf{y_v} = [(n-1)x, -x, \dots, -x].$$

Consequently, $\tilde{C}(nx, 0, \dots, 0) = g^{\leftarrow}(nx)$, proving (13). □

Remark 2. The implication $(\mathrm{B}_2)\Rightarrow(\mathrm{B}_1)$ extends the criterion of Genest and MacKay [8] for a two-dimensional copula to be Archimedean with a nonstrict additive generator.

Remark 3. To ensure the continuity of the partial derivatives $\partial C(\mathbf{u})/\partial u_k$, it suffices to assume the existence of all second order partial derivatives of C on $\operatorname{Int}(\operatorname{supp} C)$.

Corollary 1. The function ψ of Theorem 5 is decreasing.

Corollary 2. Theorem 5 also holds for Archimedean n-quasicopulas with $n > 2$.

A result close to $(\mathrm{B}_1)\Rightarrow(\mathrm{B}_2)$ for Archimedean n-quasicopulas was obtained by Nelson et al. [23, Theorem 3.1(vi)].

In Theorem 5 we do not assume the symmetry of the n-copula C, although Archimedean copulas can only be found among symmetric copulas. The symmetry of C is guaranteed by condition (10). The observation that $\tilde{C}$ in the proof of Theorem 5 only depends on the x variables suggests that an Archimedean n-copula is uniquely determined by its values on some one-dimensional subset of $\mathbb{I}^n$. This is indeed the case, as will be shown in Section 4.

The examples below illustrate the usefulness of Theorem 5.

Example 2. We will show that the n-copula defined by

$$C(\mathbf{u}) = \frac{2^n \prod_{k=1}^{n} u_k}{2\prod_{k=1}^{n}(u_k+1) - 2^n \prod_{k=1}^{n} u_k} \quad \text{for all } \mathbf{u} \in \mathbb{I}^n$$

is Archimedean. The corresponding formula (10) from Theorem 5 is

$$\begin{aligned}\psi(u_{k_1}) \frac{1}{u_{k_2}(u_{k_2}+1)} & \frac{2^{n-1}\prod_{k=1}^{n} u_k(u_k+1)}{\left[\prod_{k=1}^{n}(u_k+1) - 2^{n-1}\prod_{k=1}^{n} u_k\right]^2} \\ &= \psi(u_{k_2}) \frac{1}{u_{k_1}(u_{k_1}+1)} \frac{2^{n-1}\prod_{k=1}^{n} u_k(u_k+1)}{\left[\prod_{k=1}^{n}(u_k+1) - 2^{n-1}\prod_{k=1}^{n} u_k\right]^2}.\end{aligned}$$

Consequently, the function ψ has the form $\psi = \frac{1}{u(u+1)}$. The corresponding additive generator (11) is $g(u) = -\ln\bigl(\frac{2u}{u+1}\bigr)$. The 2-copula corresponding to this generator is a member of the family of Ali-Mikhail-Haq copulas for $\theta = 0.5$ (see Nelsen [22, Chapter 4]). Invoking (6) and Theorem 1, it is easy to check that the function $\tilde{f}(u) = \frac{u}{2-u}$ (inverse to $f(u) = \frac{2u}{u+1}$) is the multiplicative generator. Hence the n-copula $\tilde{C}$ corresponding to $\tilde{g} = -\ln \tilde{f}$ is a member of the one-parameter family of Clayton copulas for $\theta = 1$ (see Nelsen [22, Chapter 4]).

We will now illustrate the application of Theorem 5 and Corollary 2 to Archimedean n-quasicopulas.

