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A SEPARATION PRINCIPLE FOR THE STABILIZATION OF A CLASS OF TIME DELAY NONLINEAR SYSTEMS

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In this paper, we establish a separation principle for a class of time-delay nonlinear systems satisfying some relaxed triangular-type condition. Under delay independent conditions, we propose a nonlinear time-delay observer to estimate the system states, a state feedback controller and we prove that the observer-based controller stabilizes the system.

Keywords: delay system, output feedback stabilization, nonlinear observer, separation principle

Classification: 93C10, 93D15, 93D20

1. INTRODUCTION

Much attention has been paid to the theory of output feedback stabilization of nonlinear systems using high gain observers. In this regard, Atassi and Khalil [1, 2] proved a separation principle for nonlinear systems using high gain observers. [21] and [18] used a linear high gain observer to achieve global stabilization by output feedback for a class of nonlinear systems that are dominated by a triangular system satisfying a linear growth condition. In [4], a new condition is used to achieve global stabilization of nonlinear systems by a linear output feedback.

However, in various engineering systems, time delay is frequently a source of instability. Therefore, output feedback stabilization of systems with delays has been the subject of numerous papers and monographs, see e. g. [10, 11, 12, 14, 19, 20, 23] and references therein. The effective tools of the design of the output feedback controllers are the Lyapunov–Krasovskii functional approach and the LMI based design method [3]. In [15], based on linear matrix inequalities, authors developed a delay dependent method to design a linear dynamic output feedback controller which ensure global asymptotic stability for any time delay not larger than a given bound. Under a growth linear condition, [19] has proposed an observer based output feedback control such that the feedback controlled system is globally asymptotically stable and the estimated state trajectories asymptotically track the true state trajectories of the feedback controlled delay system. In [23], by constructing an appropriate Krasovskii functional and solving linear matrix inequalities, a delay dependent output feedback controller is proposed to make

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the closed loop system globally asymptotically stable. Based on the Lyapunov method, [14] derived a linear matrix inequality criterion to design an observer-based controller to stabilize the delay system. In [12], a recursive method to design output feedback controllers for block feed-forward linear systems with delays in the inputs, outputs and interconnections between the blocks is developed. [11], investigated the problem of observer based stabilization of a class of time delay nonlinear systems written in triangular form. By choosing an appropriate Lyapunov–Krasovskii functional, it is shown that a high gain parameterized linear controller achieves the global asymptotic stability. Using the input/output exact linearization method developed in [5] and [7], Germani et al. investigated the problem of observer-based control for a class of retarded systems in [8, 9].

In this paper, motivated by [4] and [11] and by constructing Lyapunov–Krasovskii functionals, we investigate the output feedback controller problem for a class of nonlinear delay systems. More specifically, we first design a nonlinear observer to estimate the system states. Then, we propose a state feedback controller to stabilize the origin of the system. Finally, we prove a separation principle, that is, the designed state feedback control law remains valid when the control law is implemented with the estimate states. In section 2, some preliminary results are summarized and the system description is given. The required assumptions and the statement of the main results are provided in section 3. In section 4, an example of application of the result is given.

2. PRELIMINARY AND PROBLEM FORMULATION

Consider a differential delay equation

$$\dot{x}(t) \quad = \quad f(x(t), x(t-\tau)), \tag{1}$$

where $\tau > 0$ is the delay time. The knowledge of x at time $t = 0$ does not allow to deduce x at time t. Thus, the initial condition is specified as a continuous function $\varphi : [-\tau, 0] \rightarrow \mathbb{R}^n$. The state of equation (1) at time t can be described as a function segment x_t defined by

$$x_t(\theta) = x(t+\theta), \ \theta \in [-\tau, 0].$$

Therefore, delay equations form a special class of functional differential equations [10]:

$$\dot{x} = F(x_t), \tag{2}$$

where $F : \mathcal{C} \rightarrow \mathbb{R}^n$; $\mathcal{C}$ denotes the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ equipped with the supremum-norm:

$$\forall \varphi \in \mathcal{C}, \quad \|\varphi\|_\infty = \max_{\theta \in [-\tau, 0]} \|\varphi(\theta)\|,$$

where $\| \ \|$ is the Euclidean norm. As system (1) is a special case of (2), we consider now system (2). We will recall the definition of asymptotic stability of the origin of system (2) and we will give a sufficient condition that assures its asymptotic stability (see [10, 17]).

Assume that F is Lipschitz on bounded sets and satisfies $F(0) = 0$. For $\varphi \in \mathcal{C}$, we denote by $x(t, \varphi)$ or shortly $x(t)$ the solution of (2) that satisfies $x_0 = \varphi$. The segment of this solution is denoted by $x_t(\varphi)$ or shortly x_t.

Definition 2.1. The zero solution of (2) is called

(i) stable, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|\varphi\|_\infty < \delta \Longrightarrow \|x(t)\| < \varepsilon,\ \forall\, t \geq 0,$$

(ii) attractive, if there exists $\sigma > 0$ such that

$$\|\varphi\|_\infty < \sigma \Longrightarrow \lim_{t\to+\infty} x(t) = 0, \tag{3}$$

(iii) asymptotically stable, if it is stable and attractive,

(iv) globally asymptotically stable, if it is stable and δ can be chosen arbitrarily large for sufficiently large ε, and (3) is satisfied for all $\sigma > 0$.

Sufficient conditions for stability of a functional differential equation are provided by the theory of Lyapunov-Krasovskii functionals [10], a generalization of the classical Lyapunov theory of ordinary differential equations [13]. For a locally Lipschitz functional $V : \mathcal{C} \to \mathbb{R}_+$, the derivative of V along the solutions of (2) is defined as

$$\dot{V} = \lim_{h\to 0^+} \frac{1}{h}(V(x_{t+h}) - V(x_t)).$$

Let us recall here that a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if it is continuous, increasing and $\alpha(0) = 0$, of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and it is unbounded. The following theorem provides sufficient Lyapunov–Krasovskii conditions for global asymptotic stability of the zero solution of system (2) (see [17]).

Theorem 2.2. Assume that there exist a locally Lipschitz functional $V : \mathcal{C} \to \mathbb{R}_+$, functions α_1, α_2 of class $\mathcal{K}_\infty$, a function α_3 of class $\mathcal{K}$, such that:

(i) $\alpha_1(\|x(t)\|) \leq V(x_t) \leq \alpha_2(\|x_t\|_\infty)$,

(ii) $\dot{V}(x_t) \leq -\alpha_3(\|x(t)\|)$,

then the zero solution of system (2) is globally asymptotically stable.

In this paper, we consider the class of nonlinear time delay systems described by the following equations:

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) + f(x(t), x(t-\tau)) \\ y(t) & = & Cx(t) \end{array} \tag{4}$$

with the state initial condition $x(t) = \varphi(t),\ t \in [-\tau, 0]$, where $\varphi : [-\tau, 0] \to \mathbb{R}^n$ is a continuous function, $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the input of the system and $y \in \mathbb{R}$ is the measured output, the matrices A, B and C are given by

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & & & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix},$$

and

$$f(x(t), x(t-\tau)) = [f_1(x(t), x(t-\tau)), \ldots, f_n(x(t), x(t-\tau))]^T.$$

We assume that the mappings $f_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $i=1,\ldots,n$, are smooth with $f_i(0,0)=0$, that is, the origin is a solution of system (4).

The main contribution of this paper is the design of a nonlinear observer based controller that stabilizes system (4) to the origin for all initial conditions.

Throughout the paper and for the sake of simplicity the time argument is omitted and the delayed state vector $x(t-\tau)$ is denoted by x^τ.

3. MAIN RESULTS

We suppose that f satisfies the following assumption:

Assumption 1. There exists functions $\gamma_1(\varepsilon) \geq 0$ and $\gamma_2(\varepsilon) \geq 0$ such that for $\varepsilon > 0$,

$$\sum_{i=1}^{n} \varepsilon^{i-1} |f_i(x, \overline{x}) - f_i(y, \overline{y})| \leq \gamma_1(\varepsilon) \sum_{i=1}^{n} \varepsilon^{i-1} |x_i - y_i| + \gamma_2(\varepsilon) \sum_{i=1}^{n} \varepsilon^{i-1} |\overline{x}_i - \overline{y}_i|.$$

We suppose also that,

Assumption 2. For $t \geq 0$, the delay τ is known and constant.