Example 3.
• For the Archimedean n-quasicopula of Example 1, the function ψ of Theorem 5 is constant, and the corresponding additive generator (11) indeed equals $g(u) = 1 - u$.
• It is easy to check that for every $n > 2$ the function

$$Q(\mathbf{u}) = \begin{cases} \dfrac{1 - \bigl[\sum_{k=1}^{n} \bigl(\frac{1-u_k}{1+u_k}\bigr)^2\bigr]^{1/2}}{1 + \bigl[\sum_{k=1}^{n} \bigl(\frac{1-u_k}{1+u_k}\bigr)^2\bigr]^{1/2}} & \text{for } \mathbf{u} \in \mathrm{H}, \\ 0 & \text{for } \mathbf{u} \in \mathbb{I}^n \setminus \mathrm{H}, \end{cases}$$

where

$$\mathrm{H} = \left\{ \mathbf{u} \in \mathbb{I}^n : \sum_{k=1}^{n} \left(\frac{1-u_k}{1+u_k} \right)^2 < 1 \right\},$$

is an n-quasicopula. Using Theorem 5, Corollary 2 and simple calculations, we find the additive generator of Q, equal to $g(u) = \bigl(\frac{1-u}{1+u}\bigr)^2$.

4. DIAGONAL GENERATORS

In this section we introduce two classes of generators: the diagonal generators and the defective diagonal generators. They are "special" multiplicative generators of Archimedean n-copulas. To a diagonal generator f corresponds the Archimedean n-copula C^f whose values on the main diagonal of $\mathbb{I}^n$ coincide with f, $C^f(u, \dots, u) = f(u)$. The copula C^f has an asymptotic representation in terms of the k-fold composition of f (or the inverse function to f).

We will give a characterization of Archimedean n-copulas in terms of diagonal generators.

For a fixed $\beta \in (1, n]$, we introduce the family $\mathscr{F}_\beta^n$ of all functions $f : \mathbb{I} \to \mathbb{I}$ satisfying:

(a) f is a strictly increasing and continuous bijection of $\mathbb{I}$ onto itself (then $f(0) = 0$ and $f(1) = 1$),

(b) $f(u) < u$ for all $u \in (0, 1)$,

(c) f is n-absolutely monotone on $(0, 1]$,

(d) $\lim_{u \to 1_-} \frac{\mathrm{d}f(u)}{\mathrm{d}u} = \beta$.

The k-fold composition of $f \in \mathscr{F}^n_\beta$ (resp. of the inverse function of f) is denoted by f^k (resp. f^{-k}).

In Theorem 6 below we will exhibit some interesting asymptotic properties of two function sequences, $g_k : \mathbb{I} \to [0, \beta^k]$ and $g_k^{\leftarrow} : [0, \beta^k] \to \mathbb{I}$, defined for $k = 1, 2, \ldots$ by

$$g_k(u) = \beta^k[1 - f^{-k}(u)] \quad \text{for } u \in \mathbb{I}, \tag{14a}$$

$$g_k^{\leftarrow}(x) = f^k\left(1 - \frac{x}{\beta^k}\right) \quad \text{for } x \in [0, \beta^k]. \tag{14b}$$

Remark 4. The functions (14a) and (14b) are mutually inverse.

In the proof of Theorem 6 we will make use of an important fact from the theory of functional equations (Kuczma [19, Theorem 6.1]), which we recall in Lemma 5 below in a modified form. Here $\mathbb{J}$ denotes any interval in $\mathbb{R}$ containing zero.

Lemma 5. Let $\breve{f} : \mathbb{J} \to \mathbb{R}$ be a C^n function ($n \geq 2$) such that:

- $u(\breve{f}(u) - u) < 0$ for all $u \in \mathbb{J} \setminus \{0\}$,
- $u\breve{f}(u) > 0$ for all $u \in \mathbb{J} \setminus \{0\}$,
- $\frac{\mathrm{d}\breve{f}}{\mathrm{d}u}(0) = s$ for some $s \in (0, 1)$.

Then for every $a \in \mathbb{R}$ there exists a unique C^n function $E : \mathbb{J} \to \mathbb{R}$ such that $a = \frac{\mathrm{d}E}{\mathrm{d}u}(0)$ and E solves the Schröder equation $E(\breve{f}(u)) = sE(u)$. The function E is given by

$$E(u) = a \lim_{k\to\infty} \frac{1}{s^k}\breve{f}^k(u). \tag{15}$$

If $a > 0$ (resp. $a < 0$), then E is strictly increasing (resp. strictly decreasing).

Before formulating and proving the announced results, we extend the functions $g_k^{\leftarrow}$ of (14b) by setting it equal to zero on the whole half-line $[0, \infty]$.

Theorem 6. Let $f \in \bigcup_{\beta\in(1,n]} \mathscr{F}^n_\beta$. Then the function sequence (14a) converges almost uniformly on $(0, 1]$ to a function $g \in \overset{*}{\mathscr{G}}{}^-_n$, while the sequence (14b) converges uniformly on $[0, \infty]$ to the inverse of g.

P r o o f. Let $f \in \mathscr{F}^n_\beta$. Define $\tilde{f}(u) = 1-f(1-u)$ and $\sigma = \frac{1}{\beta}x$. Then (14b) can be written as $g_k^{\leftarrow}(x) = 1-[\tilde{f}^k \circ \sigma^k](x)$ for $x \in [0, \beta^k]$. Suppose that there exists a C^n diffeomorphism $D : [0, 1) \to \mathbb{R}_+$ with $D(0) = 0$ and $\frac{\mathrm{d}D}{\mathrm{d}u}(0) = 1$ satisfying $\tilde{f} = D^{-1} \circ \sigma^{-1} \circ D$. Then $\tilde{f}^k \circ \sigma^k = D^{-1} \circ \sigma^{-k} \circ D \circ \sigma^k$. Note that the composition $\sigma^{-k} \circ D \circ \sigma^k$ maps $x \in [0, \beta^k)$ to $\beta^k D(x/\beta_k)$. Hence from the expansion $D(y) = y + O(y^2)$ it follows that

$$[\sigma^{-k} \circ D \circ \sigma^k](x) = \beta^k\left[\frac{x}{\beta^k} + O\left(\left(\frac{x}{\beta^k}\right)^2\right)\right] = x + O\left(\frac{x^2}{\beta^k}\right).$$

Thus

$$[\tilde{f}^k \circ \sigma^k](x) = D^{-1}\left[x + O\left(\frac{x^2}{\beta^k}\right)\right].$$

Passing to the limit we obtain $\lim_{k\to\infty}[\tilde{f}^k \circ \sigma^k](x) = D^{-1}(x)$, which implies the uniform convergence

$$g_k^{\leftarrow}(x) \to 1 - D^{-1}(x) = g^{\leftarrow}(x). \tag{16}$$

The function $g^{\leftarrow}$ is C^n.

We will show that the diffeomorphism D indeed exists. It is easy to check that the function $\breve{f} = \tilde{f}^{-1}$ (with $\mathbb{J} = [0,1)$) satisfies the assumptions of Lemma 5. Clearly, $\frac{\mathrm{d}\breve{f}}{\mathrm{d}u}(0) = 1/\beta$.

For $a = 1$ Schröder's equation has the form

$$E[\breve{f}(u)] = \frac{1}{\beta}E(u), \quad \frac{\mathrm{d}E}{\mathrm{d}u}(0) = 1, \quad E(0) = 0.$$

The equation $E[\breve{f}(u)] = \frac{1}{\beta}E(u)$ is equivalent to $\breve{f} = E^{-1} \circ \sigma \circ E$. By Lemma 5, E is a C^n diffeomorphism conjugating $\breve{f}$ and σ.

We will check that E is the desired diffeomorphism D. We have

$$\tilde{f} = [E^{-1} \circ \sigma \circ E]^{-1} = E^{-1} \circ \sigma^{-1} \circ E = D^{-1} \circ \sigma \circ D.$$

The function E is strictly increasing, and hence so is E^{-1}. From (15) it can be seen that $\lim_{u\to 1_-} D(u) = \infty$, and so $\lim_{x\to\infty} D^{-1}(x) = 1$.