Remark 3.1. Note that if the system (4) has a triangular structure (see [11]), that is each f_i depends only on $(x_1, \ldots, x_i, \overline{x}_1, \ldots, \overline{x}_i)$, and if we suppose that f_i is globally Lipschitz, which implies that there exists $k > 0$ such that,

$$|f_i(x, \overline{x}) - f_i(y, \overline{y})| \leq k \sum_{j=1}^{i} (|x_j - y_j| + |\overline{x}_j - \overline{y}_j|), \tag{5}$$

then Assumption 1 is fulfilled. Indeed,

$$\sum_{i=1}^{n} \varepsilon^{i-1} |f_i(x, \overline{x}) - f_i(y, \overline{y})| \leq \sum_{i=1}^{n} \varepsilon^{i-1} k \left(\sum_{j=1}^{i} (|x_j - y_j| + |\overline{x}_j - \overline{y}_j|) \right).$$

But we have

$$\begin{aligned}
\sum_{i=1}^{n} \varepsilon^{i-1} k \sum_{j=1}^{i} |x_j - y_j| &= k(1 + \varepsilon + \cdots + \varepsilon^{n-1})|x_1 - y_1| \\
&\quad + k(\varepsilon + \cdots + \varepsilon^{n-1})|x_2 - y_2| + \cdots + k\varepsilon^{n-1}|x_n - y_n| \\
&= k(1 + \varepsilon + \cdots + \varepsilon^{n-1})|x_1 - y_1| \\
&\quad + k\varepsilon(1 + \cdots + \varepsilon^{n-2})|x_2 - y_2| + \cdots + k\varepsilon^{n-1}|x_n - y_n| \\
&\leq k(1 + \varepsilon + \cdots + \varepsilon^{n-1}) \sum_{i=1}^{n} \varepsilon^{i-1} |x_i - y_i|
\end{aligned}$$

and thus also

$$\sum_{i=1}^{n}\varepsilon^{i-1}k\sum_{j=1}^{i}|\overline{x}_j-\overline{y}_j|\leq k(1+\varepsilon+\cdots+\varepsilon^{n-1})\sum_{i=1}^{n}\varepsilon^{i-1}|\overline{x}_i-\overline{y}_i|.$$

So Assumption 1 is satisfied with $\gamma_1(\varepsilon)=\gamma_2(\varepsilon)=k(1+\varepsilon+\cdots+\varepsilon^{n-1})$.

Remark 3.2. A system in the form (4) which satisfies Assumption 1 do not admit a time-delay matched with the control input [8, 9].

The output feedback we propose is made of a nonlinear observer and a linear controller.

3.1. Observer design

The observer design problem is studied for instance in [6, 11, 16, 22, 24] and references therein. [6] investigated the problem of state reconstruction from input and output measurements for nonlinear time delay systems and proposed a state observer. In [16], a generalized notion of linearization via output injections is studied and an observer is designed for nonlinear time delay systems. A quadratic matrix inequality approach and a linear matrix inequality approach are developed in [22] to solve the observer design problem. In [24], it is shown that the observer design problem can be formulated as a linear matrix inequality feasibility problem. [11] has built an observer for a triangular structure of system (4). In this subsection we will design an observer for system (4) under Assumption 1. We propose the following system:

$$\dot{\hat{x}}(t)=A\hat{x}+Bu(t)+f(\hat{x},\hat{x}^{\tau})+L(\varepsilon)(C\hat{x}-y),\tag{6}$$

where $L(\varepsilon)=[l_1/\varepsilon,\ldots,l_n/\varepsilon^n]^T$, with $\varepsilon>0$ and where $L=[l_1,\ldots,l_n]^T$ is selected such that $A_L:=A+LC$ is hurwitz, $\hat{x}(s)=\hat{\phi}(s),\ -\tau\leq s\leq 0$ with $\hat{\phi}:[-\tau,0]\rightarrow\mathbb{R}^n$ being any known continuous function. Let P be the symmetric positive definite solution of the Lyapunov equation

$$A_L^TP+PA_L=-I.\tag{7}$$

Theorem 3.3 gives a suitable delay-independent condition under which system (6) estimates the states of (4).

Theorem 3.3. Consider the time-delay system (4) under Assumptions 1-2. Suppose that there exists $\varepsilon>0$ such that

$$\frac{1}{2\varepsilon}-2n\gamma_1(\varepsilon)\|P\|-n\gamma_2(\varepsilon)\|P\|>0,\tag{8}$$

then (6) is a global asymptotic observer for system (4).

P r o o f. Denote $e=\hat{x}-x$ the observation error. We have

$$\dot{e}=(A+L(\varepsilon)C)e+f(\hat{x},\hat{x}^{\tau})-f(x,x^{\tau}).\tag{9}$$

For $\varepsilon > 0$, let $D(\varepsilon) = \text{diag}[1, \varepsilon, \ldots, \varepsilon^{n-1}]$. We have the following equalities:

$$\varepsilon A = D^{-1}(\varepsilon)AD(\varepsilon),\ B^T D(\varepsilon) = \varepsilon^{n-1}B^T,\ CD(\varepsilon) = C.$$

Now, let $\eta = D(\varepsilon)e$, then we get

$$\dot{\eta} = \frac{1}{\varepsilon}A_L\eta + D(\varepsilon)(f(\hat{x}, \hat{x}^\tau) - f(x, x^\tau)). \tag{10}$$

We set

$$V(\eta_t) = \eta^T P\eta + \frac{1}{2\varepsilon}\int_{t-\tau}^{t} \|\eta(s)\|^2 \,\mathrm{d}s.$$

Since P is symmetric positive definite then for all $\eta \in \mathbb{R}^n$,

$$\lambda_{\min}(P)\|\eta\|^2 \leq \eta^T P\eta \leq \lambda_{\max}(P)\|\eta\|^2,$$

where $\lambda_{\min}(P)$ (resp. $\lambda_{\max}(P)$) denotes the minimum (resp. the maximum) eigenvalue of P. This implies that on the one hand,

$$V(\eta_t) \geq \lambda_{\min}(P)\|\eta(t)\|^2,$$

and on the other hand,

$$\begin{aligned}
V(\eta_t) &= \eta^T P\eta + \frac{1}{2\varepsilon}\int_{-\tau}^{0} \|\eta(\theta + t)\|^2 \,\mathrm{d}\theta \\
&= \eta^T P\eta + \frac{1}{2\varepsilon}\int_{-\tau}^{0} \|\eta_t(\theta)\|^2 \,\mathrm{d}\theta \\
&\leq \lambda_{\max}(P)\|\eta\|^2 + \frac{1}{2\varepsilon}\int_{-\tau}^{0} \|\eta_t\|_\infty^2 \,\mathrm{d}\theta \\
&\leq (\lambda_{\max}(P) + \frac{\tau}{2\varepsilon})\|\eta_t\|_\infty^2.
\end{aligned}$$

Thus, condition (i) of Theorem 2.2 is satisfied.
By differentiation of V, we get

$$\begin{aligned}
\dot{V}(\eta_t) &= 2\eta^T P\dot{\eta} + \frac{1}{2\varepsilon}\|\eta\|^2 - \frac{1}{2\varepsilon}\|\eta^\tau\|^2 \\
&= \frac{2}{\varepsilon}\eta^T PA_L\eta + 2\eta^T PD(\varepsilon)(f(\hat{x}, \hat{x}^\tau) - f(x, x^\tau)) \\
&\quad + \frac{1}{2\varepsilon}\|\eta\|^2 - \frac{1}{2\varepsilon}\|\eta^\tau\|^2 \\
&= -\frac{1}{2\varepsilon}\|\eta\|^2 - \frac{1}{2\varepsilon}\|\eta^\tau\|^2 \\
&\quad + 2\eta' PD(\varepsilon)(f(\hat{x}, \hat{x}^\tau) - f(x, x^\tau)) \\
&\leq -\frac{1}{2\varepsilon}\|\eta\|^2 - \frac{1}{2\varepsilon}\|\eta^\tau\|^2 \\
&\quad + 2\|\eta\|\|P\|\|D(\varepsilon)(f(\hat{x}, \hat{x}^\tau) - f(x, x^\tau))\|.
\end{aligned}$$