Now we return to (16). We have $\lim_{x\to\infty} g^{\leftarrow}(x) = 0$. Clearly, $g^{\leftarrow}(0) = 1$ and $\frac{\mathrm{d}g^{\leftarrow}}{\mathrm{d}x}(0) = -1$. Moreover $g^{\leftarrow}$ is strictly decreasing.

It remains to prove the n-monotonicity of $g^{\leftarrow}$. To do so, observe that for $k = 1, 2, \ldots$, $g_k^{\leftarrow}$ is n-monotone on $[0, \beta^k)$, and these sets tend to $[0, \infty]$. It is known that the limit of an almost uniformly convergent sequence of n-monotone functions is $(n-2)$-monotone. Hence $g^{\leftarrow}$ is $(n-2)$-monotone. Moreover, by Theorem 2, the function $(-1)^{n-2}d^{n-2}g_k^{\leftarrow}/dx^{n-2}$ is also decreasing and convex. Therefore, since $g^{\leftarrow}$ is C^n, it follows that it is n-monotone and $d^n g^{\leftarrow}/dx^n$ is continuous. The almost uniform convergence (14a) follows from (14b) and Remark 4. □

We will give an application of Theorem 6 to Archimedean n-copulas. The function $\delta_C : \mathbb{I} \to \mathbb{I}$ given by

$$\delta_C(u) = C(u, \ldots, u) \quad \text{for all } u \in \mathbb{I} \tag{17a}$$

is called the *diagonal section* of the n-copula C. If C is Archimedean with additive generator g, then

$$\delta_C(u) = g^{\leftarrow}[ng(u)]. \tag{17b}$$

Remark 5. For the case of an n-quasicopula, the same definition (17a) is used.

The notion of diagonal section of a (two-dimensional) copula (or quasicopula) was implicit in the expression (8b). The diagonal section of an Archimedean copula (or quasicopula) with a strict differentiable additive generator satisfying $\lim_{u\to 1_-} \frac{\mathrm{d}g(u)}{\mathrm{d}u} = -1$ has a unique representation (Wysocki [28]; see also Section 5). This result extends to n dimensions.

Let g be the strict additive generator. If $g \in \overset{*}{\mathscr{G}}{}_n^-$, then δ_C is n times differentiable on $(0,1)$ and satisfies conditions (a), (b) and (d) in the definition of the family $\mathscr{F}_\beta^n$ for $\beta = n$. If $g \in \overset{*}{\mathscr{G}}{}_n^0$, we have the analogous fact (for some $\beta \in (1,n)$). Condition (c) in the definition of $\mathscr{F}_\beta^n$ is a kind of "regularity" of δ_C. By Theorem 6, the sequences (14a) and (14b) converge to g and $g^{\leftarrow}$ respectively. However, we do not know whether they generate an Archimedean copula C whose diagonal section is f. This is discussed in Theorem 7 below.

We define a function sequence (C_k) by

$$C_k(\mathbf{u}) = \begin{cases} f^k[f^{-k}(u_1) + f^{-k}(u_2) + \cdots + f^{-k}(u_n) - (n-1)] & \text{for } \mathbf{u} \in \mathrm{H}_k, \\ 0 \quad \text{for } \mathbf{u} \in \mathbb{I}^n \setminus \mathrm{H}_k, \end{cases} \tag{18}$$

where $\mathrm{H}_k = \{\mathbf{u} \in \mathbb{I}^n : \sum_{j=1}^n f^{-k}(u_j) > n-1\}$ for $k = 1, 2, \ldots$. Moreover, set

$$\begin{aligned} &\alpha = \frac{1}{n^k}, \quad B_{\mathbf{u}} = \sum_{j=1}^n g^{\leftarrow}(\alpha g(u_j)) - (n-1), \\ &v_C(u) = \frac{g(u)}{g^{(1)}(u)} \quad \text{for all } u \in (0,1), \end{aligned} \tag{19}$$

where $g^{(1)}$ is an alternative notation for $\mathrm{d}g/\mathrm{d}u$.