Now

$$\|D(\varepsilon)(f(\hat{x},\hat{x}^\tau)-f(x,x^\tau))\| \le \sum_{i=1}^{n}\varepsilon^{i-1}|f_i(\hat{x},\hat{x}^\tau)-f_i(x,x^\tau)|.$$

So using Assumption 1, we get

$$\begin{aligned}\|D(\varepsilon)(f(\hat{x},\hat{x}^\tau)-f(x,x^\tau))\| &\le \gamma_1(\varepsilon)\sum_{i=1}^{n}\varepsilon^{i-1}|\hat{x}_i-x_i| + \gamma_2(\varepsilon)\sum_{i=1}^{n}\varepsilon^{i-1}|\hat{x}_i^\tau - x_i^\tau| \\ &\le n\gamma_1(\varepsilon)\|D(\varepsilon)e\| + n\gamma_2(\varepsilon)\|D(\varepsilon)e^\tau\|.\end{aligned}$$

Thus

$$\|D(\varepsilon)(f(\hat{x},\hat{x}^\tau)-f(x,x^\tau))\| \le n\gamma_1(\varepsilon)\|\eta\| + n\gamma_2(\varepsilon)\|\eta^\tau\|. \tag{11}$$

So

$$\begin{aligned}\dot{V}(\eta_t) &\le -\frac{1}{2\varepsilon}\|\eta\|^2 - \frac{1}{2\varepsilon}\|\eta^\tau\|^2 + 2n\gamma_1(\varepsilon)\|P\|\|\eta\|^2 \\ &\quad +2n\gamma_2(\varepsilon)\|P\|\|\eta\|\|\eta^\tau\|.\end{aligned}$$

Using the fact that

$$2\|\eta\|\|\eta^\tau\| \le \|\eta\|^2 + \|\eta^\tau\|^2,$$

we deduce that

$$\begin{aligned}\dot{V}(\eta_t) &\le -\Big\{\frac{1}{2\varepsilon} - 2n\gamma_1(\varepsilon)\|P\| - n\gamma_2(\varepsilon)\|P\|\Big\}\|\eta\|^2 \\ &\quad -\Big\{\frac{1}{2\varepsilon} - n\gamma_2(\varepsilon)\|P\|\Big\}\|\eta^\tau\|^2.\end{aligned}$$

Let

$$\begin{aligned}a(\varepsilon) &= \frac{1}{2\varepsilon} - 2n\gamma_1(\varepsilon)\|P\| - n\gamma_2(\varepsilon)\|P\|, \\ b(\varepsilon) &= \frac{1}{2\varepsilon} - n\gamma_2(\varepsilon)\|P\|.\end{aligned}$$

Using (8), we have $a(\varepsilon) > 0$ and $b(\varepsilon) > 0$, so

$$\dot{V}(\eta_t) \le -a(\varepsilon)\|\eta\|^2.$$

This implies that condition (ii) of Theorem 2.2 is satisfied.
By applying Theorem 2.2, we conclude that the observation error is globally asymptotically stable. □

3.2. Global stabilization by state feedback

In this subsection, we establish a delay-independent condition for the asymptotic state feedback stabilization of the nonlinear system (4). The state feedback controller is given by

$$u = K(\varepsilon)x, \tag{12}$$

where $K(\varepsilon) = [k_1/\varepsilon^n, \dots, k_n/\varepsilon]$, and $K = [k_1, \dots, k_n]$ is selected such that $A_K := A + BK$ is Hurwitz.

Let S be the symmetric positive definite solution of the Lyapunov equation

$$A_K^T S + S A_K = -I. \tag{13}$$

Theorem 3.4. Suppose that Assumptions $1-2$ are satisfied and there exists $\varepsilon > 0$ such that

$$\frac{1}{2\varepsilon} - 2n\gamma_1(\varepsilon)\|S\| - n\gamma_2(\varepsilon)\|S\| > 0, \tag{14}$$

then the origin of the closed loop time-delay system $(4)-(12)$ is globally asymptotically stable.

Proof. The closed loop system is given by

$$\dot{x} = (A + BK(\varepsilon))x + f(x, x^\tau). \tag{15}$$

Let $\chi = D(\varepsilon)x$. Using the fact that $D(\varepsilon)BK(\varepsilon) = \frac{1}{\varepsilon}BKD(\varepsilon)$ we get

$$\dot{\chi} = \frac{1}{\varepsilon} A_K \chi + D(\varepsilon) f(x, x^\tau).$$

We set

$$W(\chi_t) = \chi^T S \chi + \frac{1}{2\varepsilon} \int_{t-\tau}^{t} \|\chi(s)\|^2 \, \mathrm{d}s.$$

As in the proof of Theorem 3.3, we have,

$$\lambda_{\min}(S)\|\chi(t)\|^2 \le W(\chi_t) \le (\lambda_{\max}(S) + \frac{\tau}{2\varepsilon})\|\chi\|_\infty^2.$$

($\lambda_{\min}(S)$ and $\lambda_{\max}(S)$ are respectively the minimum and the maximum eigenvalues of S). By differentiation of W, we get

$$\begin{aligned}
\dot{W}(\chi_t) &= 2\chi^T S \dot{\chi} + \frac{1}{2\varepsilon}\|\chi\|^2 - \frac{1}{2\varepsilon}\|\chi^\tau\|^2 \\
&= \frac{2}{\varepsilon}\chi^T S A_K \chi + 2\chi^T S D(\varepsilon) f(x, x^\tau) + \frac{1}{2\varepsilon}\|\chi\|^2 - \frac{1}{2\varepsilon}\|\chi^\tau\|^2 \\
&= -\frac{1}{2\varepsilon}\|\chi\|^2 - \frac{1}{2\varepsilon}\|\chi^\tau\|^2 + 2\chi' S D(\varepsilon) f(x, x^\tau) \\
&\le -\frac{1}{2\varepsilon}\|\chi\|^2 - \frac{1}{2\varepsilon}\|\chi^\tau\|^2 + 2\|\chi\|\|S\|\|D(\varepsilon) f(x, x^\tau)\|.
\end{aligned}$$

Since $f(0,0) = 0$, (11) implies that

$$\|D(\varepsilon) f(x, x^\tau)\| \le n\gamma_1(\varepsilon)\|\chi\| + n\gamma_2(\varepsilon)\|\chi^\tau\|.$$

So

$$\begin{aligned}\dot{W}(\chi_t) &\leq -\frac{1}{2\varepsilon}\|\chi\|^2 - \frac{1}{2\varepsilon}\|\chi^\tau\|^2 \\ &+ 2n\gamma_1(\varepsilon)\|S\|\|\chi\|^2 + 2n\gamma_2(\varepsilon)\|S\|\|\chi\|\|\chi^\tau\|.\end{aligned}$$

Using the fact that

$$2\|\chi\|\|\chi^\tau\| \leq \|\chi\|^2 + \|\chi^\tau\|^2,$$

we deduce that

$$\begin{aligned}\dot{W}(\chi_t) &\leq -\Big\{\frac{1}{2\varepsilon} - 2n\gamma_1(\varepsilon)\|S\| - n\gamma_2(\varepsilon)\|S\|\Big\}\|\chi\|^2 \\ &\quad -\Big\{\frac{1}{2\varepsilon} - n\gamma_2(\varepsilon)\|S\|\Big\}\|\chi^\tau\|^2.\end{aligned}$$

Let

$$\begin{aligned}c(\varepsilon) &= \frac{1}{2\varepsilon} - 2n\gamma_1(\varepsilon)\|S\| - n\gamma_2(\varepsilon)\|S\|, \\ d(\varepsilon) &= \frac{1}{2\varepsilon} - n\gamma_2(\varepsilon)\|S\|.\end{aligned}$$

Using (14) we have $c(\varepsilon) > 0$ and $d(\varepsilon) > 0$, which implies that

$$\dot{W}(\chi_t) \leq -c(\varepsilon)\|\chi\|^2.$$

By Theorem 2.2, we conclude that the origin of the closed loop system (15) is globally asymptotically stable. □

3.3. Observer-based control stabilization

In this subsection, we implement the control law with estimate states. The observer-based controller is given by:

$$u = K(\varepsilon)\hat{x}, \tag{16}$$

where $\hat{x}$ is provided by the observer (6).