Theorem 7. Let C be an n-copula whose diagonal section δ_C is in $\mathscr{F}_n^n$. Then the following conditions are equivalent:

(C_1) C is Archimedean with additive generator whose left derivative at 1 is -1.

(C_2) C is the uniform limit of the sequence (18). More precisely,

$$C(\mathbf{u}) = \begin{cases} \lim_{k\to\infty} \delta_C^k[\delta_C^{-k}(u_1) + \delta_C^{-k}(u_2) + \cdots + \delta_C^{-k}(u_n) - (n-1)] \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } \mathbf{u} \in (0,1]^n, \\ 0 \quad \text{for } \mathbf{u} \in \mathbb{I}^n \setminus (0,1]^n. \end{cases} \tag{20a}$$

Moreover, if both (C_1) and (C_2) are valid, then the additive generator of C is given by

$$g(u) = \lim_{k\to\infty} n^k[1 - \delta_C^{-k}(u)] \quad \text{for all } u \in \mathbb{I}. \tag{20b}$$

P r o o f. The convergence of (18) is well defined, since for each $\mathbf{u} \in (0,1)^n$ there exists a positive integer $K_{\mathbf{u}}$ such that $\mathbf{u} \in \mathrm{H}_k$ for all $k > K_{\mathbf{u}}$.

(C$_1$)$\Rightarrow$(C$_2$). Let g be the additive generator of C. By the uniqueness of representation of the diagonal section δ_C, the terms of the sequence (18) have the form

$$C_k(\mathbf{u}) = g^{\leftarrow}\left\{\frac{g[g^{\leftarrow}(\alpha g(u_1)) + g^{\leftarrow}(\alpha g(u_2)) + \cdots + g^{\leftarrow}(\alpha g(u_n)) - (n-1)]}{\alpha}\right\}. \tag{20c}$$

Here α is viewed as a "continuous" variable from $(0, 1/n]$. We want to let $\alpha \to 0_+$ in (20c). To apply the l'Hospital rule, we have to prove that g is C^1. Indeed, by Theorem 6 the sequence (14a) for $f = \delta_C$ converges almost uniformly to the additive generator $g \in \overset{*}{\mathscr{G}}{}_n^-$. Applying the l'Hospital rule to (20c), we obtain

$$\begin{aligned}\lim_{k\to\infty} &C_k(\mathbf{u}) \\ &= g^{\leftarrow}\left\{\lim_{\alpha\to 0_+}\frac{g[g^{\leftarrow}(\alpha g(u_1)) + g^{\leftarrow}(\alpha g(u_2)) + \cdots + g^{\leftarrow}(\alpha g(u_n)) - (n-1)]}{\alpha}\right\} \\ &= g^{\leftarrow}\Big[\sum_{k=1}^{n} g(u_k)\Big].\end{aligned}$$

It follows that C is the pointwise limit of (C_k). We will prove that the convergence is in fact uniform. To apply the Dini theorem, we need to prove that the sequence (C_k) for $f = \delta_C$ is monotone on $\mathbb{I}^n$. We will check that this sequence is indeed increasing. Differentiating the right hand side of (20c) with respect to α for fixed $\mathbf{u} \in (0,1)^n$ yields

$$\frac{\mathrm{d}g^{\leftarrow}}{\mathrm{d}x}\left[\frac{g(B_{\mathbf{u}})}{\alpha}\right]\frac{\mathrm{d}g}{\mathrm{d}u}(B_{\mathbf{u}})\left\{\frac{\alpha\sum\limits_{j=1}^{n}\dfrac{g(u_j)}{g^{(1)}[\delta_C^{-k}(u_j)]} - v_C(B_{\mathbf{u}})}{\alpha^2}\right\}. \tag{20d}$$

Note that

$$\sum_{j=1}^{n}\frac{g(u_j)}{g^{(1)}[\delta_C^{-k}(u_j)]} \to -\sum_{j=1}^{n} g(u_j), \qquad \frac{1}{n^k} \to 0$$

as $k \to \infty$. Hence for sufficiently large k we have

$$\alpha\sum_{j=1}^{n}\frac{g(u_j)}{g^{(1)}[\delta_C^{-k}(u_j)]} - v_C(B_{\mathbf{u}}) > 0,$$

so (20d) is also positive, completing the proof of (C$_1$)$\Rightarrow$(C$_2$).

(C$_2$)$\Rightarrow$(C$_1$). The function (18) for $f = \delta_C$ is the composition $g^{\leftarrow} \circ \psi_k$, where $\psi_k(\mathbf{u}) = \sum_{j=1}^{n} g_k(u_j)$, and the functions g_k, $g_k^{\leftarrow}$ are given respectively by (14a) and (14b) for $\beta = n$. Theorem 6 yields the convergences $g_k^{\leftarrow} \to g^{\leftarrow}$ and $\psi_k(\mathbf{u}) \to \psi(\mathbf{u}) = \sum_{j=1}^{n} g(u_j)$. Clearly, $g^{\leftarrow}$ is the inverse function of the generator $g \in \overset{*}{\mathscr{G}}{}_n^-$. The sequence (C_k) converges uniformly to $[g^{\leftarrow} \circ \psi](\mathbf{u}) = g^{\leftarrow}[\sum_{j=1}^{n} g(u_j)]$. $\square$

Theorem 7 characterizes the Archimedean n-copulas whose diagonal sections belong to $\mathscr{F}_n^n$. A result similar to the implication (C$_1$)$\Rightarrow$(C$_2$) for $n = 2$ was given by Sungur and Yang [26]. For Archimedean n-quasicopulas we have the following analogue of Theorem 7:

Theorem 8. Let C be an n-quasicopula whose diagonal section δ_C is in $\mathscr{F}_n^2$. Then the following conditions are equivalent:

(C_1') C is Archimedean with additive generator whose left derivative at 1 is -1.

(C_2') C has a representation (20a).

Moreover, if both (C_1') and (C_2') are valid, then the additive generator of C is given by (20b).

P r o o f. Analogous to the proof of Theorem 7. □

For functions $f \in \bigcup_{\beta\in(1,n)} \mathscr{F}_\beta^n$ the conclusion of Theorem 7 does not hold. It is easy to prove that in this case we have $g_k \to g \equiv \infty$ on $[0,1)$, $g(1) = 0$ and $g_k^{\leftarrow} \to g^{\leftarrow} \equiv 1$. However, the following turns out to be true:

Theorem 9. If $f \in \bigcup_{\beta\in(1,n)} \mathscr{F}_\beta^n$, then the function $C : \mathbb{I}^n \to \mathbb{I}$ given by (20a) is an Archimedean n-copula with strict additive generator $g \in \overset{*}{\mathscr{G}}{}_n^-$ given by $g(u) = \lim_{k\to\infty} \beta^k[1 - f^{-k}(u)]$.

P r o o f. Just modify the proof of (C_2)⇒(C_1) in Theorem 7. □

Theorems 7 and 9 suggest introducing some terminology.

The elements of $\mathscr{F}_n^n$ (respectively $\bigcup_{\beta\in(1,n)} \mathscr{F}_\beta^n$) will be called *diagonal generators* (respectively *defective diagonal generators*) of Archimedean n-copulas. To a diagonal generator f corresponds an Archimedean n-copula C_f with strict additive generator from $\overset{*}{\mathscr{F}}{}_n^-$. Its diagonal section δ_{C_f} equals f. This means that the function δ_{C_f} "represents our knowledge" about C_f. For a defective diagonal generator f, the situation is different. Although we have the corresponding Archimedean n-copula C_f with strict additive generator from $\overset{*}{\mathscr{G}}{}_n^-$, the diagonal section δ_{C_f} is not f. This justifies the term "defective". Diagonal generators of two-dimensional Archimedean copulas were introduced by Wysocki [28].