Theorem 3.5. Suppose that Assumptions 1 − 2 are satisfied, and there exists $\varepsilon > 0$ such that conditions (8) and (14) hold. Then the origin of the closed loop time-delay system (4) − (16) is globally asymptotically stable.

Proof. The closed loop system in the (χ, η) coordinates can be written as follows:

$$\begin{aligned}\dot{\chi} &= \frac{1}{\varepsilon}A_K\chi + \frac{1}{\varepsilon}BK\eta + D(\varepsilon)f(x, x^\tau), \\ \dot{\eta} &= \frac{1}{\varepsilon}A_L\eta + D(\varepsilon)(f(\hat{x}, \hat{x}^\tau) - f(x, x^\tau)).\end{aligned} \tag{17}$$

Let

$$U(\chi_t, \eta_t) = \alpha V(\eta_t) + W(\chi_t). \tag{18}$$

Using the above results, we get

$$\dot{U}(\chi_t, \eta_t) \quad \leq \quad -\alpha a(\varepsilon)\|\eta\|^2 - c(\varepsilon)\|\chi\|^2 + \frac{2}{\varepsilon}\|S\|\|K\|\|\chi\|\|\eta\|.$$

Now using the fact that for all $\theta > 0$,

$$2\|\chi\|\|\eta\| \leq \theta\|\chi\|^2 + \frac{1}{\theta}\|\eta\|^2,$$

and select $\theta = \dfrac{\varepsilon c(\varepsilon)}{4\|S\|\|K\|}$, we get

$$\dot{U}(\chi_t, \eta_t) \quad \leq \quad -\alpha a(\varepsilon)\|\eta\|^2 - \frac{c(\varepsilon)}{2}\|\chi\|^2 + \frac{8}{\varepsilon^2 c(\varepsilon)}\|S\|^2\|K\|^2\|\eta\|^2.$$

Finally we select α such that

$$\alpha a(\varepsilon) - \frac{8}{\varepsilon^2 c(\varepsilon)}\|S\|^2\|K\|^2 > 0,$$

to deduce that the origin of system (17) is globally asymptotically stable. □

Remark 3.6. As it is stated above, if system (4) satisfies (5) then Assumption 1 is satisfied with $\gamma_1(\varepsilon) = \gamma_2(\varepsilon) = k(1 + \varepsilon + \cdots + \varepsilon^{n-1})$. Let

$$\begin{aligned} c_1(\varepsilon) &= \frac{1}{2\varepsilon} - 2n\gamma_1(\varepsilon)\|P\| - n\gamma_2(\varepsilon)\|P\|, \\ c_2(\varepsilon) &= \frac{1}{2\varepsilon} - 2n\gamma_1(\varepsilon)\|S\| - n\gamma_2(\varepsilon)\|S\|. \end{aligned}$$

It is obvious that $c_1(\varepsilon)$ and $c_2(\varepsilon)$ tend to ∞ as ε tends to zero. This implies that there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$ conditions (8) and (14) are fulfilled.

Remark 3.7. A system in the form (4) is a perturbed system with a nominal linear part. This simplifies the output feedback stabilization problem since no change of coordinates is required.

Remark 3.8. Results similar to those reported in theorems 3.3, 3.4 and 3.5 can be achieved when the function f depends on the input u and satisfies Assumption 1.

3.4. Numerical example

To show the effectiveness of the control law proposed in Theorem 3.5, we apply it to the following example:

$$\begin{aligned} \dot{x}_1 &= x_2 + \tfrac{1}{120}\sin x_3 + \tfrac{1}{60}x_3(t-\tau)\cos u, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= u, \end{aligned}$$

where τ is supposed to be constant. Following the notation used throughout the paper, let $f_1(x,\overline{x},u) = \frac{1}{120}\sin x_3 + \frac{1}{60}x_3(t-\tau)\cos u$, $f_2(x,\overline{x},u) = f_3(x,\overline{x},u) = 0$. Since f_1 depends on x_3, the output feedback scheme in [11] is not applicable. However, it is easy to verify that Assumption 1 holds with $\gamma_1(\varepsilon) = \frac{1}{120\varepsilon^2}$ and $\gamma_2(\varepsilon) = \frac{1}{60\varepsilon^2}$.
Now, select $K = [-6 - 11 - 6]$ and $L = [-8 - 19 - 12]^T$, A_K and A_L are Hurwitz. The solutions of the Lyapunov equations (7) and (13) are given by