Example 4. The asymptotic representation (20a) of the copula corresponding to the defective diagonal generator $f(u) = u^{\sqrt{2}}$ is

$$C_f(u_1, u_2) = \lim_{k\to\infty} [u_1^{(\sqrt{2})^{-k}} + u_2^{(\sqrt{2})^{-k}} - 1]^{(\sqrt{2})^k} = u_1 u_2.$$

The function f is the diagonal section of the two-dimensional copula

$$C(u_1, u_2) = \exp\{-[\ln^2 u_1 + \ln^2 u_2]^{1/2}\}.$$

Remark 6. A diagonal generator (or a defective diagonal generator) is also a multiplicative generator. The Archimedean n-copulas corresponding to a diagonal generator and a defective diagonal generator are different. The only exception is the function $f(u) = u^n$, generating the independent n-copula $C^\perp(\mathbf{u}) = \prod_{k=1}^n u_k$.

To end this section, we give a method of constructing families of diagonal generators (or defective diagonal generators). Let $\check{\mathscr{F}}^n_\beta$ be the family of all functions $f \in \mathscr{F}$ with a representation $f(u) = \sum_{k=1}^{\infty} a_k u^k$, where the sequence (a_k) satisfies

$$a_k \geq 0, \quad \sum_{k=1}^{\infty} a_k = 1 \quad \text{and} \quad \sum_{k=1}^{\infty} k a_k = \beta \tag{21}$$

for some $\beta \in (1, n)$. The analogous family of functions $f \in \mathscr{F}$ satisfying (21) with $\beta = n$ will be denoted by $\check{\mathscr{F}}^n_n$. By the Main Theorem on absolutely monotone functions (Feller [7]), elements of both families are absolutely monotone. We have the inclusions $\check{\mathscr{F}}^n_\beta \subset \mathscr{F}^n_\beta$ and $\check{\mathscr{F}}^n_\beta \subset \mathscr{F}^n_n$.

Example 5. The functions

$$f_1(u) = \frac{u}{n-(n-1)u} = \frac{1}{n-1} \sum_{k=1}^{\infty} \left[\frac{(n-1)u}{n}\right]^k \quad \text{for } n \geq 2$$

and

$$f_2(u) = (n-1)\frac{u}{n-u} = (n-1) \sum_{k=1}^{\infty} \left(\frac{u}{n}\right)^k \quad \text{for } n \geq 3$$

belong respectively to $\check{\mathscr{F}}^n_n$ and $\check{\mathscr{F}}^n_{n/(n-1)}$.

It turns out that by using diagonal generators (or defective diagonal generators) one can also construct non-Archimedean copulas. Let $\delta : \mathbb{I} \to \mathbb{I}$ satisfy the following conditions:

(1) $\delta(0) = 0$, $\delta(1) = 1$,

(2) $\delta(u) \leq u$ for all $u \in \mathbb{I}$,

(3) $0 \leq \delta(v) - \delta(u) \leq n(v-u)$ for all $u, v \in \mathbb{I}$ with $u \leq v$.

The functions δ have all the properties of diagonal sections of n-copulas. Therefore, we call them *diagonal functions.* Set

$$\tilde{\delta} = \frac{nu - \delta(u)}{n-1} \quad \text{and} \quad \tau^i(k) = k + i \bmod n \quad \text{for } i, k \in \{1, \ldots, n\}.$$

Jaworski [15] proved that the function $C(\delta; \mathbf{u}) : \mathbb{I}^n \to \mathbb{I}$ corresponding to the diagonal function δ and given by

$$C(\delta; \mathbf{u}) = \frac{1}{n} \sum_{i=1}^{n} \min[\tilde{\delta}(u_{\tau^i(1)}), \ldots, \tilde{\delta}(u_{\tau^i(n)})]$$

is an n-copula with diagonal section δ. He called such copulas *diagonal.* It is easy to check that every diagonal generator (or defective diagonal generator) satisfies (1)–(3). Hence, apart from the Archimedean copulas C^f and C_f, the generator f also generates the copula $C(f; \mathbf{u})$.