$$P = \begin{bmatrix} 0.0738 & 0.0905 & 0.0417 \\ 0.0905 & 2.0845 & 1.2190 \\ 0.0417 & 1.2190 & 1.8774 \end{bmatrix}$$

and

$$S = \begin{bmatrix} 1.5333 & -0.5000 & -0.7000 \\ -0.5000 & 0.7000 & -0.5000 \\ -0.7000 & -0.5000 & 1.7000 \end{bmatrix}$$

So, $\|P\| = 3.2073$ and $\|S\| = 2.3230$. This implies that condition (8) is satisfied for $\varepsilon > 0.6415$ and condition (14) is satisfied for $\varepsilon > 0.4646$. Figure 1 shows the performance of the observer-based controller for a constant delay $\tau = 1$ and $\varepsilon = 0.7$. If we replace f_1 by $\widetilde{f}_1(x,\overline{x},u) = \frac{1}{2}\sin x_3 + x_3(t-\tau)\cos u$, then Assumption 1 is satisfied with $\gamma_1(\varepsilon) = \frac{1}{2\varepsilon^2}$ and $\gamma_2(\varepsilon) = \frac{1}{\varepsilon^2}$. Also, conditions (8) and (14) are verified for all $\varepsilon > 38.4872$.

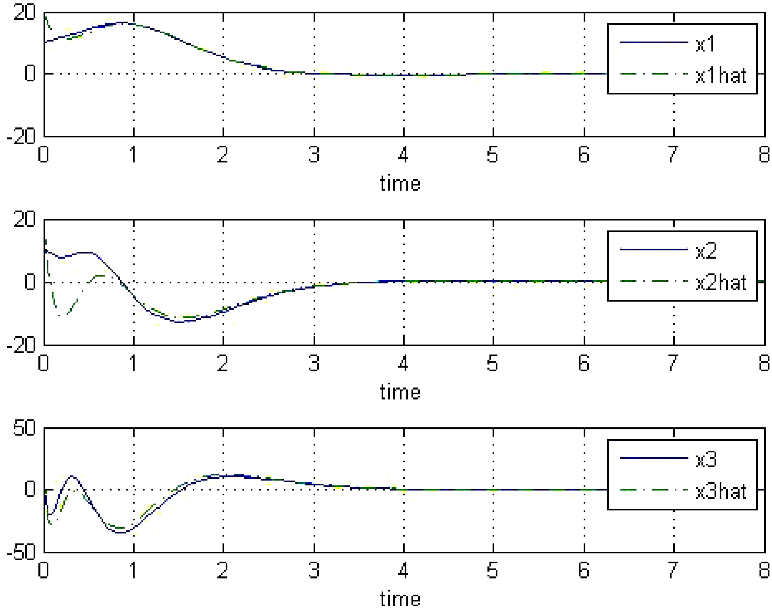

Fig. 1. State trajectories and their estimates.

4. CONCLUSION

In this paper, we have proved a separation principle for a class of time-delay nonlinear systems. This class of systems covers the systems having a triangular structure. We have shown in an illustrated example that this class of systems covers other systems. The stability conditions are delay-independent but the convergence decay can be delay dependent. To observe this dependence, we have to establish exponential stability of the closed loop system (17). This is a natural question, because if the system is delay independent, i. e., (4) is an ordinary differential equation, then U is a classical Lyapunov function that guarantees exponential stability of the closed loop system (17). In order to guarantee exponential stability of (17) we have to require of the derivative of U to satisfy the stronger condition $\dot{U} \leq -\alpha U$, for $\alpha > 0$ (see [17]). This is not possible with the function U given in (18). Using another Lyapunov–Krasovskii functional and applying Theorem 2.4 of [17], we may sharpen Theorem 3.5 and generalize it to get exponential stabilization of system (4).

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