5. CONCLUDING REMARKS

In this section we briefly discuss the uniqueness of representation of diagonal sections of Archimedean n-copulas, and we shed some light on the algebraic nature of such copulas.

Let $g_1, g_2 \in \overset{*}{\mathscr{G}}{}_1^-$ be strict additive generators of Archimedean n-copulas C_1 and C_2, respectively. Suppose that $\delta_{C_1} = \delta_{C_2} = f$, but $g_1 \neq g_2$. Differentiating the equality $g_1^{\leftarrow}(ng_1(u)) = g_2^{\leftarrow}(ng_2(u))$ and setting $\varphi(u) = g_1^{(1)}(u)/g_2^{(1)}(u)$, we obtain

$$\varphi(u) = \varphi(f(u)) \quad \text{for all } u \in (0,1]. \tag{22}$$

Fix $x \in (0,1)$. Setting in (22) consecutively $u = f^{-k}(x)$ for $k = 1, 2, \ldots$, we obtain $\varphi(f^{-k}(x)) = \varphi(x)$, and hence $\lim_{k\to\infty} \varphi(f^{-k}(x)) = \varphi(x) = \varphi(1) = 1$. Thus, by the definition of additive generator, $g_1 = g_2$, contrary to assumption. We have used the obvious pointwise convergence $f^{-k}(x) = g^{\leftarrow}\big(\frac{1}{n^k} g(x)\big) \to 1$ on $(0,1]$. For $g \in \overset{*}{\mathscr{G}}{}_1^0$ the situation is completely different.

Example 6. There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in [0, \varepsilon_0)$ the function $g_\varepsilon : \mathbb{I} \to [0, \infty]$ given by

$$g_\varepsilon(u) = \tilde{g}(u)\big(1 + \varepsilon \sin[K \ln(-\ln u)]\big), \quad K = 4\pi/\ln 2,$$

is a strict additive generator from the family $\overset{*}{\mathscr{G}}{}_1^0$; it is a sine-shaped perturbation of the generator $\tilde{g}(u) = (-\ln u)^2$ of the Gumbel 2-copula. It can be shown that the copulas C_ε corresponding to g_ε have the same diagonal section. This example was communicated to the author by the referee.

It is easy to check that every Archimedean n-copula C with additive generator g induces a one-parameter group (or semigroup) of operators $\mathscr{F}_C = \{f_t : t \in \mathbb{T}\}$ where $f_t(u) = g^{\leftarrow}[e^t g(u)]$ for all $u \in \mathbb{I}$ and $t \in \mathbb{T}$,where $\mathbb{T} = \mathbb{R}$ (resp. $\mathbb{T} = \mathbb{R}_+$). In the group (resp. semigroup) case the function (19) is $\lim_{t\to 0} \frac{\mathrm{d}f_t(u)}{\mathrm{d}t}$ (resp. $\lim_{t\to 0_+} \frac{\mathrm{d}f_t(u)}{\mathrm{d}t}$). It turns out that the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v(y)}{x}$$

(with an initial condition) for a "regular" function $v : \mathbb{I} \to [-1, 0]$ which has the properties of (19) is the n-monotone pseudoinverse $u = g^{\leftarrow}$ of the additive generator of an Archimedean n-copula C. For $n = 2$ this was investigated in Wysocki [29].

ACKNOWLEDGMENTS

The author would like to thank the referees for their valuable remarks and suggestions, which improved the paper considerably.

(Received May 6, 2013)

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